



# q-deformed Interacting Particle Systems, RSKs and Random Polymers

## Citation

Matveev, Konstantin. 2016. q-deformed Interacting Particle Systems, RSKs and Random Polymers. Doctoral dissertation, Harvard University, Graduate School of Arts & Sciences.

## Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:33493453>

## Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA>

## Share Your Story

The Harvard community has made this article openly available.  
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

$q$ -deformed Interacting Particle Systems,  
RSKs and Random Polymers

A dissertation presented

by

Konstantin Matveev

to

The Department of Mathematics

in partial fulfillment of the requirements  
for the degree of  
Doctor of Philosophy  
in the subject of  
Mathematics

Harvard University  
Cambridge, Massachusetts

April 2016

© 2016 – Konstantin Matveev  
All rights reserved.

$q$ -deformed Interacting Particle Systems, RSKs and Random Polymers

## Abstract

We introduce and study four  $q$ -randomized Robinson–Schensted–Knuth (RSK) insertion tableau dynamics. Each of them is a discrete time Markov dynamics on two-dimensional interlacing particle arrays (these arrays are in a natural bijection with semistandard Young tableaux). For  $0 < q < 1$  each dynamics provides a two-dimensional extension of the corresponding one-dimensional exactly solvable random dynamics of interacting particles. We prove that our dynamics act nicely on a certain class of probability measures on arrays, namely, on  $q$ -Whittaker processes. For  $q = 0$  these dynamics degenerate to the classical row or column RSK insertion tableau dynamics applied to a random input matrix with independent geometric or Bernoulli entries. We prove that in a scaling limit as  $q \nearrow 1$ , two of our four dynamics on interlacing arrays turn into the geometric RSK dynamics associated with log-Gamma and strict-weak directed random lattice polymers.

## CONTENTS

Acknowledgements	vi
1. Introduction	1
1.1. Overview	1
1.2. $q$ -randomized row insertion with $q$ -geometric input	6
1.3. Other dynamics and results	11
1.4. Outline	12
2. Plactic Algebra and Schur Processes	13
2.1. Preliminaries	13
2.2. Schur polynomials	15
2.3. Schensted's insertions	17
2.4. Plactic algebra	21
2.5. Schur processes	23
2.6. RSK dynamics	26
3. $q$ -Whittaker processes	32
3.1. $q$ -Whittaker and Macdonald polynomials	32
3.2. $q$ -Whittaker measures and processes	37
4. Markov dynamics	38
4.1. Univariate dynamics	40
4.2. Main equations	45
4.3. Discussion of main equations	48
4.4. Push-block dynamics	51
4.5. RSK-type dynamics	55
5. RSK-type dynamics $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ and $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ adding a dual parameter	57
5.1. Row insertion dynamics $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$	57
5.2. Bernoulli $q$ -PushTASEP	64

5.3.	Complementation	65
5.4.	Column insertion dynamics $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$	67
5.5.	Bernoulli $q$ -TASEP	71
5.6.	Small $\beta$ continuous time limit	71
6.	RSK-type dynamics $\mathcal{Q}_{\text{row}}^q[\alpha]$ and $\mathcal{Q}_{\text{col}}^q[\alpha]$ adding a usual parameter	73
6.1.	The $q$ -deformed Beta-binomial distribution	73
6.2.	Row insertion dynamics $\mathcal{Q}_{\text{row}}^q[\alpha]$	75
6.3.	Geometric $q$ -PushTASEP	81
6.4.	Column insertion dynamics $\mathcal{Q}_{\text{col}}^q[\alpha]$ . Description and discussion	82
6.5.	Column insertion dynamics $\mathcal{Q}_{\text{col}}^q[\alpha]$ . Proof	88
6.6.	Geometric $q$ -TASEP	94
6.7.	Small $\alpha$ continuous time limit	94
7.	Moments	94
7.1.	Bernoulli $q$ -PushTASEP on the line	95
7.2.	Connection to the Bernoulli $q$ -TASEP	96
7.3.	Nested contour integral formulas for $q$ -moments	97
7.4.	Remark. Geometric $q$ -PushTASEP formulas	99
8.	Polymer limits of $(\alpha)$ dynamics on $q$ -Whittaker processes	99
8.1.	Polymer partition functions	99
8.2.	Geometric RSKs	105
8.3.	Asymptotics of $q$ -deformed Beta-binomial distributions	112
8.4.	Proofs of Theorems 8.7 and 8.8	118
	References	131

## ACKNOWLEDGEMENTS

It is my pleasure to thank my advisor Alexei Borodin for introducing me to the intriguing topic of Integrable Probability, for his advice, and, foremost, for his openness and willingness to spend time answering my questions, even when I happened to appear at MIT without any prior arrangements. I am indebted to Leonid Petrov for his patience and encouragement during our joint work, and for introducing me to the exciting field of experimentation in Mathematica. I would also like to thank Ivan Corwin for valuable conversations and Christian Krattenthaler for valuable correspondence. I am grateful to the National Science and Engineering Research Council of Canada for supporting me through the PGS scholarship, and to the Harvard math department in general for creating a comfortable and pleasant research environment.

I have been exceptionally lucky to have crossed paths with Alexander S. Shtern. Without his inspiring guidance I might have never started seriously studying mathematics. I am extremely grateful to my family for all their support. Especially, I would like to thank my grandparents, Raphail and Irina, who haven't lived just a couple of months to see me defend my PhD, but who would have been truly happy to see the result to which their early contribution has been most essential.

# 1. INTRODUCTION

1.1. **Overview.** The classical Robinson–Schensted–Knuth (RSK) correspondence associates to an integer matrix a pair  $(P, Q)$  of semistandard Young tableaux of the same shape [43], [34], [70], [66]. It is informative to view an integer matrix  $M = (M_{ij})$  as a configuration of points (“balls”) in cells of the lattice  $\mathbb{Z}_{\geq 0}^2$ , with  $M_{ij}$  balls in the  $(i, j)$ -th cell (see Fig. 1, left). There are also simpler correspondences obtained from the RSK if

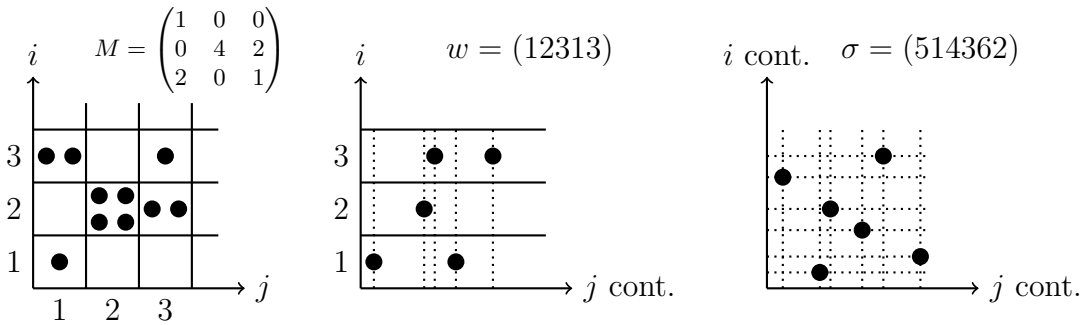


FIGURE 1. Left: an integer matrix as an input to the RSK. Center: an integer word as an input into the RS viewed as a matrix with the continuous  $j$  coordinate (at most one ball at a given horizontal position is allowed; the word encodes vertical positions of consecutive balls). Right: a permutation viewed as a matrix with both continuous coordinates (at most one ball at a given horizontal or vertical position is allowed; balls represent the graph of the permutation).

one makes one or both dimensions of the input *continuous*, see Fig. 1, center and right. In particular, the Robinson–Schensted (RS) correspondence maps integer words into pairs of Young tableaux of the same shape, but now one of them is standard.

The idea of applying the RSK correspondence to a *random* input can be traced back to [72] where it was used in connection with the asymptotic theory of characters of the infinite symmetric group (see also [16]). Together with combinatorial properties of the RSK this idea has been extensively employed in studying various stochastic systems, e.g., TASEP (= totally asymmetric simple exclusion process), the last-passage percolation [40], or longest increasing subsequences of random permutations [1], [2].



Reading the random input matrix column by column adds a dynamical perspective to random systems (with  $j$  in all three cases on Fig. 1 playing the role of time). This direction has been substantially developed in, e.g., [53], [54], [5].

The geometric version (also sometimes called “*tropical*”) of the RS and the RSK correspondences<sup>1</sup> has also been employed in the study of stochastic systems [55], [21], [59], [56]. The systems one obtains at this level are related to directed random polymers in random media, in particular, to the O’Connell–Yor [60], log-Gamma [69], and strict-weak [57], [24] random polymers. Each such polymer model can be viewed as a positive temperature version of a certain last-passage percolation-like model.

In the stochastic systems mentioned above, the RSK and related constructions provide a way to observe and understand their *integrability*. The integrability property refers to the presence of concise and exact formulas describing observables, which allows to study the asymptotic behavior of such systems, and also gives access to exact descriptions of limiting universal distributions, such as the Tracy–Widom distributions which are features of the Kardar–Parisi–Zhang (KPZ) universality class [19], [13], [15]

Random evolution of the insertion tableau  $P$  in (discrete) time  $j$  can be described by using the concept of the plactic algebra. The plactic monoid of rank  $N$  is generated by letters  $1, 2, \dots, N$  modulo certain relations, and its elements are in bijective correspondence with the semistandard Young tableaux with entries from  $1, 2, \dots, N$ , see §2.4. RSK insertion of a random column corresponds to multiplication by the random element of the plactic monoid. Multiplication on the right corresponds to the Schensted’s row insertion and multiplication on the left corresponds to the Schensted’s column insertion, see §2.3. Multiplication by a random element of a monoid can be thought of as the multiplication

---

<sup>1</sup>The geometric RSK maps arrays of positive real numbers into other such arrays in a birational way, and is obtained from the classical RSK by a certain “detropicalization”, see [42], [52].

by the corresponding linear combination from the monoid algebra. The situation is analogous to the class of problems on randomly shuffling a deck of cards, where application of a particular shuffling procedure can be thought of as multiplication by an element of  $\mathbb{C}[S_n]$ , the group algebra of the symmetric group. Both the plactic algebra and  $\mathbb{C}[S_n]$  are noncommutative, but contain some explicit commutative subalgebras. Multiplication by elements of such subalgebras in both cases leads to integrability of the corresponding stochastic dynamics, although probabilistic implications are quite different <sup>2</sup>. Random transpositions shuffling of [26] can be thought of as multiplication by the particular linear combination of the Jucys-Murphy elements of  $\mathbb{C}[S_n]$ , which commute with each other. The integrability of the corresponding Markov chain allows to prove sharp upper bound on its mixing time.

The plactic algebra contains a family of commutative subalgebras generated by the plactic Schur polynomials. Some particular products of elements from any one of these subalgebras give rise to probability measures on the semistandard Young tableaux from the family of Schur processes, see §2.5. In this case integrability allows to prove results about limit shapes and fluctuations for the corresponding models of randomly growing surfaces.

The classical RSK is deeply connected to Schur symmetric functions [51, Ch. I], while the geometric RSK is relevant to the  $\mathfrak{gl}_n$  Whittaker functions [47], [28]. Both families of functions are degenerations of more general Macdonald symmetric functions depending on two parameters  $(q, t)$  [51, Ch. VI]: the Schur functions correspond to  $q = t$ , and the Whittaker functions arise in the limit as  $t = 0$  and  $q \nearrow 1$ , [37].

The definition of the Schur processes can be given without reference to the plactic algebra. Substituting Macdonald functions instead of Schur functions in this definition leads to a more general family of probability measures, the Macdonald processes, [8].

---

<sup>2</sup>Both the symmetric group and the plactic monoid can be presented by a finite set of generators modulo finitely many relations, however, unlike the symmetric group, the plactic monoid is infinite.

However, the plactic (dynamic) point of view on Macdonald processes remains much less understood, and its further advancement is one of the main goals of the present paper.

In the recent years, there has been a progress in understanding analogues of the RS insertion tableau dynamics at other levels of the Macdonald hierarchy:  $q$ -Whittaker ( $t = 0$  and  $0 < q < 1$ ) [58], [64], [14] and Hall–Littlewood ( $q = 0$  and  $0 < t < 1$ ) [17]. At these levels, the maps become *randomized*, that is, the image of a deterministic word (as on Fig. 1, center) is no longer a fixed Young tableau, but rather a *random* one. Because of this randomness, an appropriate language for describing such RSK dynamics seems to be that of *Markov dynamics* on two-dimensional interlacing integer arrays (these arrays are in a natural bijection with semistandard tableaux, see §2.1 below for more detail). The dynamics which are analogues of the RS insertion tableau dynamics evolve in continuous time according to the  $j$  axis on Fig. 1, center.

The  $q$ -Whittaker level is relevant to integrable one-dimensional particle systems such as (continuous time)  $q$ -TASEP and the stochastic  $q$ -Boson system [67], [8], [11], [9], [29], and (continuous time)  $q$ -PushTASEP (=  $q$ -deformed pushing TASEP) [22].<sup>3</sup> In particular, continuous time Markov dynamics on interlacing arrays constructed in [58] and [14] are two-dimensional extensions of, respectively, the  $q$ -TASEP and the  $q$ -PushTASEP. That is, the latter one-dimensional processes are Markovian marginals of the dynamics on two-dimensional interlacing arrays.<sup>4</sup>

In the present paper we advance further at the  $q$ -Whittaker level, and introduce four  $q$ -randomized RSK dynamics, or, in other words, four discrete time Markov dynamics

---

<sup>3</sup>These systems are in fact quantum integrable in the sense of the coordinate Bethe ansatz [4], [48], [3], [9].

