



## K3 Surfaces, Entropy and Glue

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# K3 surfaces, entropy and glue

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# 1 Introduction

In this paper we use the gluing theory of lattices to construct K3 surface automorphisms with small entropy.

**Algebraic integers.** A *Salem number*  $\lambda > 1$  is an algebraic integer which is conjugate to  $1/\lambda$ , and whose remaining conjugates lie on  $S^1$ . There is a unique minimum Salem number  $\lambda_d$  of degree  $d$  for each even  $d$ . The smallest known Salem number is *Lehmer's number*,  $\lambda_{10}$ . These numbers and their minimal polynomials  $P_d(x)$ , for  $d \leq 14$ , are shown in Table 1.

		$P_d(x)$
$\lambda_2$	2.61803398	$x^2 - 3x + 1$
$\lambda_4$	1.72208380	$x^4 - x^3 - x^2 - x + 1$
$\lambda_6$	1.40126836	$x^6 - x^4 - x^3 - x^2 + 1$
$\lambda_8$	1.28063815	$x^8 - x^5 - x^4 - x^3 + 1$
$\lambda_{10}$	1.17628081	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$
$\lambda_{12}$	1.24072642	$x^{12} - x^{11} + x^{10} - x^9 - x^6 - x^3 + x^2 - x + 1$
$\lambda_{14}$	1.20002652	$x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1$

Table 1. The smallest Salem numbers by degree, and their minimal polynomials.

**Surface dynamics.** Now let  $F : X \rightarrow X$  be an automorphism of a compact complex surface. It is known that the topological entropy  $h(F)$  is determined by the spectral radius of  $F^*$  acting on  $H^*(X)$ . More precisely, we have

$$h(F) = \log \rho(F^*|H^2(X)), \quad (1.1)$$

and if  $h(F) > 0$ , then a minimal model for  $X$  is either a K3 surface, an Enriques surface, a complex torus or a rational surface [Ca]. The lower bound

$$h(F) \geq \log \lambda_{10} \quad (1.2)$$

holds for all surface automorphisms of positive entropy, by [Mc3].

In this paper, we will show that the lower bound (1.2) can be achieved on a K3 surface.

**Theorem 1.1** *There exists an automorphism of a K3 surface with entropy  $h(F) = \log \lambda_{10}$ .*

Although the entropy in Theorem 1.1 is the minimum possible, the associated K3 surface is not projective. For projective surfaces, we will show:

**Theorem 1.2** *There exists an automorphism of a projective K3 surface with entropy  $h(F) = \log \lambda_6$ .*

As a complement, we note:

**Theorem 1.3** *There exists an automorphism of a complex torus  $\mathbb{C}^2/\Lambda$  with  $h(F) = \log \lambda_6$ , and an automorphism of an Abelian surface with  $h(F) = \log \lambda_4$ . In each case, no smaller positive entropy is possible.*

In particular, the automorphisms provided by Theorems 1.1 and 1.2 have lower entropy than any example that can be obtained from a complex torus automorphism by passing to the associated Kummer surface (cf. [Mc2, §4]).

**Proof of Theorem 1.3.** For the first example, let  $A \in \mathrm{SL}_4(\mathbb{Z})$  be a matrix with  $\det(xI - A) = x^4 + x + 1$ . Then  $A$  gives an automorphism  $F$  of  $X = \mathbb{R}^4/\mathbb{Z}^4$  preserving a complex structure, since the roots of  $P$  occur in conjugate pairs; and the characteristic polynomial of  $\wedge^2 A$  is  $P_6(x)$ , so  $h(F) = \log \lambda_6$  (compare [Mc2, §5]). No smaller entropy can arise, since  $\exp h(F)$  must be a Salem number of degree at most  $\dim H^2(X) = 6$ .

For the second example, let  $\zeta_d = \exp(2\pi i/d)$ , let  $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ , let  $X = E \times E$ , and let  $A \in \mathrm{M}_2(\mathbb{Z}[\zeta_3])$  be any matrix with  $(\mathrm{tr} A, \det A) = (1, \zeta_6)$ . Then the largest eigenvalue of  $A$  satisfies  $|\lambda|^2 = \lambda_4$ . It follows that the induced automorphism  $F : X \rightarrow X$  has entropy  $h(F) = \log \lambda_4$ . No smaller entropy is possible because, in the projective case, the entropy is given by the log of the leading eigenvalue of  $F^*$  acting on the Néron-Severi group  $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$ , and the rank of  $\mathrm{NS}(X)$  is at most four. ■

It is known that the lower bound (1.2) can be realized on a rational surface [BK, Appendix], [Mc3], but not on an Enriques surface [Og, Thm 1.2]. At present there is no known automorphism  $F$  of a projective K3 surface with  $0 < h(F) < \log \lambda_6$ .

**Glue groups.** To explain how the examples underlying Theorems 1.1 and 1.2 were found, suppose  $F : X \rightarrow X$  is a K3 surface automorphism of positive entropy, and let  $f = F^*$  acting on the even unimodular lattice  $L = H^2(X, \mathbb{Z})$  of signature  $(3, 19)$ . Then we can write  $S(x) = \det(xI - f) = S_1(x)S_2(x)$ , where  $S_1(x)$  is a Salem polynomial and  $S_2(x)$  is a product of cyclotomic polynomials  $C_n(x)$ . There is a corresponding splitting  $f = f_1 \oplus f_2$ , leaving

invariant a sum of lattices  $L_1 \oplus L_2$  with finite index in  $L$ . Passing to the glue groups  $G(L_i) = L_i^\vee/L_i$ , we obtain an isomorphism

$$\phi : G(L_1) \rightarrow G(L_2)$$

intertwining the quotient actions of  $f_1$  and  $f_2$ . If these glue groups happen to be nontrivial vector spaces over  $\mathbb{F}_p$ , then  $S_1(x)$  and  $S_2(x)$  must have a common factor when reduced mod  $p$ . (Compare [Og, §4]).

In these terms, Theorems 1.1 and 1.2 were suggested by the fact that, when reduced modulo  $p = 3$ , the Salem polynomial  $P_{10}(x)$  is divisible by  $C_3(x) = x^2 + x + 1$ , and  $P_6(x)$  divides  $C_{13}(x) = (x^{13} - 1)/(x - 1)$ .

To actually construct examples, in §2–§4 we develop the general theory of equivariant gluing, Coxeter groups and twists. These results provide tools for producing a model  $f : L \rightarrow L$  of the desired lattice automorphism  $F^* : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ . Care must be taken to construct a candidate for the Kähler cone of  $X$  (§5). Then the strong Torelli theorem and surjectivity of the period map (reviewed in §6) show one can realize  $f : L \rightarrow L$  by a holomorphic automorphism  $F : X \rightarrow X$  of a K3 surface. Detailed constructions adapted to the Salem numbers  $\lambda_{10}$  and  $\lambda_6$  are given in §7 and §8.

Many variations on these constructions, adapted to other Salem numbers and to other properties of the resulting K3 surface, remain to be explored.

**Notes and references.** This paper is a sequel to [Mc2] and [GM], and was inspired by Oguiso’s recent example of a K3 surface automorphism with entropy  $\log \lambda_{14}$  [Og]. I would like to thank B. Gross for many useful discussions, and for pointing out the positive automorphism of  $A_2 \oplus A_2$  used in §7.

## 2 Lattices and glue

We begin by reviewing the construction of lattices and their automorphisms using glue groups. This technique goes back to Witt and Kneser [Kn]; for more details see e.g. [CoS].

**Lattices.** A *lattice*  $L$  of rank  $r$  is a free abelian group  $L \cong \mathbb{Z}^r$ , equipped with a nondegenerate inner product  $\langle x, y \rangle$  taking values in  $\mathbb{Z}$ . The inner product determines natural inclusions

$$L \subset L^\vee \subset L \otimes \mathbb{Q} \tag{2.1}$$

where

$$L^\vee = \text{Hom}(L, \mathbb{Z}) \cong \{x \in L \otimes \mathbb{Q} : \langle x, L \rangle \subset \mathbb{Z}\}.$$

We say  $L$  has signature  $(p, q)$  if the associated quadratic form

$$x^2 = \langle x, x \rangle$$

on  $L \otimes \mathbb{R}$  is equivalent to

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2. \quad (2.2)$$

**The glue group.** The finite abelian group  $G(L) = L^\vee/L$  is the *glue group* of  $L$ . It comes equipped with a nondegenerate *fractional form*  $\langle\langle x, y \rangle\rangle$  taking values in  $\mathbb{Q}/\mathbb{Z}$ , characterized by

$$\langle\langle x, y \rangle\rangle = \langle \tilde{x}, \tilde{y} \rangle \bmod 1$$

for any  $\tilde{x}, \tilde{y} \in L^\vee$  representing  $x, y \in G(L)$ .

Concretely, if  $(e_i)$  is an integral basis for  $L$  with Gram matrix  $B_{ij} = \langle e_i, e_j \rangle$ , and  $d_i \in G(L)$  are the classes represented by a dual basis for  $L^\vee$ , then the glue group has order

$$|G(L)| = \det(L) = |\det B_{ij}|,$$

and its fractional form is given by

$$\langle\langle d_i, d_j \rangle\rangle = (B^{-1})_{ij} \bmod 1.$$

**Primary decomposition** The glue group can be written canonically as an orthogonal direct sum of  $p$ -groups,

$$G(L) = \bigoplus G(L)_p,$$

where  $p$  ranges over the primes dividing  $\det(L)$ . The fractional form on  $G(L)_p$  takes values in  $\mathbb{Z}[1/p^e]/\mathbb{Z}$  for some  $e$ .