<sup>4</sup>The two-dimensional dynamics at the Hall–Littlewood level [17], however, do not seem to lead to any new one-dimensional integrable particle systems.

on interlacing arrays which act nicely on  $q$ -Whittaker processes (these are Macdonald processes with  $t = 0$ ). These dynamics unify, generalize and extend all of the above RSK-type constructions:

- When  $q = 0$ , our four  $q$ -randomized dynamics become usual or dual, row or column classical RSK insertion tableau dynamics (four classical dynamics in total). The input matrix  $M$  in the usual RSKs has  $M_{ij} \in \{0, 1, 2, \dots\}$ , and in the dual RSKs one has  $M_{ij} \in \{0, 1\}$ . When one takes  $M_{ij}$  to be independent geometric (for usual) or Bernoulli (for dual) random variables and applies a suitable classical RSK insertion tableau map, the shape of the resulting random Young diagram is distributed according to the Schur measure [62].<sup>5</sup> Similarly, our  $q$ -randomized correspondences applied to  $q$ -geometric or Bernoulli random inputs (note that the Bernoulli input needs not to be  $q$ -deformed) give rise to  $q$ -Whittaker distributed random Young diagrams. The latter property is an instance of “acting nicely” on  $q$ -Whittaker processes (see also (4.1) in §3 for more detail).

- In a limit from discrete to continuous time, our  $q$ -randomized RSKs turn into the (simpler)  $q$ -randomized RS insertion tableau dynamics introduced and studied in [58], [14].

- The two discrete time  $q$ -TASEPs (associated with  $q$ -geometric or Bernoulli random variables) studied by Borodin–Corwin [7] arise as one-dimensional marginals of our two “column” dynamics on interlacing arrays. In a similar way, our two “row” dynamics lead to discrete time  $q$ -PushTASEPs — new integrable particle systems in the KPZ universality class.

- In a scaling limit as  $q \nearrow 1$ , the dynamics on interlacing arrays associated with the  $q$ -geometric random input (these are two out of our four  $q$ -randomized RSK insertion

---

<sup>5</sup>In the present paper, the word “geometric” is attached to two separate concepts — the geometric RSKs, and the geometric and  $q$ -geometric random variables.

tableau dynamics) converge to geometric RSK dynamics. The latter dynamics (which are deterministic rational maps between arrays of positive reals) are relevant to the log-Gamma [69], [21], [59] and strict-weak [24] random lattice polymers.

In §1.2 below we describe one of our four dynamics in detail, and in §1.3 we briefly discuss other dynamics and results.

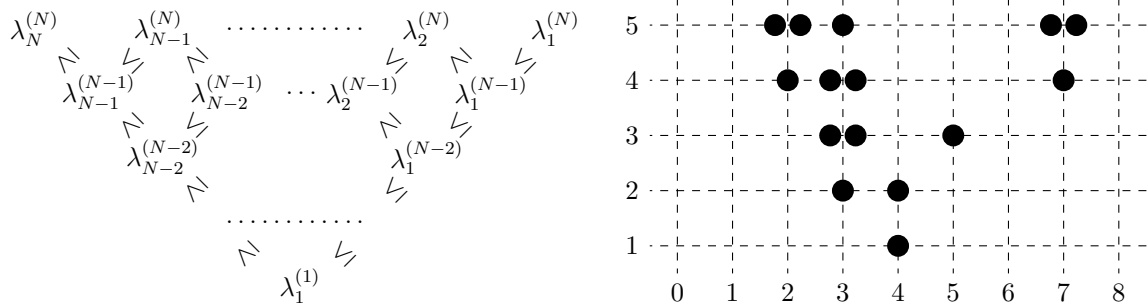


FIGURE 2. Left: An interlacing array  $\lambda$ ; we require that  $\lambda_i^{(j)} \in \mathbb{Z}_{\geq 0}$ . Right: A configuration of particles corresponding to an interlacing array of depth  $N = 5$  (right).

**1.2.  $q$ -randomized row insertion with  $q$ -geometric input.** Discrete time Markov dynamics (i.e., the  $q$ -randomized RSK insertion tableau dynamics) which we construct in the present paper live on the space of integer arrays  $\lambda$  (see Fig. 2). Neighboring levels of the array satisfy certain inequalities which we call the *interlacing property* (see 2.1 for definition). Each level  $\lambda^{(j)} = (\lambda_1^{(j)} \geq \dots \geq \lambda_j^{(j)})$  of an array can be viewed as a partition (equivalently, a Young diagram [51, I.1]), so  $\lambda$  is a sequence of interlacing Young diagrams. Each  $\lambda$  can be also viewed as a *semistandard Young tableau* of shape  $\lambda^{(N)}$  filled with numbers from 1 to  $N$ . Then each  $\lambda^{(j)}$  is the portion of the semistandard tableau filled with numbers from 1 to  $j$ , see Fig. 3.

Let us now define the ( $q$ -randomized) operation of inserting a word  $w = (1^{m_1} 2^{m_2} \dots N^{m_N})$  (i.e., the word has  $m_1$  ones,  $m_2$  twos, etc.) into an array  $\lambda$ . The result is a new, random array  $\nu$ . At the first level we have  $\nu_1^{(1)} = \lambda_1^{(1)} + m_1$ . Then, sequentially at all levels

1	1	1	1	3	4	4
2	2	2	5	5	5	5
3	3	3				
4	4					
5	5					

FIGURE 3. A semistandard Young tableau corresponding to the array on Fig. 2, right.

$j = 2, \dots, N$ , given the existing change  $\lambda^{(j-1)} \rightarrow \nu^{(j-1)}$  at the previous level and the old state  $\lambda^{(j)}$  at the current level, construct the new state  $\nu^{(j)}$  as follows. Each move  $\nu_i^{(j-1)} - \lambda_i^{(j-1)}$ ,  $i = 1, \dots, j-1$ , is *randomly* split into two pieces  $r_i^{(j-1)} + \ell_i^{(j-1)}$ , and the piece  $r_i^{(j-1)}$  is added to the new move of the upper right neighbor  $\lambda_i^{(j)}$ , while the piece  $\ell_i^{(j-1)}$  is added to the new move of the upper left neighbor  $\lambda_{i+1}^{(j)}$ . Moreover,  $\lambda_1^{(j)}$  receives an additional move of size  $m_j$ . All these splittings and moves at level  $j$  happen in parallel. That is (here and below  $\mathbf{1} \dots$  stands for the indicator),

$$\nu_i^{(j)} - \lambda_i^{(j)} = m_j \mathbf{1}_{i=1} + r_i^{(j-1)} \mathbf{1}_{i < j} + \ell_{i-1}^{(j-1)} \mathbf{1}_{i > 1}, \quad i = 1, \dots, j$$

(see Fig. 4). To complete the definition, it now remains to describe the distribution of the splitting of the move  $\nu_i^{(j-1)} - \lambda_i^{(j-1)} = r_i^{(j-1)} + \ell_i^{(j-1)}$ . This is a certain  $q$ -deformed version of the Beta-binomial distribution, namely,  $r_i^{(j-1)}$  is randomly chosen to be equal to  $r \in \{0, 1, 2, \dots, \nu_i^{(j-1)} - \lambda_i^{(j-1)}\}$  with probability

$$\varphi_{q^{-1}, q^a, q^b}(r | c) := q^{ar} \frac{(q^{b-a}; q^{-1})_r (q^a; q^{-1})_{c-r} (q^{-1}; q^{-1})_c}{(q^{-1}; q^{-1})_r (q^{-1}; q^{-1})_{c-r} (q^b; q^{-1})_c}, \quad (1.1)$$

where

$$a = \lambda_i^{(j)} - \lambda_i^{(j-1)}, \quad b = \lambda_{i-1}^{(j-1)} - \lambda_i^{(j-1)}, \quad c = \nu_i^{(j-1)} - \lambda_i^{(j-1)},$$

we adopt convention  $\lambda_0^{j-1} = \infty$ , and  $(u; q)_m = (1-u)(1-uq) \dots (1-uq^{m-1})$  are the  $q$ -Pochhammer symbols. The quantity  $\ell_i^{(j-1)}$  is simply equal to  $\nu_i^{(j-1)} - \lambda_i^{(j-1)} - r_i^{(j-1)}$ .

The quantities (1.1) define a probability distribution in  $r$  for  $a \leq b$ ,  $c \leq b$  (these conditions follow from the interlacing). Moreover, this distribution is supported on  $\{0, 1, \dots, c\} \cap \{c - a, c - a + 1, \dots, b - a - 1, b - a\}$ , which in fact ensures that the new array  $\nu$  is also interlacing (see lemma 6.2 for details).

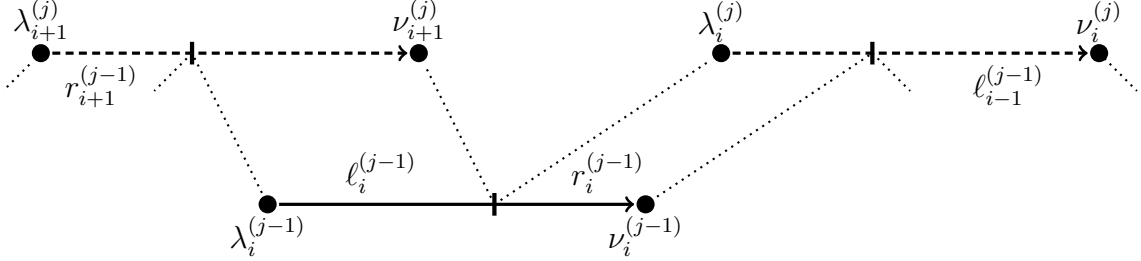


FIGURE 4. Splitting of the move at level  $j - 1$  and its propagation to the level  $j$ . Here we are using the particle interpretation of interlacing arrays as on Fig. 2, right.

Now, let us take the insertion word  $w = (1^{m_1} 2^{m_2} \dots N^{m_N})$  to be random itself. More precisely, let  $m_j$ ,  $j = 1, \dots, N$ , be independent  $q$ -geometric random variables:

$$\text{Prob}(m_j = k) = \frac{\alpha^k}{(q; q)_k} (\alpha; q)_\infty, \quad k = 0, 1, 2, \dots, \quad 0 < \alpha < 1. \quad (1.2)$$

Inserting this random word  $w$  into an array  $\lambda$  defines one step of a discrete time Markov chain on interlacing arrays. We denote this Markov chain by  $\mathcal{Q}_{\text{row}}^q[\alpha]$ .

**Theorem 1.1.** *Start the Markov dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  from the interlacing array with all  $\lambda_i^{(j)}(0) = 0$ . Then the distribution of the array  $\lambda(T)$  after  $T$  steps of this dynamics is given by the  $q$ -Whittaker process:*

$$\text{Prob}(\lambda(T) = \lambda) = (\alpha; q)_\infty^{TN} P_{\lambda^{(1)}}(1) P_{\lambda^{(2)}/\lambda^{(1)}}(1) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(1) Q_{\lambda^{(N)}}(\underbrace{\alpha, \alpha, \dots, \alpha}_T).$$

Here  $P_{\lambda/\mu}$  and  $Q_\lambda$  are the  $q$ -Whittaker polynomials, see §3 for more detail. Theorem 1.1 follows from Theorem 6.4 which we prove in §6.2.

**Remark 1.2.** In fact, we can (and will) consider a more general situation when the parameters  $\alpha$  in (1.2) may depend on  $j$  and on the time step as  $\alpha_t a_j$ . Then the  $q$ -Whittaker process above takes the form

$$\left( \prod_{j=1}^N \prod_{t=1}^T (\alpha_t a_j; q)_\infty \right) P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) Q_{\lambda^{(N)}}(\alpha_1, \alpha_2, \dots, \alpha_T).$$

We omit the dependence on  $j$  and  $t$  in Introduction.

Let us now describe three degenerations of the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$ :

- For  $q = 0$ , the splitting distributions (1.1) become supported at a single  $r \in \{0, 1, \dots, c\}$ , so the randomness in the insertion disappears, and the insertion itself turns into the classical RSK row insertion (we recall its definition in §2.6). The  $q$ -geometric random variables  $m_j$  (1.2) become geometric, and the  $q$ -Whittaker polynomials in Theorem 1.1 turn into the Schur polynomials. This justifies our treatment of the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  as the  $q$ -randomized row RSK insertion tableau dynamics.

- Fix  $0 < q < 1$ . When  $\alpha \searrow 0$  in (1.2) and one rescales time from discrete to continuous (see §6.7 for more details), the random input matrix turns into  $N$  independent Poisson processes running in parallel (i.e., we are passing from the left to the center situation on Fig. 1). Then in the splitting distributions one has  $c = 0$  or  $1$ , and the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  turns into a continuous time dynamics on  $q$ -Whittaker processes which was introduced in [14]. The latter continuous time dynamics should be viewed as a  $q$ -randomized row RS insertion tableau dynamics.

- Let  $q = e^{-\epsilon}$  and  $\alpha = e^{-\theta\epsilon}$  with  $\epsilon \searrow 0$  and  $\theta > 0$ . Define the positive random variables  $\hat{R}_k^j(t, \epsilon)$  via the following scaling:

$$\lambda_k^{(j)}(t) = (t + j - 2k + 1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log (\hat{R}_k^j(t, \epsilon)).$$



If the quantities  $\lambda_k^{(j)}$  evolve under the dynamics  $\mathcal{Q}_{\text{row}}^a[\alpha]$  started from all  $\lambda_k^{(j)}(0) = 0$ , then the rescaled quantities  $\hat{R}_k^j(t, \epsilon)$  converge to certain ratios of partition functions in the log-Gamma lattice polymer model (see §8.1 and Theorem 8.7 in particular for details). Moreover, under this scaling the randomness in the splitting (1.1) disappears, and the  $q$ -randomized insertion described above turns into the geometric RSK insertion.

**Remark 1.3.** It is worth noting that there is also a strong connection between the geometric RSK and representation theory, cf. [5], [18]. At the  $q$ -randomized level this connection does not (yet) seem to be present.

When restricted to the rightmost particles  $\lambda_1^{(j)}$ ,  $j = 1, \dots, N$ , of the interlacing array, the dynamics  $\mathcal{Q}_{\text{row}}^a[\alpha]$  induces a marginally Markovian evolution which we call the (discrete time) geometric  $q$ -PushTASEP. This is a new integrable particle system in the KPZ universality class. In the shifted coordinates  $x_i(t) := -\lambda_1^{(i)}(t) - i$  (so  $x_N < \dots < x_1$ ), the evolution of this system during time step  $t \rightarrow t + 1$  looks as follows. Sequentially for  $i = 1, 2, \dots, N$ , each particle  $x_i$  jumps to the left by  $m_i + r_1^{(i-1)}$ , where  $m_i$  is an independent  $q$ -geometric random variable (1.2), and  $r_1^{(i-1)}$  is a random variable with distribution

$$\varphi_{q^{-1}, q^a, 0}(r \mid c) = q^{ar} (q^a; q^{-1})_{c-r} \frac{(q^{-1}; q^{-1})_c}{(q^{-1}; q^{-1})_r (q^{-1}; q^{-1})_{c-r}}, \quad \begin{aligned} a &= x_{i-1}(t) - x_i(t) - 1, \\ c &= x_{i-1}(t+1) - x_{i-1}(t) \end{aligned}$$

(this is simply the splitting distribution (1.1) with  $b = +\infty$ ). Note that if  $c > a$ , then  $r_1^{(i-1)}$  chosen according to the above distribution will be at least  $c - a$ . See Fig. 5.