In the special case where every element of  $G(L)_p$  has order  $p$ , we can regard  $G(L)_p$  as a vector space over  $\mathbb{F}_p$ , and consider the fractional form as an inner product with values in  $\mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{F}_p$ ; see §3.

**Extensions of  $L$ .** The glue group provides a useful description of all the lattices  $M \supset L$  such that  $M/L$  is finite. Indeed, since  $M$  pairs integrally with  $L$ , any such extension can be regarded as a subgroup of  $L^\vee$ ; and the condition that the inner product on  $M$  is integral is equivalent to the condition that

$$\overline{M} = M/L \subset G(L)$$

is *isotropic*, i.e.  $\langle\langle x, y \rangle\rangle = 0$  for all  $x, y \in \overline{M}$ . Thus we have a bijective correspondence:

$$\left\{ \begin{array}{l} \text{Lattices } M \text{ with} \\ L \subset M \subset L^\vee \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Isotropic subgroups } \overline{M} \text{ with} \\ 0 \subset \overline{M} \subset G(L) \end{array} \right\}.$$

Note that  $[M : L] = |\overline{M}|$ ,  $\det(M) = \det(L)/[M : L]^2$ , and the glue group of the extension is given by

$$G(M) \cong \overline{M}^\perp / \overline{M}.$$

**Gluing a pair of lattices.** Now suppose  $L = L_1 \oplus L_2$ . A *gluing map* is an isomorphism  $\phi : H_1 \rightarrow H_2$  between a pair of subgroups  $H_i \subset G(L_i)$ ,  $i = 1, 2$ , satisfying

$$\langle\langle x, y \rangle\rangle = -\langle\langle \phi(x), \phi(y) \rangle\rangle. \quad (2.3)$$

This condition guarantees that

$$\overline{M} = \{(x, \phi(x)) : x \in H_1\} \subset G(L_1) \oplus G(L_2) = G(L)$$

is isotropic, and hence  $\phi$  determines a lattice

$$M = L_1 \oplus_\phi L_2$$

obtained by *gluing*  $L_1$  and  $L_2$  along  $H_1 \cong H_2$ . The extension  $L_1 \oplus L_2 \subset M$  is *primitive* in the sense that  $L_i = M \cap (L_i \otimes \mathbb{Q})$ , or equivalently  $M/L_i$  is torsion-free. Any primitive extension arises in this way, and hence we also have a natural correspondence:

$$\left\{ \begin{array}{l} \text{Primitive extensions} \\ L_1 \oplus L_2 \subset M \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Gluing maps } \phi : H_1 \rightarrow H_2 \text{ between} \\ \text{subgroups of } G(L_1) \text{ and } G(L_2) \end{array} \right\}.$$

**Even lattices.** A lattice  $L$  is *even* if  $\langle x, x \rangle \in 2\mathbb{Z}$  for all  $x \in L$ . In this case we have a natural quadratic form  $q : G(L) \rightarrow \mathbb{Q}/\mathbb{Z}$  defined by

$$q(x) = (1/2)\langle \tilde{x}, \tilde{x} \rangle \bmod 1. \quad (2.4)$$

An extension  $L \subset M$  is even iff  $q|\overline{M} = 0$ ; similarly, a gluing  $M = L_1 \oplus_\phi L_2$  of even lattices is even iff  $q(x) + q(\phi(x)) = 0$  for all  $x \in H_1$ .

Note that  $M \supset L$  is even whenever  $L$  is even and  $d = [M : L]$  is odd, for in this case we have  $(dx)^2 = x^2 \bmod 2$ .

Since  $q(x + y) = q(x) + q(y) + \langle\langle x, y \rangle\rangle$ , the fractional form determines  $q|G(L)_p$  for all odd primes  $p$  (but not for  $p = 2$ ).

**Extending isometries.** A bijective map from one lattice to another is an *isometry* if it preserves the inner product and group structure.

The *orthogonal group*  $O(L)$  consists of the isometries  $f : L \rightarrow L$ . For simplicity, we also use  $f$  to denote its linear extensions to  $L^\vee, L \otimes \mathbb{R}, L \otimes \mathbb{C}$ , etc. We let  $\bar{f}$  denote the induced isometry of  $G(L)$ .

An isometry  $f \in O(L)$  extends to  $M \supset L$  iff  $\bar{f}(\bar{M}) = \bar{M}$ . Similarly, equivariant gluing maps allow one to glue together isometries; we have a natural correspondence:

$$\left\{ \begin{array}{l} \text{Extensions } f \in O(M) \text{ of} \\ f_1 \oplus f_2 \in O(L_1 \oplus L_2) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Gluing maps } \phi : H_1 \rightarrow H_2 \\ \text{satisfying } \phi \circ f_1 = f_2 \circ \phi \end{array} \right\}.$$

**Roots and the Weyl group.** A vector  $e \in L$  is a *root* if  $\langle e, e \rangle = \pm 1$  or  $\pm 2$ . Any root determines an isometric reflection  $s \in O(L)$  by the formula

$$s(x) = x - \frac{2\langle x, e \rangle}{\langle e, e \rangle} e.$$

The subgroup generated by all such reflections is the *Weyl group*  $W(L) \subset O(L)$ . Note that  $s(x) - x$  is an integral multiple of  $e$  for all  $x \in L^\vee$ . This shows:

*The Weyl group acts trivially on the glue group.*

**Root lattices.** We say  $L$  is a *root lattice* if it has an integral basis of roots. We conclude with some examples of root lattices that will be useful later. For more details, see [CoS], [Hum].

**Odd unimodular lattices.** Let  $\mathbb{Z}^{p,q}$  denote  $\mathbb{Z}^n$  with the inner product associated to the quadratic form (2.2). This is an odd unimodular root lattice, so it has trivial glue group.

**Coxeter diagrams.** Let  $\Gamma$  be a graph with vertices labeled  $1, 2, \dots, n$ . Then  $\Gamma$  determines a symmetric form with matrix

$$B_{ij} = \langle e_i, e_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are joined by an edge, and} \\ 0 & \text{otherwise.} \end{cases}$$



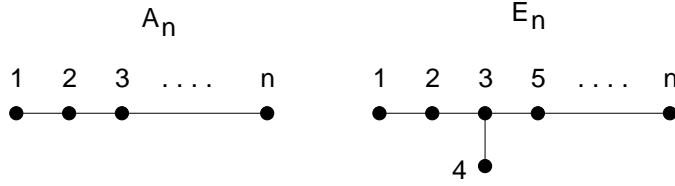


Figure 2. Diagrams for the root lattices  $A_n$  and  $E_n$ .

Provided  $\det B_{ij} \neq 0$ , this form makes  $L = \oplus \mathbb{Z}e_i$  into an even root lattice of rank  $n$ . The product of the basic reflections  $s_i$  determined by  $e_i$  yields the *Coxeter element*

$$f = s_1 s_2 \cdots s_n \in W(L) \subset O(L).$$

If  $\Gamma$  is a tree, then the conjugacy class of  $f$  is independent of the ordering of the vertices of  $\Gamma$ . Since  $f$  lies in the Weyl group,  $\bar{f}$  acts trivially on  $G(L)$ .

**$A_n$  and  $E_n$ .** The diagrams for the lattices  $A_n$  and  $E_n$  are shown in Figure 2. The  $A_n$  lattice can be regarded as the sublattice of  $\mathbb{Z}^{n+1}$  defined by the equation  $\sum x_i = 0$ . Equivalently,  $A_n$  is the orthogonal complement of  $v_n = (1, 1, \dots, 1)$ . Since  $\mathbb{Z}^{n+1}$  is unimodular, this shows

$$\mathbb{Z}^{n+1} = A_n \oplus_{\phi} (\mathbb{Z}v_n)$$

where  $\phi : G(A_n) \rightarrow G(\mathbb{Z}v_n)$  is an isomorphism. Since  $\langle v_n, v_n \rangle = n + 1$ , this implies

$$G(A_n) \cong G(\mathbb{Z}v_n) \cong \mathbb{Z}/(n + 1).$$

Similarly,  $E_n$  can be regarded as the sublattice of  $\mathbb{Z}^{n,1}$  perpendicular to

$$k_n = (1, 1, 1, \dots, 1, -3).$$

(This vector represents the canonical class on the blowup of  $\mathbb{P}^2$  at  $n$  points; cf. [Mc3, §3].) Note that  $\langle k_n, k_n \rangle = n - 9$ . Excluding the case  $n = 9$  (since the bilinear form on  $E_9$  is degenerate), we find that

$$G(E_n) \cong \mathbb{Z}/|9 - n|.$$

The signature of  $E_n$  is  $(n, 0)$  for  $n \leq 8$  and  $(n - 1, 1)$  for  $n \geq 10$ .

**Even unimodular lattices.** The inner product  $\langle e_i, e_j \rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  on  $\mathbb{Z}^2$  gives the unique even unimodular lattice  $H$  of signature  $(1, 1)$ . More generally, for any  $p, q \geq 1$  with  $p \equiv q \pmod{8}$ , there is a unique even unimodular lattice  $\Pi_{p,q}$  of signature  $(p, q)$  [MH], [Ser, §5].

We have just seen that  $E_8$  and  $E_{10}$  are unimodular, so we have  $E_{10} \cong E_8 \oplus H \cong \Pi_{9,1}$ ; and in general  $\Pi_{p,q} \cong aE_8 \oplus bH$  for suitable integers  $a, b$ .

### 3 Isometries over finite fields

In this section we give a criterion for certain lattice automorphisms to automatically glue together.