In a continuous time limit as  $\alpha \searrow 0$ , the geometric  $q$ -PushTASEP turns into the continuous time  $q$ -PushTASEP of [14], [22]. The  $q$ -moments of the form  $\mathbb{E} q^{k(x_n(t)+n)}$  (and more general such moments) of both  $q$ -PushTASEPs are given in terms of nested contour integrals. For the geometric  $q$ -PushTASEP only finitely many such moments exist (i.e., the expectation is infinite for sufficiently large  $k$ ), and for the continuous time  $q$ -PushTASEP

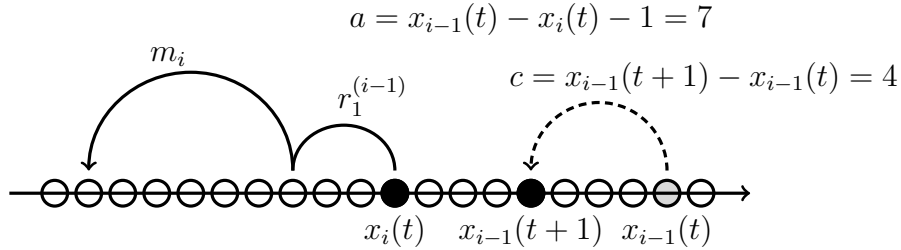


FIGURE 5. The discrete time geometric  $q$ -PushTASEP.

the moments grow too fast and also do not determine the distribution of  $x_n(t)$ . We refer to §7 for further details.

**1.3. Other dynamics and results.** Besides the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  discussed in §1.2 above, we introduce three other dynamics on  $q$ -Whittaker processes:

- $\mathcal{Q}_{\text{col}}^q[\alpha]$  (§6.4 and Theorem 6.11). At  $q = 0$  this dynamics becomes the classical RSK column insertion applied to a geometric random input  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  (§4.5). In a scaling limit as  $q \nearrow 1$ ,  $\mathcal{Q}_{\text{col}}^q[\alpha]$  turns into a geometric RSK associated with the strict-weak lattice polymer introduced in [24] (Theorem 8.8). In a continuous time limit,  $\mathcal{Q}_{\text{col}}^q[\alpha]$  turns into the  $q$ -randomized column RS insertion tableau dynamics introduced in [58]. Under  $\mathcal{Q}_{\text{col}}^q[\alpha]$ , the leftmost particles  $\lambda_j^{(j)}$  of the interlacing array evolve according to the discrete time geometric  $q$ -TASEP of [7].

- $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  (§5.1 and Theorem 5.2). At  $q = 0$  this dynamics becomes the dual RSK row insertion applied to a Bernoulli random input  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  (§4.5). In a continuous time limit,  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  turns into the  $q$ -randomized row RS insertion tableau dynamics [14]. Under  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ , the rightmost particles  $\lambda_1^{(j)}$  of the interlacing array evolve according to a new particle system, the discrete time Bernoulli  $q$ -PushTASEP (Definition 7.1).

- $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  (§5.4 and Theorem 5.7). At  $q = 0$  this dynamics becomes the dual RSK column insertion applied to a Bernoulli random input  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$  (§4.5). In a continuous time limit,  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  turns into the  $q$ -randomized column RS insertion tableau dynamics of [58].

Under  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ , the leftmost particles  $\lambda_j^{(j)}$  of the array evolve according to the discrete time Bernoulli  $q$ -TASEP [7].<sup>6</sup>

**Remark 1.4.** We believe that the four dynamics we construct are the most “natural” discrete time dynamics on  $q$ -Whittaker processes having all the desired properties and prescribed degenerations:

- The update in the dynamics is sequential, from lower to upper levels of the interlacing array.
- The dynamics act nicely on  $q$ -Whittaker measures and processes.
- The continuous time limits ( $\alpha$  or  $\beta \rightarrow 0$ ) of the dynamics coincide with continuous time RS dynamics of [58] or [14].
- For  $q = 0$ , the dynamics degenerate to the ones related to the classical RSK insertion tableau dynamics.
- In the  $q \nearrow 1$  limit, the  $(\alpha)$  dynamics converge to the ones related to the geometric RSKs.

The dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  are related to each other via a straightforward procedure we call complementation (§5.3) which shortens the proofs for  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ . Moreover, one can say that this procedure provides a direct link between the column and the row  $q$ -randomized RS insertion tableau dynamics of [58] and [14] (which are continuous time limits of  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ , respectively). This also provides a direct coupling between the Bernoulli  $q$ -TASEP and  $q$ -PushTASEP (Proposition 7.2).

**1.4. Outline.** In §2 and §3 we recall the necessary background on plactic algebra, Schensted’s insertions, Schur processes, Macdonald symmetric functions, and  $q$ -Whittaker processes. In §4 and also write down and discuss main linear equations which must be satisfied

---

<sup>6</sup>In contrast with  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$ , there is (yet) no known polymer-like limits of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  or  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ .

by our Markov dynamics on interlacing arrays and discuss two particular types of Markov dynamics, namely, push-block and RSK-type dynamics, and explain the differences between them. In §5 and §6 we explain in detail the constructions of four discrete time RSK-type dynamics on interlacing arrays, and prove that these dynamics act on the  $q$ -Whittaker processes in desired ways. In §7 we briefly discuss moment formulas for our one-dimensional interacting particle systems. In §8 we consider scaling limits as  $q \nearrow 1$  of our two  $(\alpha)$  dynamics on interlacing arrays, and show that they turn into the geometric RSK dynamics associated with log-Gamma or strict-weak directed random lattice polymers.

## 2. PLACTIC ALGEBRA AND SCHUR PROCESSES

**2.1. Preliminaries.** A *signature*<sup>7</sup> of length  $N \geq 1$  is a nonincreasing collection of integers  $\lambda = (\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N$ . We will work with signatures which have only nonnegative parts, i.e.,  $\lambda_N \geq 0$  (in which case they are also called *partitions*). Denote the set of all such objects by  $\mathbb{GT}_N^+$ . Let also  $\mathbb{GT}^+ := \bigcup_{N=1}^{\infty} \mathbb{GT}_N^+$ , with the understanding that we identify  $\lambda \cup 0 = (\lambda_1, \dots, \lambda_N, 0, 0, \dots, 0) \in \mathbb{GT}_{N+M}^+$  ( $M$  zeros) with  $\lambda \in \mathbb{GT}_N^+$  for any  $M \geq 1$ .

We will use two ways to depict signatures (see Fig. 6):

- (1) Any signature  $\lambda \in \mathbb{GT}_N^+$  can be identified with a *Young diagram* (having at most  $N$  rows) as in [51, I.1].
- (2) A signature  $\lambda \in \mathbb{GT}_N^+$  can also be represented as a configuration of  $N$  particles on  $\mathbb{Z}_{\geq 0}$  (with the understanding that there can be more than one particle at a given location).

We denote by  $|\lambda| := \sum_{i=1}^N \lambda_i$  the number of boxes in the corresponding Young diagram, and by  $\ell(\lambda)$  the number of nonzero parts in  $\lambda$  (which is finite for all  $\lambda \in \mathbb{GT}^+$ ). For  $\mu, \lambda \in \mathbb{GT}^+$

---

<sup>7</sup>These objects are also sometimes called *highest weights*, cf. [74], as they are the highest weights of irreducible representations of the unitary group  $U(N)$ .

we will write  $\mu \subseteq \lambda$  if  $\mu_i \leq \lambda_i$  for all  $i \in \mathbb{Z}_{>0}$ . In this case, the set difference of Young diagrams  $\lambda$  and  $\mu$  is denoted by  $\lambda/\mu$  and is called a *skew Young diagram*.

Two signatures  $\mu, \lambda \in \mathbb{GT}^+$  are said to *interlace* if one can append them by zeros such that  $\mu \in \mathbb{GT}_{N-1}^+$  and  $\lambda \in \mathbb{GT}_N^+$  for some  $N$ , and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N. \quad (2.1)$$

In terms of Young diagrams, this means that  $\lambda$  is obtained from  $\mu$  by adding a *horizontal strip* (or, equivalently, that *the skew diagram  $\lambda/\mu$  is a horizontal strip* which is, by definition, a skew Young diagram having at most one box in each vertical column), and we denote this by  $\mu \prec_h \lambda$ .

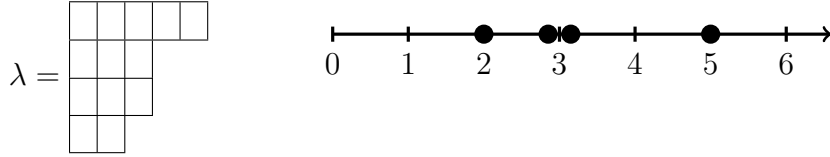


FIGURE 6. Young diagram  $\lambda = (5, 3, 3, 2) \in \mathbb{GT}_4^+$ , and the corresponding particle configuration. Note that there are two particles at location 3.

Let  $\lambda'$  denote the transposition of the Young diagram  $\lambda$ . For the diagram on Fig. 6, we have  $\lambda' = (4, 4, 3, 1, 1)$ . If  $\lambda/\mu$  is a horizontal strip, then  $\lambda'/\mu'$  is called a *vertical strip*. We will denote the corresponding relation by  $\mu' \prec_v \lambda'$ .

**Definition 2.1.** A *Gelfand–Tsetlin array* (sometimes also referred to as *scheme*, or *pattern*) of depth  $N$  is a sequence of interlacing signatures  $\boldsymbol{\lambda} = (\lambda^{(1)} \prec_h \lambda^{(2)} \prec_h \dots \prec_h \lambda^{(N)})$ , where  $\lambda^{(j)} \in \mathbb{GT}_j^+$ .

Such sequences first appeared in connection with representation theory of unitary groups [36].<sup>8</sup> We will depict sequences  $\boldsymbol{\lambda}$  as interlacing integer arrays, and also associate to them

<sup>8</sup>This justifies the notation “GT” we are using.

configurations of particles  $\{(\lambda_j^{(k)}, k) : k = 1, \dots, N, j = 1, \dots, k\}$  on  $N$  horizontal copies of  $\mathbb{Z}_{\geq 0}$ . See Fig. 2. Let us denote the set of all interlacing arrays  $\boldsymbol{\lambda}$  of depth  $N$  with top level  $\lambda$  by  $\mathbb{GT}^{(N)}(\lambda)$ . Let  $\mathbb{GT}^{(N)} := \bigcup_{\lambda \in \mathbb{GT}_N^+} \mathbb{GT}^{(N)}(\lambda)$ .

**Definition 2.2.** A *semistandard Young tableau* of shape  $\lambda$  is a filling in the boxes of the Young diagram  $\lambda$  with positive integers, which increase weakly along rows, and strictly down columns.

There is a natural correspondence between the Gelfand-Tsetlin arrays of depth  $N$  and the semistandard Young tableaux filled with numbers from 1 to  $N$ . Indeed, given  $\boldsymbol{\lambda} \in \mathbb{GT}^{(N)}$  we can produce a semistandard Young tableau of shape  $\lambda^{(N)}$  by filling  $\lambda^{(j)}/\lambda^{(j-1)}$  with numbers equal to  $j$ , see Fig. 3. Thus, by a slight abuse of notation we will also use  $\mathbb{GT}^{(N)}(\lambda)$  to denote the set of semistandard Young tableaux of shape  $\lambda$  filled with numbers from 1 to  $N$ .

**2.2. Schur polynomials.** We refer the reader to [51, Ch. I] for a more detailed overview of the Schur polynomials.

**Definition 2.3.** *Schur polynomials*  $S_\lambda(x_1, \dots, x_N)$  indexed by  $\lambda \in \mathbb{GT}_N^+$  can be defined by the following formula:

$$S_\lambda(x_1, \dots, x_N) := \sum_{\boldsymbol{\lambda} \in \mathbb{GT}^{(N)}(\lambda)} \prod_{j=1}^N x_j^{|\lambda^{(j)}| - |\lambda^{(j-1)}|}. \quad (2.2)$$

Here and thereafter we use the convention  $\lambda^{(0)} = \emptyset$ .

In representation theory Schur polynomials are the characters of polynomial irreducible representations of the general linear groups. They also admit a nice determinantal formula:

$$S_\lambda(x_1, \dots, x_N) = \frac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\det[x_i^{N-j}]_{i,j=1}^N}. \quad (2.3)$$

It is clear from (2.3) that  $S_\lambda(x_1, \dots, x_N)$  is a symmetric polynomial in  $x_1, \dots, x_N$ . The Schur polynomials are *stable* in the sense that for any  $\lambda \in \mathbb{GT}_N^+$ ,

$$S_{\lambda \cup 0}(x_1, \dots, x_N, 0) = S_\lambda(x_1, \dots, x_N).$$

Therefore, one may speak about *Schur symmetric functions*  $S_\lambda(x_1, x_2, \dots)$  in infinitely many variables, indexed by arbitrary  $\lambda \in \mathbb{GT}^+$ . These are elements of the algebra of symmetric functions, which may be viewed as a polynomial ring  $\mathbf{Sym} = \mathbb{C}[p_1, p_2, \dots]$  generated by the Newton power sums  $p_k(x_1, x_2, \dots) = \sum_{j=1}^{\infty} x_j^k$ . In other words, symmetric functions can be viewed as usual polynomials in  $p_1, p_2, \dots$ . Note that  $S_\lambda(x_1, \dots, x_N) \equiv 0$  if  $\ell(\lambda) > N$ .