**Theorem 3.1** *Let  $f_i \in O(L_i)$ ,  $i = 1, 2$  be a pair of lattice isometries, and let  $p$  be a prime. Suppose*

1. *Each glue group  $G(L_i)_p$  is a vector space over  $\mathbb{F}_p$ ;*
2. *The maps  $\bar{f}_i$  on  $G(L_i)_p$  have the same characteristic polynomial  $S(x)$ ; and*
3.  *$S(x) \in \mathbb{F}_p[x]$  is a separable polynomial, with  $S(1)S(-1) \neq 0$ .*

*Then there is a gluing map  $\phi : G(L_1)_p \rightarrow G(L_2)_p$  such that  $f_1 \oplus f_2$  extends to  $L_1 \oplus_\phi L_2$ .*

The proof is based on general properties of isometries over finite fields.

**Inner products spaces.** Let  $k$  be a field. An *inner product space* over  $k$  is a finite-dimensional vector space  $V$  equipped with nondegenerate, symmetric bilinear form  $\langle x, y \rangle : V \times V \rightarrow k$ . With respect to a basis, the form is given by a symmetric matrix  $B_{ij} = \langle e_i, e_j \rangle$ ; and the class

$$\det(V) = [\det B_{ij}] \in k^*/(k^*)^2$$

is an invariant of  $V$ .

**Example.** The sum  $W = V \oplus V^\vee$  of a vector space with its dual carries a natural *split* inner product with  $\det(W) = (-1)^{\dim V}$ . Its matrix with respect to a pair of dual bases is given by  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ .

**Polynomials.** Given a degree  $d$  monic polynomial  $S \in k[x]$  with  $S(0) \neq 0$ , let

$$S^*(x) = x^d S(x^{-1})/S(0).$$

This is again a monic polynomial, whose roots are the inverses of the roots of  $S$ . If  $S = S^*$  we say  $S$  is a *reciprocal* polynomial. In this case  $S(0) = \pm 1$ . If  $S(1)S(-1) \neq 0$ , then the degree  $d = 2e$  of  $S$  is even, and there is a unique *trace polynomial*  $R$  (of degree  $e$ ) such that

$$S(x) = x^e R(x + x^{-1}).$$

**Isometries.** Let  $f : V \rightarrow V$  be an isometry. Then  $f^\vee = f^{-1}$ , and hence the characteristic polynomial  $S(x) = \det(xI - f)$  is reciprocal. Similarly

$$(\text{Ker } P(f))^\perp = \text{Im}(P(f)^\vee) = \text{Im } P^*(f) \quad (3.1)$$

for any  $P \in k[x]$ .

**Finite fields.** Now let  $f : V \rightarrow V$  be an isometry of an inner product space over a *finite* field  $k$ .

We first note that  $V$  is almost determined, up to isometry, by its dimension. In fact:

1. If  $\text{char } k$  is odd, then  $V$  is uniquely determined by  $\dim(V)$  and by  $\det(V) \in k^*/(k^*)^2 \cong \mathbb{Z}/2$ ; while
2. If  $\text{char } k$  is 2, then  $V$  is uniquely determined by  $\dim(V)$  and the parity of  $V$  (which is even if  $\langle x, x \rangle = 0$  for all  $x \in V$ , and otherwise odd). Even forms exists only in even dimensions.

See e.g. [MH, App. 2], [Ger, §2.8].

We now turn to the problem of classifying the *pair*  $(V, f)$  up to isometry.

**Proposition 3.2** *If  $S(x) = \det(xI - f)$  is irreducible and  $\dim V > 1$ , then  $(V, f)$  is determined up to isometry by  $S$ .*

**Proof.** We claim  $(V, f)$  is isometric to  $(K, g)$ , where  $K = k[t]/S(t)$ ,  $g(x) = tx$  and the inner product on  $K$  is given by  $\langle x, y \rangle_K = \text{Tr}_k^K(xy')$ . Here  $x \mapsto x'$  is the Galois involution on  $K$  sending  $t$  to  $t^{-1}$ , whose existence is guaranteed by the fact that  $S(t)$  is a reciprocal polynomial.

To make this identification, first observe that  $t \mapsto f$  gives an isomorphism  $K \cong k[f] \subset \text{End}_k(V)$  sending the Galois involution to the adjoint involution (since  $f^\vee = f^{-1}$ ). Upon choosing a nonzero vector  $v \in V$ , we obtain an isomorphism  $V \cong Kv \cong K$  sending  $f$  to  $g$ . By nondegeneracy of the trace form, there is then a unique  $k$ -linear map  $\xi : K \rightarrow K$  such that

$$\langle x, y \rangle = \text{Tr}_k^K(\xi(x)y').$$

Using the fact that  $\langle f(x), f(y) \rangle = \langle x, y \rangle = \langle y, x \rangle$ , we find that  $\xi(x) = bx$  where  $b = b' \in K$ . Since  $\deg(S) > 1$ , the Galois involution is nontrivial, and hence  $b = aa'$  for some  $a \in K$  (as a counting argument shows). But then we can simply replace  $v$  by  $av$  to obtain a new identification  $V \cong Kav \cong K$  such that  $\langle x, y \rangle = \text{Tr}_k^K(xy')$ . ■

**Proposition 3.3** *If  $\det(xI - f) = Q(x)Q^*(x)$ , where  $Q(x)$  and  $Q^*(x)$  are distinct irreducible monic polynomials, then  $(V, f)$  is determined up to isometry by  $Q(x)$ .*

**Proof.** In this case  $V = \text{Ker } Q(f) \oplus \text{Ker } Q^*(f) = W \oplus W^\vee$ , where the inner product identifies the second summand with the dual of the first. Since the linear map  $f|_W$  is determined by  $Q(x)$ , the pair  $(V, f)$  is determined up to isometry by the same information. ■

**Proposition 3.4** *If  $S(x) = \det(xI - f)$  is separable and  $S(1)S(-1) \neq 0$ , then  $(V, f)$  is determined up to isometry by  $S$ .*

**Proof.** Since  $S$  is a separable, reciprocal polynomial, it factors as a product of distinct irreducible polynomials

$$S(x) = S_1(x) \cdots S_r(x) Q_1(x) Q_1^*(x) \cdots Q_s(x) Q_s^*(x)$$

where  $S_i = S_i^*$ . Thus  $V$  splits as an  $f$ -invariant orthogonal direct sum

$$V = \left( \bigoplus_1^r \text{Ker } S_i(f) \right) \oplus \left( \bigoplus_1^s \text{Ker } Q_i(f) Q_i^*(f) \right).$$

(Orthogonality follows from (3.1).) The assumption  $S(1)S(-1) \neq 0$  insures  $\dim \text{Ker } S_i(f) > 1$  for each  $i$ . Thus the preceding two propositions can be applied, term to term, to show that  $(V, f)$  is determined up to isometry by  $S$ . ■

**Proof of Theorem 3.1.** The fractional form makes  $G(L_i)_p$  into an inner product space over  $\mathbb{F}_p \cong \mathbb{Z}[1/p]/\mathbb{Z}$ . Since  $\bar{f}_i$  acts isometrically, we may apply the preceding result (after reversing the sign of one of the forms) to obtain the desired gluing map  $\phi$ . ■

**The glue group of  $A_{p-1}$ .** In the absence of an automorphism, the isometry type of a glue group may need to be determined directly. For later use, we record a particular case:

**Proposition 3.5** *The fractional form makes  $V = G(A_{p-1})$  into an inner product space over  $k = \mathbb{F}_p$  with  $\det(V) = [-1] \in k^*/(k^*)^2$ .*

**Proof.** The vector  $x = (1, 1, \dots, 1, 1 - p)/p \in A_{p-1}^\vee \subset \mathbb{R}^p$  satisfies  $\langle x, x \rangle = (p - 1)/p = -1/p \pmod{1}$ . ■

**Notes and references.** The results above can be regarded as special cases of the fact that a Hermitian space over a finite field is determined up to isomorphism by its dimension; see [MH, App. 2].

## 4 Twists

In this section we discuss the twists of a lattice  $L$  by a self-adjoint endomorphism  $a : L \rightarrow L$ . Twisting allows one to adjust the signature and glue group of  $L$  while respecting the action of a given isometry.

**Twisting lattices.** Let  $L$  be a lattice of rank  $r$ . Suppose  $a \in \text{End}(L)$  satisfies  $a = a^\vee$  and  $\det(a) \neq 0$ . Then

$$\langle x, y \rangle_a = \langle ax, y \rangle$$

defines a new inner product on  $L$ , giving us a new lattice  $L(a)$  called the *twist* of  $L$  by  $a$ .

It is easy to see that  $G(L(a)) = L^\vee/aL$  and  $\det(L(a)) = \det(L)|\det(a)|$ . More precisely, we have an exact sequence

$$0 \rightarrow L/aL \rightarrow G(L(a)) \rightarrow G(L) \rightarrow 0, \quad (4.1)$$

which splits if  $\det(a)$  and  $\det(L)$  are relatively prime.

**Twisting isometries.** Now suppose  $L$  is equipped with an isometry  $f : L \rightarrow L$ . Let  $\mathbb{Z}[f] \subset \text{End}(L)$  be the ring generated by  $f$ , and suppose  $a \in \mathbb{Z}[f + f^{-1}]$  and  $\det(a) \neq 0$ . Then  $a = a^\vee$  and  $af = fa$ , so  $f \in O(L(a))$  as well. Thus we can regard  $L$ ,  $L(a)$  and their glue groups as modules over  $\mathbb{Z}[f]$ . With this understanding, (4.1) is an exact sequence of  $\mathbb{Z}[f]$ -modules.