**Definition 2.4.** A *specialization* of the algebra of symmetric functions  $\mathbf{Sym}$  is an algebra morphism  $\mathbf{Sym} \rightarrow \mathbb{C}$ . This is a generalization of the operation of taking the value of a symmetric function at a point. A specialization  $\mathbf{A}$  is said to be *Schur-nonnegative*, if  $S_\lambda(\mathbf{A}) \geq 0$  for any  $\lambda \in \mathbb{GT}^+$ .

Schur-nonnegative specializations are completely described by the Thoma's theorem [71], see also [41]. Namely, these are specializations  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)$ , where  $\boldsymbol{\alpha} = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0)$ ,  $\boldsymbol{\beta} = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$ ,  $\gamma \geq 0$ , and  $\sum_i (\alpha_i + \beta_i) < \infty$ , which may be defined via the generating function corresponding to signatures  $(n) \in \mathbb{GT}_1^+$ :

$$\sum_{n=0}^{\infty} S_{(n)}(\mathbf{A}) \cdot u^n = e^{\gamma u} \prod_{i=1}^{\infty} \frac{1 + \beta_i u}{1 - \alpha_i u} := \Pi(u; \mathbf{A}). \quad (2.4)$$

The left hand side of (2.4) is equal to  $\exp(\sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{A}) u^k)$ , so alternatively we can say that  $\mathbf{A}$  is defined by setting

$$p_1(\mathbf{A}) = \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i + \gamma, \quad p_k(\mathbf{A}) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k \quad \text{for } k = 2, 3, \dots$$

**Remark 2.5.** The specialization with all  $\beta_i = 0$  and  $\gamma = 0$  is the same as assigning values to the formal variables,  $x_j = \alpha_j$ , and we refer to the parameters  $\alpha_j$  as *usual parameters*.

In this case, if there are only finitely many nonzero  $a_j$ , we refer to the corresponding specialization as a *finite length specialization*. The specialization with all  $\alpha_j = 0$  and  $\gamma = 0$  sends  $S_\lambda$  to  $S_{\lambda'}(\beta)$  (value of the usual  $(\beta)$  specialization at  $S_{\lambda'}$ ), so we refer to the parameters  $\beta_i$  as the *dual parameters*.  $\gamma$  is the *Plancherel parameter*.

Let  $\mathbf{A} \cup \mathbf{B}$  denote the union of specializations (a generalization of concatenating the sets of variables). Formally it is defined as  $p_k(\mathbf{A} \cup \mathbf{B}) = p_k(\mathbf{A}) + p_k(\mathbf{B})$ ,  $k \geq 1$ .

We will also need the Cauchy identity

$$\sum_{\lambda \in \mathbb{GT}^+} S_\lambda(\vec{x}) S_\lambda(\vec{y}) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} p_k(\vec{x}) p_k(\vec{y})\right) \quad (2.5)$$

for  $\vec{x} = (x_1, x_2, \dots)$  and  $\vec{y} = (y_1, y_2, \dots)$ . By applying a  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)$  specialization to functions in  $y$ -variables and setting  $x_{N+1} = x_{N+2} = \dots = 0$  we can get the following corollary:

$$\begin{aligned} \sum_{\lambda \in \mathbb{GT}^+} S_\lambda(x_1, \dots, x_N) S_\lambda(\mathbf{A}) &= \Pi(x_1; \mathbf{A}) \cdots \Pi(x_N; \mathbf{A}) \\ &= e^{\gamma(x_1 + \dots + x_N)} \left( \prod_{i=1}^{\infty} \sum_{m=0}^{\infty} \alpha_i^m S_{(m)}(x_1, \dots, x_N) \right) \left( \prod_{i=1}^{\infty} \sum_{m=0}^N \beta_i^m S_{1^m}(x_1, \dots, x_N) \right). \end{aligned} \quad (2.6)$$

**2.3. Schensted's insertions.** Schensted's row and column insertions ([68]) are combinatorial constructions serving as the building blocks of the RSK algorithms, see [43], [34]. Each insertion can be described in the language of semistandard Young tableaux as a sequence of row and column bumpings. In the language of interlacing arrays these bumpings correspond to elementary operations of deterministic long-range pulling and pushing, which involve only two consecutive levels of an array.

**Definition 2.6.** (Deterministic long-range pulling, Fig. 7)



Let  $j = 2, \dots, N$ , and signatures  $\bar{\lambda}, \bar{\nu} \in \mathbb{GT}_{j-1}^+$ ,  $\lambda \in \mathbb{GT}_j^+$  satisfy  $\bar{\lambda} \prec_h \lambda$ ,  $\bar{\nu} = \bar{\lambda} + \bar{e}_i$ , where  $\bar{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$  (for some  $i = 1, \dots, j-1$ ) is the  $i$ th basis vector of length  $j-1$ . Define  $\nu \in \mathbb{GT}_j^+$  to be

$$\nu = \text{pull}(\lambda \mid \bar{\lambda} \rightarrow \bar{\nu}) := \begin{cases} \lambda + e_i, & \text{if } \bar{\lambda}_i = \lambda_i; \\ \lambda + e_{i+1}, & \text{otherwise.} \end{cases}$$

Here  $e_i$  and  $e_{i+1}$  are basis vectors of length  $j$ .

In words, the particle  $\bar{\lambda}_i$  at level  $j-1$  which moved to the right by one generically pulls its upper left neighbor  $\lambda_{i+1}$ , or pushes its upper right neighbor  $\lambda_i$  if the latter operation is needed to preserve the interlacing. Note that the long-range pulling mechanism does not encounter any blocking issues.



FIGURE 7. An example of pulling mechanism for  $i = 2$  at levels 2 and 3 (i.e.,  $j = 3$ ). Left:  $\bar{\lambda}_2 = \lambda_2$ , which forces the pushing of the upper right neighbor. Right: in the generic situation  $\bar{\lambda}_2 < \lambda_2$  the upper left neighbor is pulled.

**Definition 2.7.** (Deterministic long-range pushing, Fig. 8) As in the previous definition, let  $j = 2, \dots, N$ ,  $\bar{\lambda}, \bar{\nu} \in \mathbb{GT}_{j-1}^+$ ,  $\lambda \in \mathbb{GT}_j^+$  be such that  $\bar{\lambda} \prec_h \lambda$  and  $\bar{\nu} = \bar{\lambda} + \bar{e}_i$ . Define  $\nu \in \mathbb{GT}_j^+$  to be

$$\nu = \text{push}(\lambda \mid \bar{\lambda} \rightarrow \bar{\nu}) := \lambda + e_m, \quad \text{where } m = \max\{p: 1 \leq p \leq i \text{ and } \lambda_p < \bar{\lambda}_{p-1}\}.$$

In words, the particle  $\bar{\lambda}_i$  at level  $j-1$  which moved to the right by one, pushes its first upper right neighbor  $\lambda_m$  which is not blocked (and therefore is free to move without

violating the interlacing). Generically (when all particles are sufficiently far apart)  $\lambda_m = \lambda_i$ , so the immediate upper right neighbor is pushed.

**Remark 2.8** (Move donation). It is useful to equivalently interpret the mechanism of Definition 2.7 in a slightly different way. Namely, let us say that when the particle  $\bar{\lambda}_i$  at level  $j - 1$  moves, it gives the particle  $\lambda_i$  at level  $j$  a *moving impulse*. If  $\lambda_i$  is blocked (i.e., if  $\lambda_i = \bar{\lambda}_{i-1}$ ), this moving impulse is *donated* to the next particle  $\lambda_{i-1}$  to the right of  $\lambda_i$ . If  $\lambda_{i-1}$  is blocked, too, then the impulse is donated further, and so on. Note that the particle  $\lambda_1$  cannot be blocked, so this moving impulse will always result in an actual move.

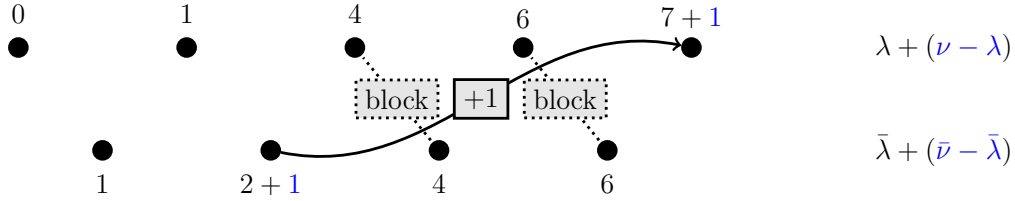


FIGURE 8. An example of pushing mechanism for  $i = 3$  at levels 4 and 5 (i.e.,  $j = 5$ ). Since the particles  $\lambda_3 = \bar{\lambda}_2$  and  $\lambda_2 = \bar{\lambda}_1$  are blocked, the first particle which can be pushed is  $\lambda_1$ .

**Definition 2.9.** The *Schensted's row insertion* is an algorithm that takes a semistandard tableau  $\lambda \in \mathbb{GT}^{(N)}$ , and an integer  $1 \leq x \leq N$ , and constructs a new tableau  $\lambda \leftarrow x$  according to the following procedure:

- If  $x$  is at least as large as all the entries in the first row of  $\lambda$ , add  $x$  in a new box to the end of the first row. In this case the algorithm terminates.
- Otherwise find the leftmost entry  $y$  in the first row that is strictly larger than  $x$  and replace it by  $x$ .
- Repeat the same steps with  $y$  and the second row, then with the replaced entry  $z$  and the third row, ..., and so on until the replaced entry can be put in the end of the next row, possibly by forming a new row of one entry. Then the algorithm terminates.

In terms of arrays and long-range pulling we can describe this insertion in the following way:

- Levels  $\lambda^{(1)}, \dots, \lambda^{(x-1)}$  remain unchanged.
- Rightmost particle on the level  $x$  moves by 1 to the right, i.e  $\lambda^{(x)} \rightarrow \lambda^{(x)} + \bar{e}_1$ .
- Then pull operations are consecutively performed for  $j = x + 1, \dots, N$ .

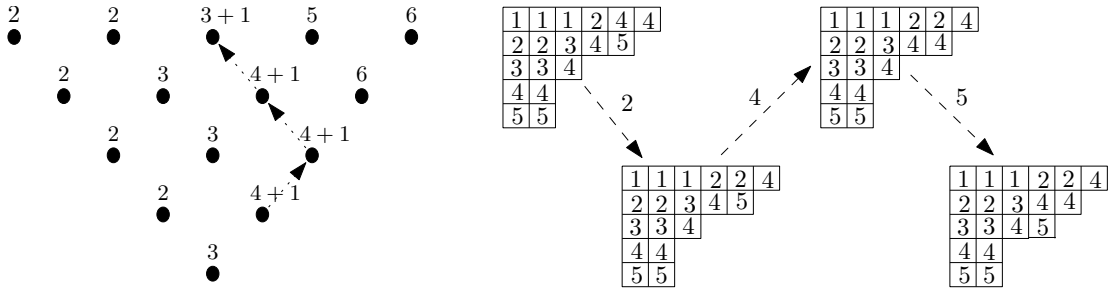


FIGURE 9. An example of Schensted's row insertion in terms of semistandard tableaux and particle arrays for  $N = 5$ .

In words, this push-pull chain of movements starts on the right edge of the array and progresses upwards until it reaches the top level. This is the same as saying that shape of a tableau is augmented by one cell after row insertion of a single entry. One can also row-insert a word  $X = x_1 x_2 \dots x_\ell$  in a tableau  $\lambda$  by consecutively inserting its entries one by one:

$$\lambda \leftarrow X := \lambda \leftarrow x_1 \leftarrow x_2 \leftarrow \dots \leftarrow x_\ell$$

**Definition 2.10.** The *Schensted's column insertion* is an algorithm that takes a semistandard tableau  $\lambda \in \mathbb{GT}^{(N)}$ , and an integer  $1 \leq x \leq N$ , and constructs a new tableau  $x \rightarrow \lambda$  according to the following procedure:

- If  $x$  is strictly larger than all the entries in the first column of  $\lambda$ , add  $x$  in a new box at the bottom of the first column. In this case the algorithm terminates.
- Otherwise find the topmost entry  $y$  in the first column that is at least large as  $x$  and replace it by  $x$ .

- Repeat the same steps with  $y$  and the second column, then with the replaced entry  $z$  and the third column, ..., and so on until the replaced entry can be put at the bottom of the next column, possibly by forming a column of one entry. Then the algorithm terminates.

In terms of arrays and long-range pushing we can describe this insertion in the following way:

- Levels  $\lambda^{(1)}, \dots, \lambda^{(x-1)}$  remain unchanged.
- Leftmost particle on the level  $x$  moves by 1 to the right, i.e  $\lambda^{(x)} \rightarrow \lambda^{(x)} + \bar{e}_x$ .
- Then push operations are consecutively performed for  $j = x + 1, \dots, N$ .

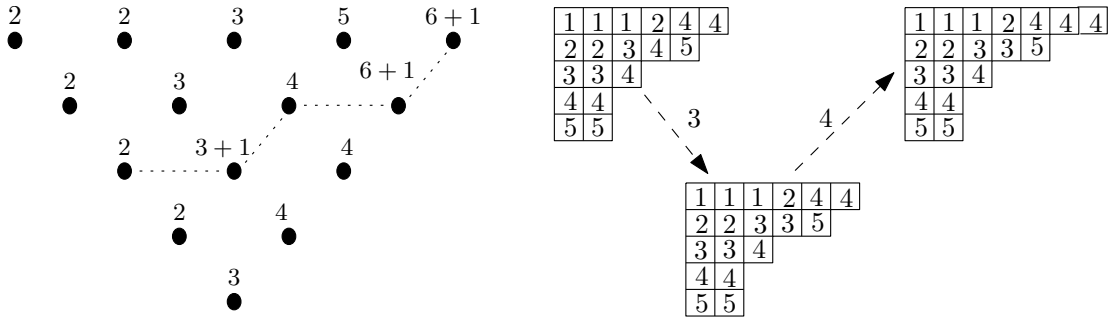


FIGURE 10. An example of Schensted's column insertion in terms of semistandard tableaux and particle arrays for  $N = 5$ . Only steps that change the tableau are shown.

As in the case of the row insertion, on each of the levels from  $x$ -th to  $N$ -th precisely one particle moves to the right by 1. The sequence of moves progresses upwards and to the right until it reaches the top level. One can also column-insert a word  $X = x_1 x_2 \dots x_\ell$  in a tableau  $\lambda$  by consecutively inserting its entries one by one in reverse order:

$$X \rightarrow \lambda := x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_\ell \rightarrow \lambda$$

**2.4. Plactic algebra.** The plactic monoid discovered by Knuth provides one natural framework for thinking about Schensted's insertions, see [50] for an excellent exposition.

**Definition 2.11.** The *plactic monoid*  $Pl_N$  of rank  $N$  is a monoid generated by letters  $1, 2, \dots, N$  modulo the *Knuth relations*:

$$xzy \equiv zxy \quad (x \leq y < z), \quad yxz \equiv yzx \quad (x < y \leq z). \quad (2.7)$$

We define the *plactic algebra*  $\mathbb{C}[[Pl_N]]$  to be the algebra of formal countable linear combinations over  $\mathbb{C}$  of elements of  $Pl_N$  with absolutely summable sets of nonzero coefficients. The algebra multiplication is extended from the monoid multiplication.

**Remark 2.12.** The plactic algebra is usually defined by allowing only finite linear combinations, as in the general definition of the monoid algebra, however it will be useful in our setting to sometimes consider infinite sums.

To a semistandard tableau  $\lambda \in \mathbb{GT}^{(N)}$  one can associate an element  $w(\lambda) \in Pl_N$  represented by a word obtained by reading entry letters of  $\lambda$  first in the bottom row from left to right, then in the second row from the bottom from left to right, and so on. For instance, for a tableau on Fig. 3 the corresponding word will be 554433322255551111344. The following proposition explains basic connections between the plactic monoid and Schensted insertions.

**Proposition 2.13.** (see [50]).

- (1) For every  $a \in Pl_N$  there exists a unique tableau  $\lambda$ , such that  $a = w(\lambda)$ .
- (2)  $w(\lambda \leftarrow x) = w(\lambda) \cdot x$ .
- (3)  $w(x \rightarrow \lambda) = x \cdot w(\lambda)$ .

Hence consecutively inserting  $x_1, x_2, \dots, x_r$  via the Schensted's row insertion into a tableau  $\lambda$  leads to the the same result as multiplication of  $w(\lambda)$  by  $X = x_1 x_2 \cdots x_r$  on the right, while multiplication of  $w(\lambda)$  by  $X$  on the left amounts to the same result as consecutively inserting  $x_r, \dots, x_2, x_1$  in  $\lambda$  via the Schensted's column insertion.

The plactic algebra is noncommutative for  $N \geq 2$ , but it contains nice families of commuting elements. More precisely, for  $\lambda \in \mathbb{GT}_N^+$  and  $a_1, a_2, \dots, a_N \in \mathbb{C}$  define the *plactic Schur polynomial*

$$S_\lambda^{Pl}(a_1 \cdot \mathbf{1}, a_2 \cdot \mathbf{2}, \dots, a_N \cdot \mathbf{N}) := \sum_{\lambda \in \mathbb{GT}^{(N)}(\lambda)} w(\boldsymbol{\lambda}) \cdot a_1^{|\lambda^{(1)}|} a_2^{|\lambda^{(2)}| - |\lambda^{(1)}|} \dots a_N^{|\lambda^{(N)}| - |\lambda^{(N-1)}|}. \quad (2.8)$$

**Proposition 2.14.** (see [50]).  $S_\lambda(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N})$  and  $S_\mu(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N})$  commute for arbitrary  $\lambda$  and  $\mu$ , and their product can be expressed as

$$S_\lambda^{Pl}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}) S_\mu^{Pl}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}) = \sum_{\nu} c_{\lambda, \mu}^{\nu} S_\nu^{Pl}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}), \quad (2.9)$$

where  $c_{\lambda, \mu}^{\nu}$  is the Littlewood-Richardson coefficient, i.e the coefficient of  $S_\nu$  in the expansion of  $S_\lambda S_\mu$  in the basis of Schur functions.

**Remark 2.15.** A homomorphism  $\mathbb{C}[[Pl_N]] \rightarrow \mathbb{C}$  defined by sending every generator  $k$  to 1 sends (2.9) to

$$S_\lambda(a_1, \dots, a_N) S_\mu(a_1, \dots, a_N) = \sum_{\nu} c_{\lambda, \mu}^{\nu} S_\nu(a_1, \dots, a_N),$$

which is the defining relation for the Littlewood-Richardson coefficients.

**Remark 2.16.** If  $b_i = ba_i$  for some  $b \in \mathbb{C}$  and all  $1 \leq i \leq N$ , then  $S_\lambda^{Pl}(b_1 \cdot \mathbf{1}, \dots, b_N \cdot \mathbf{N}) = b^{|\lambda|} S_\lambda^{Pl}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N})$ , so  $S_\mu(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N})$  and  $S_\lambda(b_1 \cdot \mathbf{1}, \dots, b_N \cdot \mathbf{N})$  commute also for proportionate  $\vec{a}$  and  $\vec{b}$ . For non-proportionate  $\vec{a}$  and  $\vec{b}$  this is in general not true.

**2.5. Schur processes.** We will say that an element  $U \in \mathbb{C}[[Pl_N]]$ ,  $U = \sum_{\lambda \in \mathbb{GT}^{(N)}} u_\lambda w(\boldsymbol{\lambda})$  is *stochastic* if  $u_\lambda \geq 0$  for all  $\boldsymbol{\lambda}$  and  $\sum_{\lambda \in \mathbb{GT}^{(N)}} u_\lambda = 1$ . Clearly, a product of several stochastic elements is also stochastic. Each such  $U$  gives rise to two infinite stochastic matrices  $L[U]$  and  $R[U]$  with rows and columns indexed by elements of  $\mathbb{GT}^{(N)}$ .  $R[U]$  is defined as a matrix representing operator of plactic multiplication by  $U$  on the right, while

$L[U]$  is defined as a matrix representing operator of plactic multiplication by  $U$  on the left. Matrices  $L[U]$  and  $R[U]$  give rise to *Markov dynamics* on interlacing arrays, which will be of interest for us for certain special  $U$ .

**Remark 2.17.** Matrices  $L[U_1]$  and  $R[U_2]$  commute for all stochastic  $U_1, U_2 \in \mathbb{C}[[Pl_N]]$ . Matrices  $L[U_1]$  and  $L[U_2]$  ( $R[U_1]$  and  $R[U_2]$ ) commute if and only if  $U_1$  and  $U_2$  commute.

**Definition 2.18.** For  $a_1, \dots, a_N > 0$

(1) For  $\alpha \geq 0$  such that  $\alpha a_j < 1$  for all  $j = 1, 2, \dots, N$ , let

$$U(\vec{a}, \alpha) := \left( \prod_{j=1}^N (1 - \alpha a_j) \right) \sum_{m=0}^{\infty} \alpha^m S_{(m)}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}). \quad (2.10)$$

(2) For  $\beta \geq 0$ , let

$$U(\vec{a}, \hat{\beta}) := \left( \prod_{j=1}^N \frac{1}{1 + \beta a_j} \right) \sum_{m=0}^{\infty} \beta^m S_{1^m}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}). \quad (2.11)$$

(3) For  $\gamma \geq 0$ , let

$$U_{Plancherel}(\vec{a}, \gamma) = e^{-\gamma a_1 - \dots - \gamma a_n} \exp(\gamma a_1 \cdot \mathbf{1} + \dots + \gamma a_n \cdot \mathbf{n}) := \quad (2.12)$$

$$e^{-\gamma a_1 - \dots - \gamma a_n} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} S_{(1)^k}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N})^k \quad (2.13)$$

$U(\vec{a}, \alpha)$ ,  $U(\vec{a}, \hat{\beta})$  and  $U_{Plancherel}(\vec{a}, \gamma)$  are stochastic elements of the plactic algebra, which commute with each other for fixed  $\vec{a}$  and different  $\alpha, \beta, \gamma$ . This follows from the Proposition (2.14). Then the Proposition (2.6) implies the following corollary:

**Proposition 2.19.** For  $a_1, \dots, a_N > 0$  and a Schur-nonnegative specialization  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)$ , such that  $a_j \alpha_i < 1$  for all  $i, j$

$$\frac{1}{\Pi(\vec{a}; \mathbf{A})} \sum_{\lambda \in \text{GT}_N^+} S_{\lambda}^{Pl}(a_1 \cdot \mathbf{1}, \dots, a_N \cdot \mathbf{N}) S_{\lambda}(\mathbf{A}) = U_{Plancherel}(\vec{a}, \gamma) \cdot \prod_{i=1}^{\infty} U(\vec{a}, \alpha_i) \cdot \prod_{i=1}^{\infty} U(\vec{a}, \hat{\beta}_i),$$

$$\text{where } \Pi(\vec{a}; \mathbf{A}) = \Pi(a_1, \mathbf{A}) \cdots \Pi(a_N, \mathbf{A}) \quad (2.14)$$

The left hand side of (2.14) is a stochastic element of  $\mathbb{C}[[Pl_N]]$ , so it can be viewed as a probability measure on  $\mathbb{GT}^{(N)}$  depending on  $\vec{a}$  and  $\mathbf{A}$ . Such measure is called a *Schur process* and was first defined in [63]. It attaches to  $\lambda$  the probability weight <sup>9</sup>

$$\mathcal{M}_{\mathbf{A}}^{\vec{a}; q=0}(\lambda) = \frac{1}{\Pi(\vec{a}; \mathbf{A})} S_{\lambda^{(N)}}(\mathbf{A}) \prod_{j=1}^N a_j^{|\lambda^{(j)}| - |\lambda^{(j-1)}|}. \quad (2.15)$$

For fixed  $\vec{a}$  Markov dynamics

$$\begin{aligned} \mathcal{Q}_{\text{row}}^{q=0}[\alpha] &:= R[U(\vec{a}; \alpha)], & \mathcal{Q}_{\text{col}}^{q=0}[\alpha] &:= L[U(\vec{a}; \alpha)], \\ \mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}] &:= R[U(\vec{a}; \hat{\beta})], & \mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}] &:= L[U(\vec{a}; \hat{\beta})], \end{aligned}$$

$$\mathcal{Q}_{\text{Plancherel, row}}^{q=0}[\gamma] := R[U_{\text{Plancherel}}(\vec{a}, \gamma)], \quad \mathcal{Q}_{\text{Plancherel, col}}^{q=0}[\gamma] := L[U_{\text{Plancherel}}(\vec{a}, \gamma)]$$

preserve the family of Schur processes. More precisely, we will deal with matrices  $\mathcal{Q}[\mathbf{B}] \in \mathbb{GT}^{(N)}$  such that

$$\mathcal{M}_{\mathbf{A}}^{\vec{a}; q=0} \mathcal{Q}[\mathbf{B}] = \mathcal{M}_{\mathbf{A} \cup \mathbf{B}}^{\vec{a}; q=0}, \quad \sum_{\lambda \in \mathbb{GT}^{(N)}} \mathcal{M}_{\mathbf{A}}^{\vec{a}; q=0}(\lambda) \mathcal{Q}[\mathbf{B}](\lambda \rightarrow \nu) = \mathcal{M}_{\mathbf{A} \cup \mathbf{B}}^{\vec{a}; q=0}(\nu), \quad \nu \in \mathbb{GT}^{(N)}, \quad (2.16)$$

where the specialization  $\mathbf{B}$  is as in one of the following elementary cases:

- (1)  $\mathbf{B} = (\alpha)$  is a specialization into one usual parameter  $\alpha$ .
- (2)  $\mathbf{B} = (\hat{\beta})$  is a specialization into one dual parameter  $\beta$ . (2.17)
- (3)  $\mathbf{B}$  is a specialization with  $\alpha = \beta \equiv 0$  and  $\gamma > 0$ .

---

<sup>9</sup>The reason for such notation will become clear from the next section, in which we define the more general  $q$ -Whittaker processes.



**Remark 2.20.** Operators  $\mathcal{Q}_{\text{Plancherel, row}}^{q=0}[\gamma]$  (respectively  $\mathcal{Q}_{\text{Plancherel, col}}^{q=0}[\gamma]$ ) form a Markov semigroup for  $\gamma \in \mathbb{R}_{\geq 0}$ . We can regard the Plancherel parameter  $\gamma$  as time, so these operators give rise to the continuous-time Markov processes, which can be described in the following way. Each letter  $j$  has an exponential clock with rate  $a_j$  for  $1 \leq j \leq N$ . Once the clock rings (with probability 1 two clocks never ring at the same time), the corresponding letter is inserted into the tableau via the row (respectively column) insertion. This continuous time dynamics can be seen as a continuous time limits of the discrete time dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  (respectively  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ ). More precisely, take  $\alpha = \beta = \Delta$ , let each discrete time step correspond to the continuous time interval  $\Delta$ , and take  $\Delta \rightarrow 0$ .

**2.6. RSK dynamics.** Let us now discuss four dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$  in more detail. The former two dynamics arise from the *row RSK algorithm*<sup>10</sup> applied to geometric or Bernoulli random input, respectively (cf. Remark 2.22 below). All of these four dynamics belong to the following class of dynamics on interlacing arrays.

**Definition 2.21.** A dynamics  $\mathcal{Q}$  on interlacing arrays will be called a *sequential update dynamics* if its one-step transition probabilities from  $\lambda$  to  $\nu$ ,  $\lambda, \nu \in \mathbb{GT}^{(N)}$ , have a product form

$$\begin{aligned} \mathcal{Q}(\lambda \rightarrow \nu) = \\ \mathcal{U}_1(\lambda^{(1)} \rightarrow \nu^{(1)}) \mathcal{U}_2(\lambda^{(2)} \rightarrow \nu^{(2)} \mid \lambda^{(1)} \rightarrow \nu^{(1)}) \dots \mathcal{U}_N(\lambda^{(N)} \rightarrow \nu^{(N)} \mid \lambda^{(N-1)} \rightarrow \nu^{(N-1)}), \end{aligned} \tag{2.18}$$

---

<sup>10</sup>The row RSK is the most classical version of the Robinson–Schensted–Knuth algorithm.

where  $\mathcal{U}_j$ 's are conditional probabilities of transitions at levels  $j = 1, \dots, N$  satisfying<sup>11</sup>

$$\mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) \geq 0, \quad \sum_{\nu^{(j)} \in \mathbb{GT}_j^+} \mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) = 1. \quad (2.19)$$

In words, the transition  $\lambda \rightarrow \nu$  looks as follows. First, update  $\lambda^{(1)} \rightarrow \nu^{(1)}$  at the bottom level  $\mathbb{GT}_1^+$  according to the distribution  $\mathcal{U}_1$ . Then for each  $j = 2, \dots, N$ , given the transition  $\lambda^{(j-1)} \rightarrow \nu^{(j-1)}$  at the previous level, update  $\lambda^{(j)} \rightarrow \nu^{(j)}$  at level  $\mathbb{GT}_j^+$  according to the conditional distribution  $\mathcal{U}_j$ . We see that the evolution of several first levels  $\lambda^{(1)}, \dots, \lambda^{(k)}$  of the interlacing array does not depend on what is happening at the upper levels  $\lambda^{(k+1)}, \dots, \lambda^{(N)}$ .