**Proposition 4.1** *If  $a \in \mathbb{Z}[f + f^{-1}]$  and  $L$  is even, then so is  $L(a)$ .*

**Proof.** Write  $a = \sum_0^n a_i(f^i + f^{-i})$  with  $a_i \in \mathbb{Z}$ , and observe that  $\langle f^{-1}x, x \rangle = \langle fx, x \rangle$ ; thus for all  $x \in L$ , we have

$$\langle ax, x \rangle = a_0 \langle x, x \rangle + \sum_1^n 2 \langle f^i x, x \rangle \in 2\mathbb{Z}.$$

■

**Primes and divisors.** For more detailed results, we fix a prime  $p$  not dividing  $\det(L)$ , and let  $P \mapsto \overline{P}$  denote the natural map  $\mathbb{Z}[x] \rightarrow \mathbb{F}_p[x]$ . Then the twist  $M = L(p)$  of a lattice of rank  $r$  satisfies

$$G(M)_p \cong \mathbb{F}_p^r \quad \text{and} \quad \det(xI - \overline{f}|G(M)_p) = \overline{S}(x),$$

where  $S(x) = \det(xI - f)$ .

By twisting with a divisor of  $p$  in the ring  $\mathbb{Z}[f + f^{-1}]$ , we can sometimes arrange that the characteristic polynomial of  $\overline{f}|G(M)_p$  is a given divisor of  $\overline{S}(x)$ . To state a result in this direction, assume that:

1. The polynomial  $\overline{S}(x) \in \mathbb{F}_p[x]$  is separable; and
2. We have  $pL \subset aL$ , where  $a \in \mathbb{Z}[f + f^{-1}]$ .

Then the result of twisting by  $a$  can be described as follows.

**Theorem 4.2** *The lattice  $M = L(a)$  has glue group*

$$G(M) \cong G(M)_p \oplus G(L) \quad (4.2)$$

as a  $\mathbb{Z}[f]$ -module. Moreover  $G(M)_p$  is a vector space over  $\mathbb{F}_p$ , and the characteristic polynomial of  $\overline{f}|G(M)_p$  is given by

$$\overline{Q}(x) = \gcd(\overline{A}(x), \overline{S}(x)) \in \mathbb{F}_p[x],$$

where  $a = A(f) \in \mathbb{Z}[f]$ .

**Proof.** Since  $pL \subset aL$ ,  $\det(a)$  is a power of  $p$ ; and since  $p$  does not divide  $\det(L)$ , the exact sequence (4.1) splits, which gives (4.2). The assumption  $pL \subset aL$  also implies that  $G(M)_p \cong L/aL$  is a quotient of  $V = L/pL \cong \mathbb{F}_p^r$ , so it is a vector space over  $\mathbb{F}_p$ . By separability, we have  $V \cong \mathbb{F}_p[x]/(\overline{S})$ , and hence

$$G(M)_p \cong V/aV \cong \mathbb{F}_p[x]/(\overline{A}, \overline{S}) \cong \mathbb{F}_p[x]/(\overline{Q})$$

as modules over  $\mathbb{Z}[f]$ . ■

**Dedekind domains.** The existence of a desired twist is guaranteed in certain situations by the following result.

**Theorem 4.3** *Suppose  $\mathcal{O} = \mathbb{Z}[f + f^{-1}]$  is a Dedekind domain of class number one,*

$$\overline{S}(x) = \det(xI - f) \bmod p$$

*is separable,  $\overline{S}(1)\overline{S}(-1) \neq 0$  and  $\gcd(p, \det L) = 1$ . Let  $\overline{S}_1(x)$  be a reciprocal factor of  $\overline{S}(x)$ . Then there exists a twist  $M = L(a)$ , with  $a \in \mathbb{Z}[f + f^{-1}]$  dividing  $p$ , such that*

$$\overline{S}_1(x) = \det(xI - \overline{f}|G(M)_p). \quad (4.3)$$

**Proof.** Let  $R(y)$  be the trace polynomial associated to  $S(x)$ , so  $S(x) = x^e R(x + x^{-1})$ . Let  $\overline{R} = \overline{R}_1 \overline{R}_2$  be the factorization of  $\overline{R}$  corresponding to

the given factorization  $\overline{S} = \overline{S}_1 \overline{S}_2$ . Then  $\mathcal{O} \cong \mathbb{Z}[y]/(R)$ , so by basic number theory (see e.g. [La, I, §8]), there is a factorization  $p = a_1 a_2$  in  $\mathcal{O}$  such that

$$\mathcal{O}/(a_i \mathcal{O}) \cong \mathbb{F}_p[y]/(\overline{R}_i)$$

for  $i = 1, 2$ . Equivalently, if  $a_i = A_i(f + f^{-1})$  with  $A_i \in \mathbb{Z}[y]$ , then  $(\overline{A}_i) = (\overline{R}_i)$  as ideals in  $\mathbb{F}_p[y]/(\overline{R})$ .

Now we can also write  $a_1 = A(f)$ , since  $\mathbb{Z}[f^{-1}] = \mathbb{Z}[f]$ . Then  $A(x) = A_1(x + x^{-1})$  in the ring  $\mathbb{Z}[x]/(S)$ . Consequently

$$(\overline{A}(x)) = (\overline{A}_1(x + x^{-1})) = (\overline{R}_1(x + x^{-1})) = (\overline{S}_1(x))$$

as ideals in  $\mathbb{F}_p[x]/(\overline{S})$ , and hence  $\overline{S}_1 = \gcd(\overline{A}, \overline{S})$ , which gives (4.3) for  $M = L(a_1)$ .  $\blacksquare$

**Signature.** To conclude, we relate the signatures of  $L$  and  $L(a)$ .

Let  $f : L \rightarrow L$  be an isometry of a lattice of signature  $(p, q)$  such that  $S(x) = \det(xI - f)$  is separable and  $S(1)S(-1) \neq 0$ .

Since  $S(x)$  is reciprocal, it has  $2s$  roots outside  $S^1$  and  $2t$  roots on  $S^1$ . The map  $\lambda \mapsto \lambda + \lambda^{-1}$  sends the roots on  $S^1$  to a set  $T \subset (-2, 2)$  with  $|T| = t$ . As in [GM], we define the *sign invariant*  $\epsilon_L : T \rightarrow \langle \pm 1 \rangle$  by

$$\epsilon_L(\tau) = \begin{cases} +1 & \text{if } E_\tau \text{ has signature } (2, 0), \\ -1 & \text{if } E_\tau \text{ has signature } (0, 2), \end{cases}$$

where

$$E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L \otimes \mathbb{R} \cong \mathbb{R}^{p,q}.$$

Then the signature of  $L$  is given by

$$(p, q) = (s, s) + \sum_T \begin{cases} (2, 0) & \text{if } \epsilon_L(\tau) = +1, \\ (0, 2) & \text{if } \epsilon_L(\tau) = -1. \end{cases} \quad (4.4)$$

Now for any twisting parameter  $a = A(f + f^{-1}) \in \mathbb{Z}[f + f^{-1}]$ , define  $\epsilon_a : T \rightarrow \langle \pm 1 \rangle$  so that  $\epsilon_a(\tau)A(\tau) > 0$ . We then have

$$\epsilon_{L(a)}(\tau) = \epsilon_L(\tau)\epsilon_a(\tau), \quad (4.5)$$

and by (4.4) this determines the signature of  $L(a)$ .

## 5 Positivity

In this section we discuss the notion of a positive automorphism of a Euclidean or Lorentzian lattice. (The Lorentzian case is not needed until §8).

**The Euclidean case.** To begin with, assume  $L$  is an even, positive-definite lattice. Let

$$\Phi = \{y \in L : y^2 = 2\} \tag{5.1}$$

be the finite set of roots in  $L$ . We say  $\Phi^+ \subset \Phi$  is a *system of positive roots* if there is an  $x \in L$  such that

$$\Phi = \Phi^+ \cup (-\Phi^+) \quad \text{and} \quad \langle x, y \rangle > 0 \quad \forall y \in \Phi^+. \tag{5.2}$$

Such an  $x$  exists iff the convex hull of  $\Phi^+$  does not contain the origin.

We say an isometry  $f \in O(L)$  is *positive* if it preserves a system of positive roots.

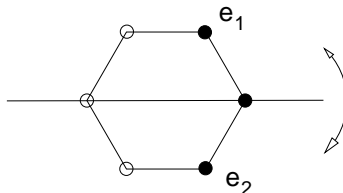


Figure 3. Reflection through the  $x$ -axis preserves a set of positive roots in the  $A_2$  lattice.

**Example.** The hexagonal root lattice  $A_2$  admits a positive involution  $f$  which interchanges the basic roots  $e_1$  and  $e_2$ ; see Figure 3. This map is not in the Weyl group  $W(L)$ ; it comes from a symmetry of the  $A_2$  diagram. We have  $\bar{f}(x) = -x$  on the glue group  $G(A_2) \cong \mathbb{Z}/3$ , and in fact  $f$  generates  $O(L)/W(L) \cong \mathbb{Z}/2$ .

**Basic properties.** If  $L = L_1 \oplus L_2$ , and  $f_i \in O(L_i)$  are positive, then so is  $f = f_1 \oplus f_2$ . This is because every root of  $L$  lies in  $L_1$  or  $L_2$ .

So long as  $L$  has at least one root, any positive map  $f \in O(L)$  has 1 as an eigenvalue, since  $f$  must fix  $\sum\{y : y \in \Phi^+\}$ . By splitting along this eigenspace, we can present  $(L, f)$  as a gluing  $(L_1 \oplus_\phi L_2, f_1 \oplus f_2)$  where  $f_1$  is the identity map and  $L_2$  has no roots. Conversely, any map of this form is positive.