Under each of these RSK dynamics at each step of the discrete time corresponding to an update  $\lambda \rightarrow \nu$ , new randomness is introduced via  $N$  independent random variables  $V_1, \dots, V_N$ , which are either geometric random variables (belonging to  $\mathbb{Z}_{\geq 0}$ ) with parameters  $\alpha a_1, \dots, \alpha a_N$  in the case of  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ , or Bernoulli random variables  $\in \{0, 1\}$  with parameters  $\beta a_1, \dots, \beta a_N$  in the case of  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  and  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ . These random variables are resampled during each time step.

**Remark 2.22.** We see that all randomness in each of the four RSK-type dynamics can be organized into a matrix  $(V_j^{(t)})_{1 \leq j \leq N, t=1,2,\dots}$  (with appropriate distribution of the  $V_j^{(t)}$ 's). Such matrices containing nonnegative integers are usually thought of as inputs for classical Robinson–Schensted–Knuth correspondences.

Under each of the four dynamics, the particle at the first level of the array is updated as  $\nu_1^{(1)} = \lambda_1^{(1)} + V_1$ . Then, for each  $j = 2, \dots, N$ , assume that we are given signatures

---

<sup>11</sup>By agreement, for  $j = 1$  we mean  $\mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) \equiv \mathcal{U}_1(\lambda^{(1)} \rightarrow \nu^{(1)})$ .

$\bar{\lambda}, \bar{\nu} \in \mathbb{GT}_{j-1}^+$ ,  $\lambda \in \mathbb{GT}_j^+$  satisfying relations as on Fig. 15 (note that these relations depend on the type  $(\alpha)$  or  $(\hat{\beta})$  of the dynamics). Let us represent the movement  $\bar{\lambda} \rightarrow \bar{\nu}$  at level  $j-1$  as

$$\bar{\nu} - \bar{\lambda} = \sum_{i=1}^{j-1} c_i \bar{e}_i, \quad \begin{cases} c_i \in \mathbb{Z}_{\geq 0} & \text{in the case of } \mathcal{Q}_{\text{row}}^{q=0}[\alpha] \text{ and } \mathcal{Q}_{\text{col}}^{q=0}[\alpha]; \\ c_i \in \{0, 1\} & \text{in the case of } \mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}] \text{ and } \mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}] \end{cases}$$

(recall that  $\bar{e}_i$  is the  $i$ th basis vector of length  $j-1$ ). Also denote  $|c| := \sum_{i=1}^{j-1} c_i$ .

Depending on the dynamics, we will construct the signature  $\nu \in \mathbb{GT}_j^+$  (which also fits into relations on Fig. 15) as follows:

- ( $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ , Fig. 11) First, do  $|c|$  operations **pull** (Definition 2.6) in order *from left to right*, starting from position  $j-1$  all the way up to position 1. In more detail, let  $\mu(j-1, 0) := \lambda$  and for  $p = 1, \dots, c_{j-1}$  let

$$\mu(j-1, p) := \text{pull}(\mu(j-1, p-1) \mid \bar{\lambda} + (p-1)\bar{e}_{j-1} \rightarrow \bar{\lambda} + p\bar{e}_{j-1}),$$

then let  $\mu(j-2, 0) := \mu(j-1, c_{j-1})$  and for  $p = 1, \dots, c_{j-2}$  let

$$\mu(j-2, p) := \text{pull}(\mu(j-2, p-1) \mid \bar{\lambda} + c_{j-1}\bar{e}_{j-1} + (p-1)\bar{e}_{j-2} \rightarrow \bar{\lambda} + c_{j-1}\bar{e}_{j-1} + p\bar{e}_{j-2}),$$

etc., all the way up to  $\mu(1, c_1) := \text{pull}(\mu(1, c_1-1) \mid \bar{\nu} - \bar{e}_1 \rightarrow \bar{\nu})$ . (Clearly, if some  $c_i = 0$ , then the steps corresponding to  $\mu(i, \cdot)$  should be omitted.)

After these  $|c|$  operations, define  $\nu := \mu(1, c_1) + V_j e_1$ . That is, let the rightmost particle at level  $j$  jump to the right by  $V_j$  (which is a geometric random variable with parameter  $\alpha a_j$ ).

- ( $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$ , Fig. 12) First, define  $\mu(1, 0) := \lambda + V_j e_1$ . That is, let the rightmost particle at level  $j$  jump to the right by  $V_j$  (which is a Bernoulli random variable with parameter  $\beta a_j$ ).

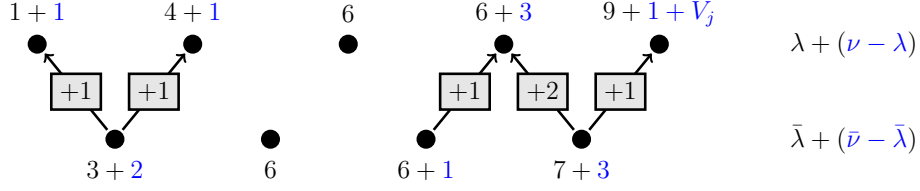


FIGURE 11. An example of a step of  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  at levels 4 and 5. Propagation steps (represented by numbers on arrows) are performed from left to right, according to **pull** operation. After that, the rightmost particle at level  $j$  jumps to the right by  $V_j$ .

After that, perform  $|c|$  operations **pull** (Definition 2.6) in order *from right to left*, starting from position 1 all the way up to position  $j-1$  (details are analogous to the above dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ ). Then set  $\nu := \mu(j-1, c_{j-1})$ .

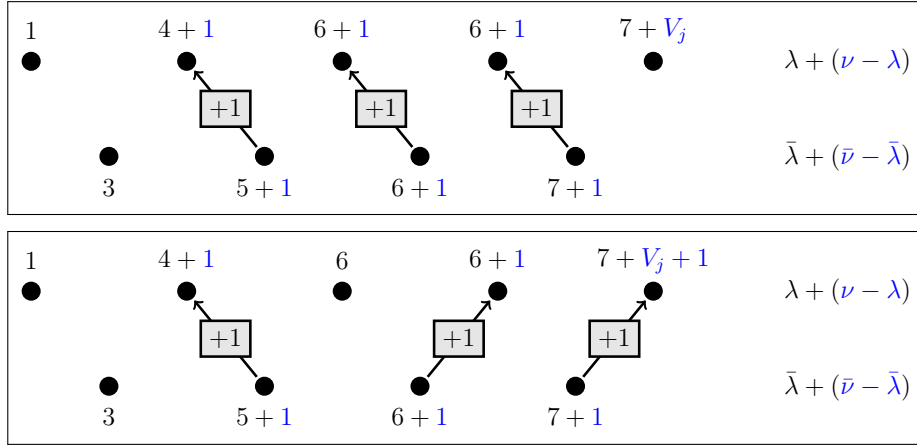


FIGURE 12. An example of a step of  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  at levels 4 and 5. Propagation steps are performed from right to left, according to **pull** operation. Above:  $V_j = 1$ , below:  $V_j = 0$ .

- ( $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ , Fig. 13) First, the leftmost particle  $\lambda_j$  at level  $j$  receives  $V_j$  moving impulses (here  $V_j$  is a geometric random variable with parameter  $\alpha a_j$ ). Each moving impulse means that  $\lambda_j$  tries to jump to the right by one, and if it is blocked (i.e., if  $\lambda_j = \bar{\lambda}_{j-1}$ ), then the moving impulse is donated to  $\lambda_{j-1}$ , etc. (see Remark 2.8). Denote the signature at level  $j$  arising after these  $V_j$  moving impulses by  $\mu(j-1, 0)$ .

After that, perform  $|c|$  operations **push** (Definition 2.7), in order *from left to right*, starting from position  $j - 1$  all the way up to position 1 (details are analogous to the above). Then we set  $\nu := \mu(1, c_1)$ .

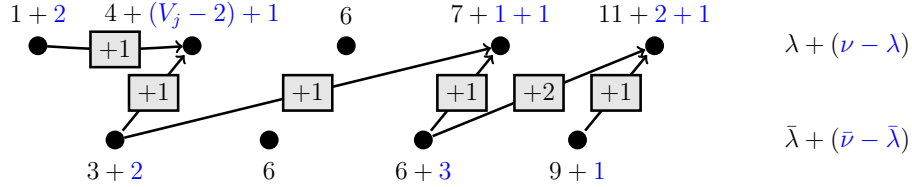


FIGURE 13. An example of a step of  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  at levels 4 and 5. We have  $V_j = 3$ , which means that initially the particle  $\lambda_5$  jumps to the right by 2 and the particle  $\lambda_4$  jumps by 1 (because of move donation). After that, propagation steps are performed from left to right, according to **push** operation.

- ( $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ , Fig. 14) First, perform  $|c|$  operations **push** (Definition 2.7), in order *from right to left*, starting from position 1 all the way up to position  $j - 1$  (details are analogous to what is done above). Let  $\mu(j - 1, c_{j-1})$  be the signature at level  $j$  arising after these  $|c|$  operations.

After that, let the leftmost particle at level  $j$  receives  $V_j$  moving impulses (here  $V_j$  is a Bernoulli random variable with parameter  $\beta a_j$ ). That is, if  $V_j = 0$ , then set  $\nu := \mu(j - 1, c_{j-1})$ . Otherwise, if  $V_j = 1$ , the  $j$ th particle at level  $j$  tries to jump to the right by one. If it is blocked, the impulse is donated to the  $(j - 1)$ th particle at level  $j$ , etc. In this case, denote by  $\nu$  the signature at level  $j$  arising after this moving impulse.

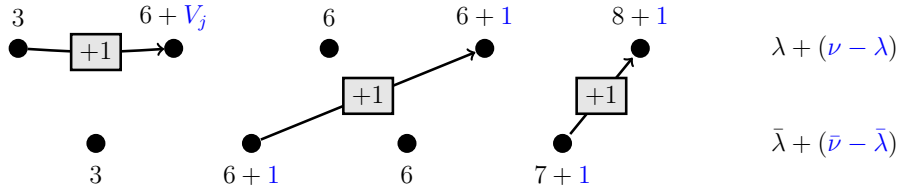


FIGURE 14. An example of a step of  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$  at levels 4 and 5. Propagation steps are performed from right to left, according to **push** operation. We have  $V_j = 1$ , and the jump of the rightmost particle at level  $j$  is donated to the right.

The above four rules of constructing the signature  $\nu \in \mathbb{GT}_j^+$  complete the description of the RSK-type dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ , and  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ , respectively.

**Remark 2.23.** By the very construction, at each step of any of the four above RSK-type dynamics the quantity  $|\lambda^{(N)}|$  is increased by  $V_1 + \dots + V_N$ , as it should be (cf. the discussion before Remark 4.8).

**Remark 2.24.** In RSK-type dynamics on  $q$ -Whittaker processes considered in §5 and §6 below, a part of new randomness at each step also comes from independent random variables  $V_1, \dots, V_N$  (having  $q$ -geometric or Bernoulli distribution, cf. Remark 4.2). Moreover, for  $q > 0$  the mechanisms of particle interactions will be  $q$ -randomized (i.e. will no longer be deterministic). This would lead to four  $q$ -randomized RSK insertion tableau dynamics: the row and column ( $\alpha$ ), and the row and column ( $\hat{\beta}$ ). In fact, for  $q > 0$  the step-by-step nature of the  $q = 0$  case (when push or pull operations are performed one at a time) will be broken, and certain series of push or pull operations will be clumped together and  $q$ -randomized as a whole. This will make the dynamics at the  $q$ -Whittaker level more complicated.

Each of the four RSK-type dynamics possesses a marginally Markovian projection (onto the leftmost or the rightmost particles of the interlacing array) leading to a certain discrete time particle system on  $\mathbb{Z}$ . Namely,  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  give rise to the *geometric and Bernoulli PushTASEPs*, respectively, on the rightmost particles  $\lambda_1^{(j)}$ ,  $j = 1, \dots, N$ . Similarly,  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$  lead to the *geometric and Bernoulli TASEPs*, respectively, on the leftmost particles  $\lambda_j^{(j)}$ . The  $q$ -deformed dynamics of §5 and §6 below would lead to  $q$ -deformations of these four particle systems.

### 3. $q$ -WHITTAKER PROCESSES

Schur processes can be generalized to the Macdonald processes, of which the  $q$ -Whittaker processes are a special case when we set the value of the deformation parameter  $t$  to 0. The definition of the Macdonald processes is based on Macdonald polynomials. Let us briefly recall their definition and properties which are essential for us. An excellent exposition and much more details may be found in [51, Ch. VI].

#### 3.1. $q$ -Whittaker and Macdonald polynomials.

**Definition 3.1.** Let  $q, t$  be two parameters. Consider the first order  $q$ -difference operator acting on functions in  $N$  variables:

$$(\mathcal{D}^{(1)}f)(x_1, \dots, x_N) := \sum_{i=1}^N \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} f(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_N).$$

This operator preserves the space  $\mathbb{C}(q, t)[x_1, \dots, x_N]^{S(N)}$  of symmetric polynomials with coefficients which are rational functions in  $q$  and  $t$ .

Eigenfunctions of  $\mathcal{D}^{(1)}$  are given by the *Macdonald symmetric polynomials*  $P_\lambda(x_1, \dots, x_N \mid q, t)$  indexed by  $\lambda \in \mathbb{GT}_N^+$ , with eigenvalues

$$\mathcal{D}^{(1)}P_\lambda = (q^{\lambda_1}t^{N-1} + q^{\lambda_2}t^{N-2} + \dots + q^{\lambda_{N-1}}t + q^{\lambda_N})P_\lambda$$

(which are pairwise distinct for generic  $q, t$ ). The polynomials  $P_\lambda$  are homogeneous, and form a linear basis for  $\mathbb{Q}(q, t)[x_1, \dots, x_N]^{S(N)}$ .