**The Lorentzian case.** We now turn to the case where  $L$  is an even lattice of signature  $(n, 1)$ .



In this case we say  $\Phi^+$  is a *positive root system* if it satisfies (5.2) for some  $x \in L$  with  $x^2 < 0$ . Geometrically, the roots  $y \in \Phi$  define a locally finite system of hyperplanes  $y^\perp$  in the hyperbolic space  $\mathbb{H}^n \subset \mathbb{P}(L \otimes \mathbb{R})$ , cutting it into open chambers. The choice of a positive root system (up to sign) is the same as the choice of one of those chambers; and a chamber can be specified by giving a representative point  $[x] \in \mathbb{H}^n$  satisfying  $\langle x, \Phi^+ \rangle > 0$ .

Note that  $\Phi$  excludes any roots of  $L$  with  $y^2 = -2$ .

**Example.** Let  $L$  be the Lorentz lattice  $\mathbb{Z}^2$  with  $(a, b)^2 = 2(a^2 + ab - b^2)$ . Its roots  $\Phi$  include the Fibonacci pairs  $(1, 1), (2, 3), (5, 8)$ , etc. Let  $e_1 = (1, 0), e_2 = (0, 1)$ , and let  $\Phi_i = \{y \in \Phi : \langle e_i, y \rangle > 0\}$ . Then  $\Phi_2$  is a positive root system but  $\Phi_1$  is not, essentially because  $e_2^2 < 0$  but  $e_1^2 > 0$ .

**The light cone condition.** In the Lorentzian case, we say  $f : L \rightarrow L$  is *positive* if it preserves a positive root system *and* it stabilizes each component of the light cone defined by  $x^2 < 0$  in  $L \otimes \mathbb{R} \cong \mathbb{R}^{n,1}$ . (Thus  $f(x) = -x$  is never positive).

**Gluing.** Let  $L = L_1 \oplus_\phi L_2$  be an even lattice obtained by gluing a Lorentzian lattice  $L_1$  to a Euclidean lattice  $L_2$ . We wish to investigate when an automorphism  $f = f_1 \oplus f_2$  of  $L$  is positive.

We remark that the timelike vectors in any Lorentzian lattice satisfy a reverse Cauchy-Schwarz inequality:

$$x^2, y^2 \leq 0 \implies |x^2 y^2| \leq |\langle x, y \rangle|^2, \quad (5.3)$$

as is easily verified by reducing to the case  $x = e_{n+1}$  in  $\mathbb{R}^{n,1}$ .

**Theorem 5.1** *Suppose  $x^2 \in 2a_i \mathbb{Z}$  for all  $x \in L_i$ ,  $a_i > 1$ ,  $bL \subset L_1 \oplus L_2$ , and  $b^2 \notin \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$ . Then  $f_1 \oplus f_2$  is a positive automorphism of  $L$  provided  $f_1$  has an eigenvalue  $\lambda > 1$ .*

Here  $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$  denotes the set of positive integers.

**Proof.** Let  $y \in \Phi$  be a root of  $L$ . Then  $y = (y_1, y_2) \in L_1^\vee \oplus L_2^\vee$  satisfies  $y^2 = y_1^2 + y_2^2 = 2$ , and  $by \in L_1 \oplus L_2$ . Therefore

$$(by)^2 = 2b^2 = (by_1)^2 + (by_2)^2 = 2(a_1 n_1 + a_2 n_2)$$

with  $n_1, n_2 \in \mathbb{Z}$  and  $n_2 \geq 0$ . By assumption, this equation has no solutions with  $n_1, n_2 > 0$ . Our assumptions also imply that  $L_1$  and  $L_2$  have no roots, so we cannot have  $n_1 = 0$  or  $n_2 = 0$ . Thus we must have  $n_1 < 0$  and hence  $y_1^2 < 0$ ; more precisely, we have  $y_1^2 \leq -a_1/b^2$ .

Pick any  $x_1 \in L_1$  with  $x_1^2 < 0$ . Then by (5.3), we have

$$|\langle x_1, y \rangle|^2 = |\langle x_1, y_1 \rangle|^2 \geq |x_1^2 y_1^2| \geq a_1 |x_1^2| / b_2 > 0 \quad \forall y \in \Phi.$$

Thus  $\Phi^+ = \{y \in \Phi : \langle x_1, y \rangle > 0\}$  is a positive root system for  $L$ . It consists exactly of the roots which project into the component of the light cone of  $L_1 \otimes \mathbb{R}$  as  $x_1$ . Since  $f_1$  has an eigenvalue  $\lambda > 1$ , it preserves this component, and hence  $f$  preserves  $\Phi^+$  and is positive.  $\blacksquare$

Geometrically, the proof shows that the inclusion  $L_1 \subset L$  determines an  $f$ -invariant hyperbolic subspace  $\mathbb{H}^{n_1} \subset \mathbb{H}^n$  lying completely inside one of the chambers defined by  $\Phi$ .

**Theorem 5.2** *Suppose  $f_1$  and  $f_2$  are positive, and every root  $(y_1, y_2) \in L$  which is not in  $L_1 \oplus L_2$  satisfies  $y_2^2 \geq 2$ . Then  $f_1 \oplus f_2$  is a positive automorphism of  $L$ .*

**Proof.** Let  $\Phi$  denote the roots of  $L$ , let  $\Phi_i = \Phi \cap L_i$  for  $i = 1, 2$ , and let

$$\Phi_3 = \{(y_1, y_2) \in \Phi : y_1^2 \leq 0, y_1 \neq 0\}.$$

Consider any root  $y = (y_1, y_2)$  of  $L$  that is not in  $\Phi_1 \cup \Phi_2$ . Since  $L_1$  and  $L_2$  are primitive, neither  $y_1$  nor  $y_2$  is zero. If  $(y_1, y_2) \in L_1 \oplus L_2$ , this implies  $y_1^2 = 2 - y_2^2 \leq 0$ ; and the same is true, by assumption, if  $(y_1, y_2) \notin L_1 \oplus L_2$ . Therefore

$$\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3.$$

Let  $\Phi_i^+ \subset \Phi_i$  be an  $f_i$ -invariant system of positive roots for  $i = 1, 2$ , and choose  $x_i \in L_i$  such that  $\langle x_i, y \rangle > 0$  for all  $y \in \Phi_i^+$ . We may assume  $x_1^2 < 0$ .

Note that any  $(y_1, y_2) \in \Phi_3$  satisfies  $\langle x_1, y_1 \rangle \neq 0$ , since  $y_1^2 \leq 0$  and  $y_1 \neq 0$ . Let

$$\Phi_3^+ = \{(y_1, y_2) \in \Phi_3 : \langle x_1, y_1 \rangle > 0\}.$$

This subset just consists of the elements of  $\Phi_3$  that project to the component of the lightcone in  $L_1 \otimes \mathbb{R}$  that contains  $-x_1$ . Since  $f_1$  preserves this component, we have  $f(\Phi_3^+) = \Phi_3^+$ .

We claim that  $\Phi_+ = \Phi_1^+ \cup \Phi_2^+ \cup \Phi_3^+$  is a positive root system for  $L$ . Since  $f(\Phi_+) = \Phi_+$ , this will complete the proof of positivity of  $f = f_1 \oplus f_2$ .

Evidently  $\Phi^+ \cup (-\Phi^+) = \Phi$ . Let  $x = (Mx_1, x_2)$  for some integer  $M > 0$ . For all  $M$  large enough, we have  $x^2 < 0$ . It remains only to show that for all  $M$  large enough, we also have  $\langle x, y \rangle > 0$  for all  $y \in \Phi^+$ .

The desired inequality is immediate for all  $y \in \Phi_1^+ \cup \Phi_2^+$ . Now suppose  $y = (y_1, y_2) \in \Phi_3^+$ . Choose  $d > 0$  such that  $dL \subset L_1 \oplus L_2$ . Then  $dy_1 \in L_1$ , and hence  $\langle x_1, y_1 \rangle \geq 1/d$ . Similarly, we have  $|y_1^2| \geq 1/d^2$  provided  $y_1^2 \neq 0$ .

We claim that

$$y_2^2 \leq (2d^2 + 1)\langle x_1, y_1 \rangle^2 \quad (5.4)$$

for all  $(y_1, y_2) \in \Phi_3^+$ . Indeed, if  $y_1^2 \neq 0$  then  $|y_1^2| \geq 1/d^2$  and hence

$$y_2^2 = 2 + |y_1^2| \leq (2d^2 + 1)|y_1^2|.$$

The reverse Cauchy-Schwarz inequality together with the fact that  $|x_1^2| \geq 1$  then gives

$$|y_1^2| \leq |x_1^2 y_1^2| \leq \langle x_1, y_1 \rangle^2,$$

which yields (5.4). For the case  $y_1^2 = 0$  we just observe that  $\langle x_1, y_1 \rangle^2 \geq 1/d^2$ , and hence

$$y_2^2 = 2 \leq 2d^2 \langle x_1, y_1 \rangle^2,$$

so (5.4) holds in this case as well.

Now the usual Cauchy-Schwarz inequality implies

$$\langle x_2, y_2 \rangle^2 \leq x_2^2 y_2^2 \leq x_2^2 (2d^2 + 1) \langle x_1, y_1 \rangle^2.$$

So for any  $M$  large enough that  $M^2 > (2d^2 + 1)x_2^2$ , we have  $\langle Mx_1, y_1 \rangle > |\langle x_2, y_2 \rangle|$ , and hence  $\langle x, y \rangle = \langle Mx_1, y_1 \rangle + \langle x_2, y_2 \rangle > 0$  for all  $y = (y_1, y_2) \in \Phi_3^+$ . Thus  $\Phi^+$  is a positive,  $f$ -invariant root system, and  $f$  is a positive automorphism of  $L$ .  $\blacksquare$

## 6 Automorphisms of K3 surfaces

In this section we relate automorphisms of lattices to automorphisms of K3 surfaces.