For  $q = t$  Macdonald polynomials become the Schur polynomials, and, in particular, their coefficients no longer depend on  $q$ . Similarly to the Schur polynomials, the Macdonald polynomials are *stable* in the sense that for any  $\lambda \in \mathbb{GT}_N^+$ ,

$$P_{\lambda \cup 0}(x_1, \dots, x_N, 0 \mid q, t) = P_\lambda(x_1, \dots, x_N \mid q, t),$$

so one may also speak about *Macdonald symmetric functions*  $P_\lambda(x_1, x_2, \dots \mid q, t)$ . The Macdonald symmetric functions admit an equivalent alternative definition:

**Definition 3.2.** Let  $(\cdot, \cdot)_{q,t}$  be the scalar product on  $\text{Sym}^{12}$  defined on products of power sums  $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$  as

$$(p_\lambda, p_\mu)_{q,t} = \mathbf{1}_{\lambda=\mu} z_\lambda(q, t), \quad z_\lambda(q, t) := \left( \prod_{i \geq 1} i^{m_i} (m_i)! \right) \cdot \left( \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \right),$$

where  $\lambda = (1^{m_1} 2^{m_2} \dots)$  means that  $\lambda$  has  $m_1$  parts equal to 1,  $m_2$  parts equal to 2, etc.

The  $P_\lambda$ 's form a unique family of homogeneous symmetric functions such that:

- (1) They are pairwise orthogonal with respect to the scalar product  $(\cdot, \cdot)_{q,t}$ .
- (2) For every  $\lambda$ , we have

$$P_\lambda(x_1, x_2, \dots \mid q, t) = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}} + \text{lower monomials in lexicographic order.}$$

The dependence on the parameters  $(q, t)$  is in coefficients of the lexicographically lower monomials.<sup>13</sup>

Set  $b_\lambda(q, t) := 1/(P_\lambda, P_\lambda)_{q,t}$ ; this is an explicit quantity determined via the shape of the Young diagram  $\lambda$ . Then the symmetric functions  $Q_\lambda(\cdot \mid q, t) := b_\lambda(q, t) P_\lambda(\cdot \mid q, t)$  are biorthonormal with the  $P_\lambda$ 's:  $(P_\lambda, Q_\mu)_{q,t} = \mathbf{1}_{\lambda=\mu}$ .

**Definition 3.3.** The skew Macdonald symmetric functions  $Q_{\lambda/\mu}$ ,  $\mu, \lambda \in \mathbb{GT}^+$ , are defined as the only symmetric functions for which  $(Q_{\lambda/\mu}, P_\nu)_{q,t} = (Q_\lambda, P_\mu P_\nu)_{q,t}$  for any  $\nu \in \mathbb{GT}^+$ .

---

<sup>12</sup>In this definition the algebra  $\text{Sym}$  of symmetric functions is considered over the field  $\mathbb{C}(q, t)$ , rather than  $\mathbb{C}$ , as we had before.

<sup>13</sup>Lexicographic order means that, for example,  $x_1^2$  is higher than  $\text{const} \cdot x_1 x_2$  which is in turn higher than  $\text{const} \cdot x_2^2$ .



The “ $P$ ” versions are given by  $P_{\lambda/\mu} = (b_\mu(q, t)/b_\lambda(q, t))Q_{\lambda/\mu}$ . These skew functions are identically zero unless  $\mu \subseteq \lambda$ . By setting  $q = t$  we can also talk about skew Schur functions.

The skew Macdonald symmetric functions enter the following recurrence relations:

$$P_\lambda(x_1, \dots, x_N) = \sum_{\mu \in \mathbb{GT}_{N-K}^+} P_{\lambda/\mu}(x_1, \dots, x_K) P_\mu(x_{K+1}, \dots, x_N), \quad \lambda \in \mathbb{GT}_N^+, \quad 1 \leq K \leq N \quad (3.1)$$

(and similarly for the  $Q_\lambda$ 's). This may be viewed as an alternative definition of the skew Macdonald polynomials  $P_{\lambda/\mu}$  in finitely many variables. If  $K = 1$  in (3.1), then the summation is over the interlacing signatures  $\mu \prec_h \lambda$ . In this case  $P_{\lambda/\mu}(x_1)$  is proportional to  $x_1^{|\lambda|-|\mu|}$  by homogeneity (cf. (3.3) below), and (3.1) is also sometimes referred to as the branching rule for the Macdonald polynomials.

From now on let us set the second Macdonald parameter  $t$  to zero. Then  $P_\lambda(\cdot \mid q, 0)$  are known as the  $q$ -Whittaker functions, i.e., the  $q$ -deformed  $\mathfrak{gl}_n$  Whittaker functions, cf. [37] and [8, §3]. This degeneration should also include changing the ground field from  $\mathbb{C}(q, t)$  to  $\mathbb{C}(q)$ . In fact, from now on we will just take  $q$  to be a number from  $[0, 1)$  and change the ground field back to  $\mathbb{C}$ .

**Remark 3.4.** Other notable degenerations of the Macdonald polynomials include the Hall–Littlewood polynomials (for  $q = 0, t \neq 0$ ). We refer to [51] and [41] for details.

We will use  $q$ -binomial coefficients and  $q$ -Pochhammer symbols

$$\binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad (a; q)_m := \begin{cases} (1-a)(1-aq) \dots (1-aq^{m-1}), & m > 0; \\ 1, & m = 0; \\ (1-aq^{-1})^{-1} (1-aq^{-2})^{-1} \dots (1-aq^m)^{-1}, & m < 0 \end{cases} \quad (3.2)$$

to record certain explicit  $q$ -dependent quantities related to  $q$ -Whittaker functions.<sup>14</sup> We have

$$P_{\lambda/\mu}(x_1 | q, 0) = \psi_{\lambda/\mu} x_1^{|\lambda| - |\mu|}, \quad \psi_{\lambda/\mu} = \psi_{\lambda/\mu}(q) := \mathbf{1}_{\mu \prec_h \lambda} \prod_{i=1}^{\ell(\mu)} \binom{\lambda_i - \lambda_{i+1}}{\lambda_i - \mu_i}_q; \quad (3.3)$$

$$Q_{\lambda/\mu}(x_1 | q, 0) = \phi_{\lambda/\mu} x_1^{|\lambda| - |\mu|}, \quad \phi_{\lambda/\mu} = \phi_{\lambda/\mu}(q) := \frac{\mathbf{1}_{\mu \prec_h \lambda}}{(q; q)_{\lambda_1 - \mu_1}} \prod_{i=1}^{\ell(\lambda)} \binom{\mu_i - \mu_{i+1}}{\mu_i - \lambda_{i+1}}_q. \quad (3.4)$$

By iteratively applying (3.1) we can get the following summation formula:

$$P_{\lambda}(x_1, \dots, x_N | q, 0) = \sum_{\lambda \in \mathbb{GT}^{(N)}(\lambda)} \psi_{\lambda} \prod_{j=1}^N x_j^{|\lambda^{(j)}| - |\lambda^{(j-1)}|}, \quad \psi_{\lambda} := \prod_{j=1}^N \psi_{\lambda^{(j)}/\lambda^{(j-1)}}. \quad (3.5)$$

**Definition 3.5.** A specialization  $\mathbf{A}$  of the algebra  $\mathbf{Sym}$  is said to be  $q$ -Whittaker nonnegative if  $P_{\lambda}(\mathbf{A}) \geq 0$  for any  $\lambda \in \mathbb{GT}^+$ . Such are the specializations  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)_q$ , where  $\boldsymbol{\alpha} = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0)$ ,  $\boldsymbol{\beta} = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$ ,  $\gamma \geq 0$ , and  $\sum_i (\alpha_i + \beta_i) < \infty$ , which may be defined via the generating function corresponding to signatures  $(n) \in \mathbb{GT}_1^+$ :

$$\sum_{n=0}^{\infty} Q_{(n)}(\mathbf{A}) \cdot u^n = e^{\gamma u} \prod_{i=1}^{\infty} \frac{1 + \beta_i u}{(\alpha_i u; q)_{\infty}} := \Pi_q(u; \mathbf{A}). \quad (3.6)$$

The left hand side of (3.6) is equal to  $\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1-q^k} p_k(\mathbf{A}) u^k\right)$ , so alternatively we can say that  $\mathbf{A}$  is defined by setting

$$p_1(A) = \sum_{i=1}^{\infty} \alpha_i + (1-q) \left( \sum_{i=1}^{\infty} \beta_i + \gamma \right), \quad p_k(A) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} (1-q^k) \left( \sum_{i=1}^{\infty} \beta_i^k \right)$$

for  $k = 2, 3, \dots$ . The *Kerov's conjecture* (see [41], section 2.9.3) states that the specializations of the form  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)_q$  exhaust all  $q$ -Whittaker nonnegative specializations. In [41] this conjecture is stated at the Macdonald level.

---

<sup>14</sup>In the  $q$ -Pochhammer symbol,  $m$  may be  $+\infty$  since  $0 \leq q < 1$ . Note also that in all cases,  $(a; q)_m = (a; q)_{\infty} / (aq^m; q)_{\infty}$

As before, we will refer to parameters  $\alpha_i$ ,  $\beta_i$  and  $\gamma$  as usual, dual and Plancherel parameters respectively. We will also stop specifying the dependence on  $q$  in the subscript of a specialization. Note that  $P_{\lambda/\mu}(\hat{\beta}) = Q_{\lambda'/\mu'}(\beta)$ . An obvious generalization of the recurrence relation (3.1) allows to express  $P_\lambda(\mathbf{A} \cup \mathbf{B})$  through  $P_{\lambda/\mu}(\mathbf{A})$  and  $P_\mu(\mathbf{B})$ . Thus, we can equivalently say that the specialization into usual parameters is *completely determined* by (3.3) (or (3.4)) and (3.1). Similarly, the specialization into dual parameters is determined by the same recurrence (3.1), but with a different one-parameter formula:

$$Q_{\lambda/\mu}(\hat{\beta}_1 \mid q, 0) = \psi'_{\lambda/\mu} \beta_1^{|\lambda| - |\mu|}, \quad \psi'_{\lambda/\mu} = \psi'_{\lambda/\mu}(q) := \mathbf{1}_{\mu \prec_\nu \lambda} \prod_{i \geq 1: \lambda_i = \mu_i, \lambda_{i+1} = \mu_{i+1} + 1} (1 - q^{\mu_i - \mu_{i+1}}). \quad (3.7)$$

We will also need Cauchy identities for  $q$ -Whittaker symmetric functions recorded below. Similar identities (involving  $t$ ) also exist for the general Macdonald symmetric functions.

$$\sum_{\lambda \in \text{GT}^+} P_\lambda(a_1, \dots, a_N) Q_\lambda(\mathbf{A}) = \Pi_q(a_1; \mathbf{A}) \dots \Pi_q(a_N; \mathbf{A}); \quad (3.8)$$

$$\sum_{\varkappa \in \text{GT}^+} P_{\varkappa/\lambda}(\mathbf{A}) Q_{\varkappa/\nu}(\mathbf{B}) = \Pi_q(\mathbf{A}; \mathbf{B}) \sum_{\mu \in \text{GT}^+} Q_{\lambda/\mu}(\mathbf{B}) P_{\nu/\mu}(\mathbf{A}). \quad (3.9)$$

In (3.9),  $\Pi_q(\mathbf{A}; \mathbf{B})$  is given by

$$\Pi_q(\mathbf{A}; \mathbf{B}) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1 - q^n} p_n(\mathbf{A}) p_n(\mathbf{B}) \right). \quad (3.10)$$

For the proofs see [51], VI.2.6 and VI.7.(Example 6). This definition agrees with (3.6) when one of the specializations is into a single usual parameter. Note also that  $\Pi_q(\mathbf{A} \cup \mathbf{B}; \mathbf{C}) = \Pi_q(\mathbf{A}; \mathbf{C}) \Pi_q(\mathbf{B}; \mathbf{C})$ .

Finally, we will need the Pieri rules: For any  $r \geq 1$ ,

$$P_{(1^r)} P_\mu = \sum_{\lambda: \lambda/\mu \text{ is a vertical } r\text{-strip}} \psi'_{\lambda/\mu} P_\lambda, \quad Q_{(r)} P_\mu = \sum_{\lambda: \lambda/\mu \text{ is a horizontal } r\text{-strip}} \phi_{\lambda/\mu} P_\lambda$$

$$P_{(r)}P_\mu = \sum_{\lambda: \lambda/\mu \text{ is a horizontal } r\text{-strip}} \psi_{\lambda/\mu} P_\lambda \quad (3.11)$$

(an  $r$ -strip means a strip consisting of  $r$  boxes). Here  $P_{(1r)} = e_r$  is in fact equal to the  $r$ -th elementary symmetric function  $e_r(x_1, x_2, \dots) = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}$  (note that  $e_1 = p_1$ ), and the  $Q_{(r)}$ 's are the quantities entering the generating function (3.6).

**3.2.  $q$ -Whittaker measures and processes.** The (depth  $N$ )  $q$ -Whittaker processes (first introduced in [8]) are probability measures on  $\boldsymbol{\lambda} \in \mathbb{GT}^{(N)}$  defined in the following way. The  $q$ -Whittaker process  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}$  depends on a  $q$ -Whittaker nonnegative specialization<sup>15</sup>  $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \gamma)_q$  (Definition 3.5) and on additional parameters  $\vec{a} = (a_1, \dots, a_N)$  with  $a_j > 0$ , satisfying  $\alpha_i a_j < 1$  for all possible  $i$  and  $j$  (this ensures the finiteness of the normalizing constant  $\Pi_q(\vec{a}; \mathbf{A})$  in (3.12) below). It attaches to  $\boldsymbol{\lambda} \in \mathbb{GT}^{(N)}$  the probability weight

$$\mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda}) = \frac{1}{\Pi_q(\vec{a}; \mathbf{A})} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) Q_{\lambda^{(N)}}(\mathbf{A}), \quad (3.12)$$

These weights sum to one as a corollary of (3.8). For  $q = 0$  we recover the Schur processes.