**K3 structures.** Fix an even, unimodular lattice  $L$  of signature  $(3, 19)$ . A *K3 structure* on  $L$  consists of following data:

1. A *Hodge decomposition*

$$L \otimes \mathbb{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$$

such that  $L^{i,j} = \overline{L^{j,i}}$ , and the Hermitian spaces  $L^{1,1}$  and  $L^{2,0} \oplus L^{0,2}$  have signatures  $(1, 19)$  and  $(2, 0)$  respectively;

2. A *positive cone*  $\mathcal{C} \subset L^{1,1} \cap (L \otimes \mathbb{R})$ , forming one of the two components of the locus  $x^2 > 0$ ; and
3. A set of *effective roots*

$$\Psi^+ \subset \Psi = \{x \in L \cap L^{1,1} : x^2 = -2\},$$

satisfying  $\Psi = \Psi^+ \cup (-\Psi^+)$ .

We require that the *Kähler cone*

$$\mathcal{C}^+(L) = \{x \in \mathcal{C} : \langle x, y \rangle > 0 \forall y \in \Psi^+\}$$

defined by this data is nonempty.

**Realizability.** A K3 structure on  $L$  is *realized* by a K3 surface  $X$  if there exists an isomorphism

$$\iota : L \rightarrow H^2(X, \mathbb{Z})$$

sending  $L^{i,j}$  to  $H^{i,j}(X)$  and sending  $\mathcal{C}^+(L)$  to the Kähler cone in  $H^{1,1}(X, \mathbb{R})$ . Similarly, an isometry  $f : L \rightarrow L$  is *realized* an automorphism  $F : X \rightarrow X$  if  $\iota$  can be chosen so that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\iota} & H^2(X, \mathbb{Z}) \\ f \downarrow & & \downarrow F^* \\ L & \xrightarrow{\iota} & H^2(X, \mathbb{Z}) \end{array}$$

commutes.

The following fundamental theorem [BPV, VIII] encapsulates the strong Torelli theorem for K3 surfaces as well as surjectivity of the period map:

**Theorem 6.1** *Any K3 structure on  $L$  is realized by a unique K3 surface  $X$ , and any  $f \in \text{O}(L)$  preserving a given K3 surface structure is realized by a unique automorphism  $F : X \rightarrow X$ .*

**Remarks.** The Hodge structure on  $L$  determines  $X$  up to isomorphism, while the Kähler cone  $\mathcal{C}^+(L)$  pins down the isomorphism  $\iota$ . The Néron-Severi group  $\text{NS}(X) \cong \text{Pic}(X) \subset H^2(X, \mathbb{Z})$  is given by

$$\text{NS}(X) = \iota(L \cap L^{1,1}).$$

To conclude, we address the problem of finding an  $f$ -invariant K3 structure.

**Theorem 6.2** *Let  $f \in \mathrm{O}(L)$  be an isometry with spectral radius  $\rho(f) > 1$ . Suppose  $\rho(f)$  is an eigenvalue of  $f$ , and there is a unique  $\tau \in (-2, 2)$  such that*

$$E_\tau = \mathrm{Ker}(f + f^{-1} - \tau I) \subset L \otimes \mathbb{R}$$

*has signature  $(2, 0)$ . Then  $f$  is realizable by a K3 surface automorphism iff  $f|M(-1)$  is positive, where  $M = L \cap E_\tau^\perp$ .*

**Proof.** Our assumptions imply there is an  $f$ -invariant Hodge structure with  $L^{2,0} \oplus L^{0,2} \cong E_\tau \otimes \mathbb{C}$ . Moreover, this is the unique  $f$ -invariant Hodge structure of K3 type, up to interchanging the roles of  $L^{2,0}$  and  $L^{0,2}$ . In particular,  $L^{1,1}$  is uniquely determined, as is the candidate Néron-Severi group

$$M = L \cap L^{1,1} = L \cap E_\tau^\perp$$

and the set of roots  $\Psi = \{x \in M : x^2 = -2\}$ .

Suppose  $f|M(-1)$  is positive. Let  $\Psi^+ \subset \Psi$  be a system of positive roots preserved by  $f$ . If  $M$  has signature  $(1, n)$ , then (by the definitions in §5, taking into account the reversal of signs) there is an  $x \in M$  with  $x^2 > 0$  such that  $\langle x, y \rangle > 0$  for all  $y \in \Psi^+$ . Choose  $\mathcal{C}$  so  $x \in \mathcal{C}^+$ ; then  $\mathcal{C}^+$  is nonempty, and  $f(\mathcal{C}) = \mathcal{C}$  because the leading eigenvalue of  $f$  is positive. Thus  $f$  is realizable. If  $M$  has signature  $(0, n)$ , then the same argument applies, except  $x^2 < 0$ . But we can then simply replace  $x$  with  $x' = x + z$ , where  $z \in L^{1,1} \cap M^\perp$  and  $z^2 + x^2 > 0$ ; then  $(x')^2 > 0$ , so  $\mathcal{C}^+ \neq \emptyset$ , and hence  $f$  is realizable in this case as well.

Conversely, if  $f$  is realizable by  $F : X \rightarrow X$ , then  $F$  preserves the Kähler cone in  $H^{1,1}(X, \mathbb{R})$ , and hence  $f$  preserves the dual system of positive roots  $\Psi^+ \subset \mathrm{NS}(X)(-1) \cong M(-1)$ , so it is positive. ■

The proof shows the pair  $(X, F)$  realizing  $f$  is unique up to complex conjugation, and that  $M \cong \mathrm{NS}(X)$ .

## 7 Minimum entropy

In this section we will show:

**Theorem 7.1** *There exists an automorphism  $F : X \rightarrow X$  of a non-algebraic K3 surface with entropy  $h(F) = \log \lambda_{10} \approx \log 1.17628$ .*

**Building blocks.** To exhibit  $F$ , we will construct a lattice automorphism  $f : L \rightarrow L$  with characteristic polynomial

$$\det(xI - f) = P_{10}(x)(x - 1)^9(x + 1)(x^2 + 1)$$

satisfying the realizability criterion stated in Theorem 6.2. The pair  $(L, f)$  will in turn be obtained as a gluing of  $(L_1, f_1)$  and  $(L_2, f_2)$ . We begin by describing these two constituents of  $f$ .

**The Salem factor.** Recall from §2 that  $E_{10}$  is an even, unimodular lattice of signature  $(9, 1)$ . The Salem polynomial  $P_{10}(x)$  arises naturally as the characteristic polynomial

$$S_1(x) = P_{10}(x) = \det(xI - f_1)$$

of the Coxeter automorphism  $f_1 : E_{10} \rightarrow E_{10}$  (see e.g. [Mc1]). Reducing mod  $p$  for  $p = 3$ , we find

$$\overline{S}_1(x) = (x^2 + 1)(x^8 + x^7 + 2x^6 + x^5 + x^3 + 2x^2 + x + 1) \quad (7.1)$$

in  $\mathbb{F}_3[x]$ . This factorization suggests that with suitable twisting, we may be able to arrange that  $\overline{f}_1$  acts with characteristic polynomial  $(x^2 + 1)$  on a glue group isomorphic to  $\mathbb{F}_3^2$ . And indeed this is the case: if we let

$$a = 2(f_1 + f_1^{-1}) + 3 \in \text{End}(E_{10}),$$

then  $|\det(a)| = 9$  and  $3E_{10} \subset aE_{10}$ , as can be checked by a matrix computation (e.g.  $3a^{-1}$  is integral). It then follows from Theorem 4.2 that for  $L_1 = E_{10}(a)$ , we have

$$G(L_1) \cong \mathbb{F}_3^2 \quad \text{and} \quad \overline{Q}_1(x) = \det(xI - \overline{f}_1) = (x^2 + 1) \bmod 3.$$

To determine the signature of  $L_1$ , let  $R_1(y) = y^5 + y^4 - 5y^3 - 5y^2 + 4y + 3$  be the trace polynomial of  $S_1(x)$ , and let

$$T = \{\tau_1, \tau_2, \tau_3, \tau_4\} \approx \{-1.886, -1.468, -0.584, 0.913\}$$

denote the roots of  $R_1(y)$  which lie in  $(-2, 2)$ . The associated eigenspaces of  $f + f^{-1}$  all have signature  $(2, 0)$ , since  $E_{10}$  has signature  $(9, 1)$  (see equation 4.4). On the other hand, the polynomial  $P(y) = 2y + 3$  is negative for  $y = \tau_1$  but positive for  $y = \tau_2, \tau_3, \tau_4$ ; since  $a = P(f + f^{-1})$ , this implies  $L_1 = E_{10}(a)$  has signature  $(7, 3)$  (by equation 4.5). Finally  $L_1$  is an even lattice, by Proposition 4.1.

**The cyclotomic factor.** Now recall from §5 that there is a positive automorphism  $g : A_2 \rightarrow A_2$  such that  $\bar{g}$  acts with order two on  $G(A_2) \cong \mathbb{F}_3$ . Let  $L_2 = E_8 \oplus A_2 \oplus A_2$ . Note that  $L_2$  has signature  $(12, 0)$ , and every root of  $L_2$  lies in one of its summands. It follows that the order four map  $f_2 : L_2 \rightarrow L_2$  given by

$$f_2(x, y, z) = (x, g(z), y)$$

is also positive. Its characteristic polynomial is given by

$$S_2(x) = (x - 1)^9(x + 1)(x^2 + 1),$$

while its action on  $G(L_2) \cong \mathbb{F}_3^2$  has characteristic polynomial

$$\bar{Q}_2(x) = \det(xI - \bar{f}_2) = x^2 + 1.$$

**Proof of Theorem 7.1.** Since  $\bar{Q}_1(x) = \bar{Q}_2(x) = x^2 + 1 \in \mathbb{F}_3[x]$  is a separable polynomial, nonvanishing at  $x = \pm 1$ , there is a gluing map  $\phi : G(L_1) \rightarrow G(L_2)$  conjugating  $\bar{f}_1$  to  $\bar{f}_2$  by Theorem 3.1.

Let  $L = (L_1 \oplus_\phi L_2)(-1)$ . This is a lattice of signature  $(3, 7) + (0, 12) = (3, 19)$ . Since we are gluing at the odd prime  $p = 3$ ,  $L$  is still even. By construction,  $f_1 \oplus f_2$  extends to an isometry  $f : L \rightarrow L$ , with characteristic polynomial  $S(x) = S_1(x)S_2(x)$ .

Since  $S_1(x)$  is a Salem polynomial and  $S_2(x)$  is a product of cyclotomic polynomials, the spectral radius

$$\rho(f) = \max\{|\lambda| : S(\lambda) = 0\} = \lambda_{10}$$

is an eigenvalue of  $f$ . Moreover, the twist by  $a$  provides us with a unique eigenspace

$$E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L_1 \otimes \mathbb{R} \cong \mathbb{R}^{3,19}$$

with signature  $(2, 0)$ , coming from  $\tau = \tau_1$ .

Since  $S_1(x)$  is irreducible, no element of  $L_1$  lies in  $E_\tau^\perp$ ; thus  $M = L \cap E_\tau^\perp = L_2(-1)$ . By construction,  $f_2|_{L_2} \cong f|M(-1)$  is positive. Thus by Theorem 6.2 there is a K3 surface automorphism  $F : X \rightarrow X$  realizing  $f$ ; and  $h(F) = \rho(f) = \lambda_{10}$  by equation (1.1). ■

**Remarks.**

1. We have  $\text{NS}(X)(-1) \cong E_8 \oplus A_2 \oplus A_2$ . Thus by Grauert's criterion [BPV, III.2], the exceptional curves on  $X$  can be blown down to yield a singular complex manifold  $Y$  with no curves at all. The map  $F$  descends to an automorphism of  $Y$  which exchanges its two singular points of type  $A_2$ , and fixes its unique singular point of type  $E_8$ .

2. The ring  $\mathbb{Z}[f_1 + f_1^{-1}]$  is in fact a Dedekind domain of class number one, so by Theorem 4.3, the existence of the desired twist of  $E_{10}$  is automatic once one has the factorization (7.1).
3. Oguiso gives an example of a K3 surface automorphism with entropy  $\log \lambda_{14} \approx \log 1.20002$  [Og]. This example is analogous to the one above, but with  $L_1 = \Pi_{11,3}$  and  $L_2 = E_8$ . Here both lattices are unimodular, so no glue is necessary, and one can take  $f_2(x) = x$ . The existence of an  $f_1 \in O(L_1)$  with characteristic polynomial  $P_{14}(x)$  is guaranteed by [GM].
4. Many examples of K3 surface automorphisms based on Salem numbers of degree 22 are given in [Mc2]. In these examples no gluing takes place (there is no cyclotomic factor), and positivity is automatic, because the Néron-Severi group is trivial.
5. We remark that gluing theory is also useful for describing the Kummer surface  $X$  associated to a complex torus  $A$ : indeed,  $H^2(X, \mathbb{Z})$  is obtained by gluing  $H^2(A, \mathbb{Z})(2)$  along  $\mathbb{F}_2^6$  to a fixed lattice of rank 16 (see [BPV, VIII.5]).

## 8 A projective example

In this section we will show:

**Theorem 8.1** *There exists an automorphism  $F : X \rightarrow X$  of a projective K3 surface with entropy  $h(F) = \log \lambda_6 \approx \log 1.40126$ .*

For the proof, we will construct a model for  $F|H^2(X, \mathbb{Z})$  by gluing together four lattice automorphisms  $f_i : L_i \rightarrow L_i$ ,  $i = 1, \dots, 4$  (see Figure 4), and twisting by  $-1$ . The result will be an automorphism  $f : L \rightarrow L$  of an even, unimodular lattice of signature  $(3, 19)$  with characteristic polynomial

$$S(x) = P_6(x)C_{13}(x)(x^2 + x + 1)(x - 1)^2, \quad (8.1)$$

where  $P_6(x)$  is the Salem polynomial for  $\lambda_6$  (see Table 1) and  $C_{13}(x) = (x^{13} - 1)/(x - 1)$ . Theorem 6.2 will imply that  $f$  is realizable.

We now turn to the construction of the building blocks  $f_i : L_i \rightarrow L_i$ . For each  $i$  we will determine the signature of  $L_i$ , the glue group  $G(L_i)$ , and the characteristic polynomial

$$\overline{Q}_{i,p}(x) = \det(xI - \overline{f}_i|G(L_i)_p) \in \mathbb{F}_p[x]$$



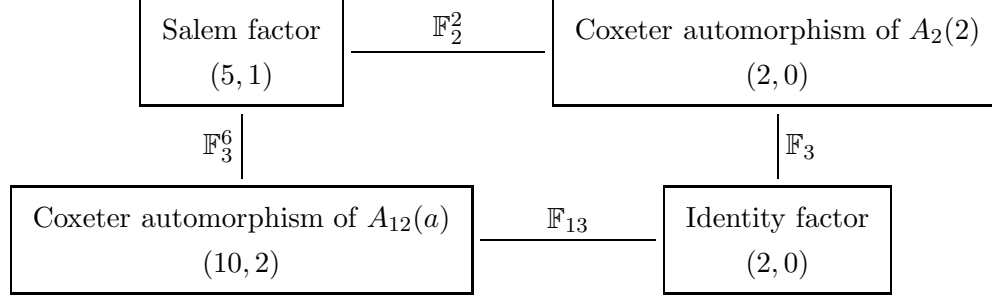


Figure 4. Assembly of an algebraic K3 automorphism with entropy  $\log \lambda_6$ .

of  $\bar{f}_i$  for each prime  $p$  where it is nontrivial.

**1. The Salem factor.** Let  $W = \wedge^2 \mathbb{Z}^4$ , with the natural inner product

$$\langle \alpha, \beta \rangle = \alpha \wedge \beta \in \wedge^4 \mathbb{Z}^4 \cong \mathbb{Z}.$$

Then  $W \cong \text{II}_{3,3}$  is an even, unimodular lattice of signature  $(3, 3)$ , and any  $g \in \text{SL}_4(\mathbb{Z})$  gives rise to an isometry  $f = \wedge^2 g \in \text{O}(W)$ . In particular, if we take  $g = g_1$  to be the companion matrix for  $x^4 + x + 1$ , then we obtain a map  $f_1 \in \text{O}(W)$  with characteristic polynomial

$$S_1(x) = \det(I - f_1) = P_6(x).$$

(This is related to the fact that  $\log \lambda_6$  can be realized as the entropy of an automorphism of a complex 2-torus [Mc2, §5].) Reducing mod  $p = 2$ , we find

$$\bar{S}_1(x) = (1 + x + x^2)(1 + x + x^2 + x^3 + x^4)$$

in  $\mathbb{F}_2[x]$ . This factorization suggests we can find a twist  $W(a)$  such that

$$G(W(a)) = \mathbb{F}_2^2 \quad \text{and} \quad \det(xI - \bar{f}_1|W(a)) = (1 + x + x^2).$$

Indeed, this is the case for  $a = -(1 + f + f^{-1})$ , as can be verified with the help of Theorem 4.2. Similarly, if we set  $L_1 = W(3a)$ , then we find

$$\begin{aligned} G(L_1) &= G(L_1)_2 \oplus G(L_1)_3 = \mathbb{F}_2^2 \oplus \mathbb{F}_3^6, \\ \bar{Q}_{1,2}(x) &= (1 + x + x^2) \quad \text{and} \\ \bar{Q}_{1,3}(x) &= (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3). \end{aligned}$$

(Note that  $\overline{Q}_{1,3}(x)$  is a reciprocal polynomial, even though its irreducible factors are not). Since  $W$  is even, and we have twisted by  $3a$ , we have  $\langle x, x \rangle \in 6\mathbb{Z}$  for all  $x \in L_1$ . In particular,  $L_1$  is an even lattice with *no roots*. Its signature is  $(5, 1)$ , as can be checked using equation (4.5).

**2. The order 13 factor.** Next consider the Coxeter automorphism  $f_2 : A_{12} \rightarrow A_{12}$ . This map has order 13, so its characteristic polynomial is given by  $S_2(x) = C_{13}(x)$ , which factors modulo  $p = 3$  as

$$\overline{S}_2(x) = (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3)(2 + 2x + x^3)(2 + x^2 + x^3).$$

The map  $\overline{f}_2|G(A_{12}) \cong \mathbb{Z}/13$  is the identity, since  $f_2$  belongs to the Weyl group of  $A_{12}$  (see §2). If we set  $L_2 = A_{12}(a)$  where  $a = 1 + (f_2 + f_2^{-1}) - (f_2 + f_2^{-1})^3$ , then  $aA_{12} \subset 3A_{12}$  and we find

$$\begin{aligned} G(L_2) &= G(L_2)_3 \oplus G(L_1)_{13} = \mathbb{F}_3^6 \oplus \mathbb{F}_{13}, \\ \overline{Q}_{2,3}(x) &= (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3), \quad \text{and} \\ \overline{Q}_{2,13}(x) &= (x - 1). \end{aligned}$$

(Note that we have chosen  $a$  so that  $\overline{Q}_{1,3} = \overline{Q}_{2,3}$ .) In this case  $L_2$  is an even lattice of signature  $(10, 2)$ .

The action of  $\overline{f}_2$  determines the fractional form on  $G(L_2)_3$ , but not on  $G(L_2)_{13}$  (where it acts by the identity). For later use we note:

*The determinant of  $G(L_2)_{13}$  is not a square.*

That is, the nonzero values of the form  $13\langle\langle x, x \rangle\rangle$  consist of the non-square numbers  $\{2, 4, 6, 7, 8, 11\} \bmod 13$ . This can be verified by a direct matrix computation, or by noting that  $\det G(A_{12}) = -1$  (as computed in Proposition 3.5) is a square mod 13 but 8 is not, and  $ax = 8x$  for  $x \in G(A_{12})$ , since  $\overline{f}_2(x) = x$ .

**3. The order 3 factor.** Let  $f_3 : A_2 \rightarrow A_2$  be the Coxeter automorphism, and let  $L_3 = A_2(2)$ . Then the characteristic polynomial of  $f_3$  is given by  $S_3(x) = x^2 + x + 1$ , and  $\overline{f}_3|G(A_2)$  is the identity. It follows that

$$\begin{aligned} G(L_3) &= G(L_3)_2 \oplus G(L_3)_3 = \mathbb{F}_2^2 \oplus \mathbb{F}_3, \\ \overline{Q}_{2,2}(x) &= x^2 + x + 1, \quad \text{and} \\ \overline{Q}_{2,3}(x) &= (x - 1). \end{aligned}$$

The lattice  $L_2$  has signature  $(2, 0)$ , and  $\langle x, x \rangle \in 4\mathbb{Z}$  for all  $x \in L_2$ . Since  $-1$  is not a square mod 3, and neither is the twisting parameter 2, using Proposition 3.5 again we find:

The determinant of  $G(L_3)_3$  is a square.

**4. The identity factor.** Finally let  $L_4 \cong \mathbb{Z}^2$  be the even lattice of signature  $(2, 0)$  and determinant 39 with inner product matrix  $B = \begin{pmatrix} 2 & 1 \\ 1 & 20 \end{pmatrix}$ . Then

$$G(L_4) = G(L_4)_3 \oplus G(L_4)_{13} = \mathbb{F}_3 \oplus \mathbb{F}_{13}.$$

To control later gluings, the following two properties are important.

*If  $x \in L_4^\vee$  projects to a nontrivial element of  $\mathbb{F}_3 \subset G(L_4)$ , then  $x^2 > 2$ .*

In fact,  $(1, 1)/3$  and  $(-1, -1)/3$  are the minimal norm vectors in  $L_4^\vee$  representing the nonzero elements of  $\mathbb{F}_3$ , and each satisfies  $x^2 = 8/3$ .

*Neither  $\det G(L_4)_3$  nor  $\det G(L_4)_{13}$  is a square.*

To see this, note that  $2/39$  appears on the diagonal of  $B^{-1}$ . Thus  $\langle x, x \rangle = 2/39 \pmod{1}$  for some  $x$  in  $G(L_4)$ ; this implies  $\det G(L_4)_3 = [3\langle 13x, 13x \rangle] = [2 \pmod{3}]$  and  $\det G(L_4)_{13} = [13\langle 3x, 3x \rangle] = [6 \pmod{13}]$ , and neither class is a square.

As an automorphism of  $L_4$ , we simply take  $f_4(x) = x$ .

**Assembly.** By construction, we have gluing isometries

$$\begin{aligned} \phi_{12} & : G(L_1)_3 \rightarrow G(L_2)_3 \cong \mathbb{F}_3^6, \\ \phi_{13} & : G(L_1)_2 \rightarrow G(L_3)_2 \cong \mathbb{F}_2^2, \\ \phi_{24} & : G(L_2)_{13} \rightarrow G(L_4)_{13} \cong \mathbb{F}_{13}, \quad \text{and} \\ \phi_{34} & : G(L_3)_3 \rightarrow G(L_4)_3 \cong \mathbb{F}_3, \end{aligned}$$

satisfying  $\phi_{ij}\bar{f}_i = \bar{f}_j\phi_{ij}$ . The first glue map  $\phi_{12}$  exists by Theorem 3.1, since  $\bar{Q}_{1,3} = \bar{Q}_{2,3}$ . Similar reasoning applies to the second. The third map  $\phi_{24}$  exists because both the domain and range have non-square determinant, and because  $(-1)$  is a square mod 13. (Recall that a gluing map must reverse the sign of the bilinear form.) The last map exists because its domain has square determinant, while its range does not, and because  $-1$  is *not* a square mod 3.

Let  $\bigoplus_\phi L_i$  denote the unimodular lattice of signature  $(19, 3)$  obtained by gluing together all four lattices, as shown Figure 4. Let  $L = (\bigoplus_\phi L_i)(-1)$ , and let  $f : L \rightarrow L$  denote the linear extension of  $\bigoplus f_i$ . Then the characteristic polynomial of  $f$  is given by  $S(x) = \prod_1^4 S_i(x)$ , which agrees with equation (8.1).

**Evenness.** We claim  $L$  is even. This is almost automatic, since its constituents  $L_i$  are even and since almost all the gluings take place over groups of odd order. The one exception comes from  $\phi_{13}$ . To verify evenness, we must check that the  $\mathbb{Q}/\mathbb{Z}$ -valued quadratic forms  $q_i(x)$  on  $G(L_i)_2$ ,  $i = 1, 3$ , defined by equation (2.4), satisfy  $q_1(x) + q_3(\phi_{13}(x)) = 0 \pmod{1}$ . But  $q_i$  is invariant under  $\bar{f}_i$ , which cyclically permutes the three nonzero vectors in  $G(L_i)_2 \cong \mathbb{F}_2^2$ . This easily implies that  $q_i(x) = 1/2$  for all  $x \neq 0$ , so the sum vanishes and hence  $L$  is even.

**Proof of Theorem 8.1.** Clearly  $\rho(f) = \lambda_6 > 1$ , since all other roots of  $S(x)$  are inside or on the unit circle. By construction,  $f + f^{-1}$  has a unique eigenspace of signature  $(2, 0)$ , namely

$$E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L_2 \otimes \mathbb{R}$$

where  $\tau = 2 \cos(2\pi/13)$ . Since the action of  $f$  on  $L_2$  is irreducible over  $\mathbb{Q}$ , the candidate Néron-Severi group  $M = L \cap E_\tau^\perp$  satisfies

$$M(-1) = L_1 \oplus_{\phi_{13}} L_3 \oplus_{\phi_{34}} L_4.$$

We claim  $f_{13} = f_1 \oplus f_3$  is positive on the Lorentzian lattice

$$L_{13} = L_1 \oplus_{\phi_{13}} L_3.$$

Indeed, we have  $x^2 \in 2a_i\mathbb{Z}$  for all  $x \in L_i$ , where  $a_1 = 3$  and  $a_3 = 2$ ; and  $bL \subset L_1 \oplus L_3$  for  $b = 2$ , since we are gluing along  $\mathbb{F}_2^2$ . Since  $b^2 = 4 \notin \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_3$ , positivity follows from Theorem 5.1.

Note that, since  $f_4$  is the identity, it gives a positive automorphism of the Euclidean lattice  $L_4$ .

We now claim that the sum of positive automorphisms,  $f_{134} = f_{13} \oplus f_4$ , is positive on  $L_{134} = L_{13} \oplus_{\phi_{34}} L_4 = M(-1)$ . Here the gluing takes place over  $\mathbb{F}_3$ . In fact, the desired positivity follows from Theorem 5.2. For if  $(x, y) \in L_{134}$  is a root but  $(x, y) \notin L_{13} \oplus L_4$ , then  $y \in L_4^\vee$  represents a nonzero element  $[x_4] \in \mathbb{F}_3 \subset G(L_4)$ , and hence (as we have seen above)  $y^2 \geq 8/3 > 2$ .

Thus  $f|M(-1)$  is positive. By Theorem 6.2, there is a K3 surface automorphism  $F : X \rightarrow X$  realizing  $f : L \rightarrow L$ ; by construction,  $h(F) = \log \rho(f) = \log \lambda_6$ ; and since  $M \cong \text{NS}(X)$  has signature  $(1, 9)$ ,  $X$  is projective. ■

### Remarks.

1. Since  $P_6(x)$  is an *unramified* Salem polynomial, the fact that it can be realized by an isometry  $f_1 \in \mathrm{O}(\mathrm{II}_{3,3})$  also follows from general results [GM].
2. The existence of the twists producing  $L_1$  and  $L_2$  can, as in §7, be explained by the fact that  $\mathbb{Z}[f_1 + f_1^{-1}]$  and  $\mathbb{Z}[f_2 + f_2^{-1}]$  are Dedekind domains of class number one.
3. We note that  $\mathrm{NS}(X)$  has signature  $(1, 9)$  and determinant  $3^6 \cdot 13 = 9477$ , and  $F^*|H^{2,0}(X)$  has order 13.
4. The Coxeter polynomial for  $Eh_8$  (the ‘hyperbolic extension of  $E_6$ ’, with diagram  $Y_{3,3,4}$ ) is the same as the characteristic polynomial for  $f|L_1 \oplus L_3$ , namely  $P_6(x)(1 + x + x^2)$  (see [Mc1, Table 5]). In fact the Coxeter automorphism of  $Eh_8$  could have been taken as the starting point for the construction of  $f$ , just as the Coxeter automorphism of  $E_{10}$  (the hyperbolic extension of  $E_8$ ) was the starting point for the construction in §7.

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