Alternatively, the probability weights  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda})$  may be defined via the generating function<sup>16</sup>

$$\sum_{\boldsymbol{\lambda}=(\lambda^{(1)} \prec_h \dots \prec_h \lambda^{(N)})} \mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda}) \left(\frac{u_1}{a_1}\right)^{|\lambda^{(1)}|} \left(\frac{u_2}{a_2}\right)^{|\lambda^{(2)}| - |\lambda^{(1)}|} \dots \left(\frac{u_N}{a_N}\right)^{|\lambda^{(N)}| - |\lambda^{(N-1)}|} = \frac{\Pi_q(\vec{u}; \mathbf{A})}{\Pi_q(\vec{a}; \mathbf{A})}, \quad (3.13)$$

<sup>15</sup>In the rest of the paper, we will speak only about  $q$ -Whittaker nonnegative specializations, and omit the words “ $q$ -Whittaker nonnegative”.

<sup>16</sup>In (3.13),  $\Pi_q(\vec{u}; \mathbf{A}) = \Pi_q(u_1; \mathbf{A}) \dots \Pi_q(u_N; \mathbf{A})$ , and similarly for the denominator (cf. (3.6), (3.10)). Here the  $a_j$ 's are regarded as constants, and the  $u_j$ 's as variables.

plus a certain *q-Gibbs property* requiring that the quantities

$$\frac{\mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda})}{P_{\lambda^{(1)}}(a_1)P_{\lambda^{(2)}/\lambda^{(1)}}(a_2)\dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)} \quad (3.14)$$

depend only on the top row  $\lambda^{(N)}$ , and not on  $\lambda^{(1)}, \dots, \lambda^{(N-1)}$ . Note that setting  $\vec{u} = \vec{a}$  turns (3.13) into an identity stating that the sum of all probability weights is 1.

**Remark 3.6.** It is natural to call the property involving quantities (3.14) “*q-Gibbs*” because for  $q = 0$  and  $a_1 = \dots = a_N = 1$  it reduces to the following *Gibbs property*: The conditional distribution of the interlacing array  $\boldsymbol{\lambda}$  under  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda})|_{q=0, a_j \equiv 1}$  obtained by fixing the top row  $\lambda^{(N)} \in \mathbb{GT}_N^+$  is the *uniform distribution* on the set of all interlacing arrays  $\boldsymbol{\lambda} \in \mathbb{GT}^{(N)}$  with fixed top row  $\lambda^{(N)}$  (note that the latter set is finite). For general  $q$  and  $\vec{a}$ , the conditional distribution will not be uniform, but instead each interlacing array will have the conditional weight proportional to  $P_{\lambda^{(1)}}(a_1)P_{\lambda^{(2)}/\lambda^{(1)}}(a_2)\dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)$ .

By the Cauchy identity (3.8) and the fact that the  $q$ -Whittaker polynomials form a linear basis, both definitions (3.13)–(3.14) are equivalent. To see this, one also has to note that  $\frac{P_{\lambda^{(1)}}(u_1)\dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(u_N)}{P_{\lambda^{(1)}}(a_1)\dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)}$  is equal to the product of  $(u_j/a_j)^{|\lambda^{(j)}| - |\lambda^{(j-1)}|}$  in the left-hand side of (3.13) (provided that the  $\lambda^{(j)}$ ’s satisfy the interlacing constraints).

The marginal distribution of the top row  $\lambda^{(N)}$  under  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}$  is the *q-Whittaker measure*  $\mathcal{MM}_{\mathbf{A}}^{\vec{a}}$  which is defined by either of the following equivalent ways:

$$\sum_{\lambda \in \mathbb{GT}_N^+} \mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) \frac{P_{\lambda}(\vec{u})}{P_{\lambda}(\vec{a})} = \frac{\Pi_q(\vec{u}; \mathbf{A})}{\Pi_q(\vec{a}; \mathbf{A})}, \quad (3.15)$$

$$\mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) = \frac{P_{\lambda}(\vec{a})Q_{\lambda}(\mathbf{A})}{\Pi_q(\vec{a}; \mathbf{A})}. \quad (3.16)$$

#### 4. MARKOV DYNAMICS

One of the main goals of the present paper is the construction of Markov dynamics preserving the family of  $q$ -Whittaker processes. More precisely, we will deal with infinite

matrices  $\mathcal{Q}[\mathbf{B}]$  (with rows and columns indexed by interlacing arrays) such that

$$\mathcal{M}_{\mathbf{A}}^{\vec{a}} \mathcal{Q}[\mathbf{B}] = \mathcal{M}_{\mathbf{A} \cup \mathbf{B}}^{\vec{a}}, \quad \sum_{\boldsymbol{\lambda}} \mathcal{M}_{\mathbf{A}}^{\vec{a}}(\boldsymbol{\lambda}) \mathcal{Q}[\mathbf{B}](\boldsymbol{\lambda} \rightarrow \boldsymbol{\nu}) = \mathcal{M}_{\mathbf{A} \cup \mathbf{B}}^{\vec{a}}(\boldsymbol{\nu}), \quad \boldsymbol{\nu} \in \mathbb{GT}^{(N)}, \quad (4.1)$$

where the specialization  $\mathbf{B}$  is as in one of the following cases:

- (1)  $\mathbf{B} = (\alpha)$  is a specialization into one usual parameter  $\alpha$ .
- (2)  $\mathbf{B} = (\hat{\beta})$  is a specialization into one dual parameter  $\beta$ . (4.2)
- (3)  $\mathbf{B}$  is a specialization with  $\boldsymbol{\alpha} = \boldsymbol{\beta} \equiv 0$  and  $\gamma > 0$ .

In the Schur case  $q = 0$  we already know such operators  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ ,  $\mathcal{Q}_{\text{Plancherel, row}}^{q=0}[\gamma]$ ,  $\mathcal{Q}_{\text{Plancherel, col}}^{q=0}[\gamma]$ , see (2.16). Our goal is to construct their  $q$ -deformations in the  $q$ -Whittaker case. As in Remark (2.20), the third case in (4.2) leads to continuous time Markov dynamics, in which the parameter  $\gamma$  plays the role of time. These continuous time dynamics  $\mathcal{Q}_{\text{Plancherel, row}}^q[\gamma]$  and  $\mathcal{Q}_{\text{Plancherel, col}}^q[\gamma]$  were constructed for the first time in [58] and [14] respectively. They are simpler than the discrete time processes (corresponding to the first two cases in (4.2)) considered in the present paper, and in fact arise as their continuous time limits (similar to the way it happens in Remark (2.20) for the Schur case), see subsections (5.6) and (6.7).

We will thus not focus on continuous time dynamics, and will deal with construction of matrices  $\mathcal{Q}[\alpha]$  and  $\mathcal{Q}[\hat{\beta}]$  whose elements  $\mathcal{Q}[\alpha](\boldsymbol{\lambda} \rightarrow \boldsymbol{\nu})$  and  $\mathcal{Q}[\hat{\beta}](\boldsymbol{\lambda} \rightarrow \boldsymbol{\nu})$  are transition probabilities from  $\boldsymbol{\lambda}$  to  $\boldsymbol{\nu}$  (where  $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathbb{GT}^{(N)}$ ) in one step of the discrete time. It is also helpful to view  $\mathcal{Q}[\alpha]$  and  $\mathcal{Q}[\hat{\beta}]$  as (*Markov operators*) acting on functions in the spatial variables  $\boldsymbol{\lambda}$  (e.g., these operators act in the space of bounded functions).

Adding a specialization  $\mathbf{B} = (\alpha)$  or  $(\hat{\beta})$  to  $\mathbf{A}$  as in (4.1) corresponds to multiplying the right-hand side of (3.13) by

$$\prod_{j=1}^N \frac{(\alpha a_j; q)_{\infty}}{(\alpha u_j; q)_{\infty}} \quad \text{or} \quad \prod_{j=1}^N \frac{1 + \beta u_j}{1 + \beta a_j}, \quad (4.3)$$

respectively, since  $\Pi_q(u; A)\Pi_q(u; \alpha) = \Pi_q(u, A \cup(\alpha))$  and  $\Pi_q(u; A)\Pi_q(u; \hat{\beta}) = \Pi_q(u, A \cup(\hat{\beta}))$ . (Factors containing  $a_j$  correspond to normalization, and it is the dependence on  $u_j$  in these expressions which is crucial.) The problem of finding Markov operators  $\mathcal{Q}[\alpha]$  and  $\mathcal{Q}[\hat{\beta}]$  can thus be informally restated as the problem of *turning* (by virtue of (3.13)) the *multiplication operators* in the variables  $\vec{u}$  (4.3) into operators *acting in the spatial variables*  $\lambda$ .

**4.1. Univariate dynamics.** A similar problem of turning multiplication operators (4.3) into operators acting in the spatial variables  $\lambda \in \mathbb{GT}_N^+$  may be posed for the generating function for the  $q$ -Whittaker measures (3.15), (3.16). In this case, the problem of finding the corresponding matrices  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  (with rows and columns indexed by signatures  $\lambda \in \mathbb{GT}_N^+$ ) has a *unique* solution:

**Proposition 4.1.** *There exist unique transition matrices  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  which add specializations  $(\alpha)$  or  $(\hat{\beta})$ , respectively, to the  $q$ -Whittaker measure  $\mathcal{MM}_{\mathbf{A}}^{\vec{a}}$  for every nonnegative specialization  $\mathbf{A}$ , in the sense similar to (4.1):*

$$\mathcal{MM}_{\mathbf{A}}^{\vec{a}} \mathcal{P}[\alpha] = \mathcal{MM}_{\mathbf{A} \cup(\alpha)}^{\vec{a}}, \quad \mathcal{MM}_{\mathbf{A}}^{\vec{a}} \mathcal{P}[\hat{\beta}] = \mathcal{MM}_{\mathbf{A} \cup(\hat{\beta})}^{\vec{a}}.$$

Their matrix elements are given by

$$\mathcal{P}[\alpha](\lambda \rightarrow \nu) = \prod_{j=1}^N (\alpha a_j; q)_{\infty} \frac{P_{\nu}(\vec{a})}{P_{\lambda}(\vec{a})} \phi_{\nu/\lambda} \alpha^{|\nu| - |\lambda|} \quad (4.4)$$

$$\mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu) = \prod_{j=1}^N \frac{1}{1 + \beta a_j} \frac{P_{\nu}(\vec{a})}{P_{\lambda}(\vec{a})} \psi'_{\nu/\lambda} \beta^{|\nu| - |\lambda|}, \quad (4.5)$$

where  $\phi_{\nu/\lambda}$  and  $\psi'_{\nu/\lambda}$  are explicit quantities given in (3.4) and (3.7), respectively.

Transition operators  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  were introduced in [8], see also [6] for a similar construction for the Schur measures (cf. §4.1 below).

*Proof.* Let us consider only the case of  $(\hat{\beta})$ , the case of  $(\alpha)$  is analogous.

Multiply both sides of (3.15) by  $\prod_{j=1}^N \frac{1+\beta u_j}{1+\beta a_j}$ . By the very definition of the  $q$ -Whittaker measures, the right-hand side can be rewritten as

$$\frac{\Pi_q(\vec{u}; \mathbf{A} \cup (\hat{\beta}))}{\Pi_q(\vec{a}; \mathbf{A} \cup (\hat{\beta}))} = \sum_{\nu \in \mathbb{GT}_N^+} \mathcal{MM}_{\mathbf{A} \cup (\hat{\beta})}^{\vec{a}}(\nu) \frac{P_\nu(\vec{u})}{P_\nu(\vec{a})}.$$

In the left-hand side, use the well-known property  $\prod_{j=1}^N (1 + \beta u_j) = \sum_{r=0}^N e_r(u_1, \dots, u_N) \beta^r$  of the elementary symmetric functions [51, I.(2.2)] together with the first Pieri rule (3.11) to write

$$P_\lambda(\vec{u}) \prod_{j=1}^N (1 + \beta u_j) = \sum_{\nu: \lambda \prec_\nu} P_\nu(\vec{u}) \psi'_{\nu/\lambda} \beta^{|\nu| - |\lambda|}.$$

(In the  $(\alpha)$  case, one needs to use the generating function (3.6) and the second Pieri rule.)

Then the left hand side becomes

$$\prod_{j=1}^N \frac{1}{1 + \beta a_j} \sum_{\lambda \in \mathbb{GT}_N^+} \sum_{\nu: \lambda \prec_\nu} \mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) \frac{P_\nu(\vec{u}) \psi'_{\nu/\lambda} \beta^{|\nu| - |\lambda|}}{P_\lambda(\vec{a})}.$$

Collecting the coefficients by  $P_\nu(\vec{u})/P_\nu(\vec{a})$  in the left-hand side, one can rewrite it as

$$\sum_{\nu \in \mathbb{GT}_N^+} \frac{P_\nu(\vec{u})}{P_\nu(\vec{a})} \sum_{\lambda: \lambda \prec_\nu} \mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu),$$

where the operator  $\mathcal{P}[\hat{\beta}]$  is given by (4.5). Since  $P_\nu(\vec{u})/P_\nu(\vec{a})$  are linearly independent as polynomials in  $\vec{u}$

$$\mathcal{MM}_{\mathbf{A} \cup (\hat{\beta})}^{\vec{a}}(\nu) = \sum_{\lambda: \lambda \prec_\nu} \mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu) = \sum_{\lambda} \mathcal{MM}_{\mathbf{A}}^{\vec{a}}(\lambda) \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu)$$

for all  $\nu \in \mathbb{GT}_N^+$ . To show uniqueness suppose there is another operator  $\mathcal{P}[\hat{\beta}]'$  that satisfies

$$\mathcal{MM}_{\mathbf{A}}^{\vec{a}} \mathcal{P}[\hat{\beta}]' = \mathcal{MM}_{\mathbf{A} \cup (\hat{\beta})}^{\vec{a}}.$$



Pick  $\lambda_0$  and  $\nu_0$ , such that  $\mathcal{P}[\hat{\beta}](\lambda_0 \rightarrow \nu_0) \neq \mathcal{P}[\hat{\beta}'](\lambda_0 \rightarrow \nu_0)$ . For any specialization  $\mathbf{A}$

$$\sum_{\lambda \in \mathbb{GT}_N^+} \mathcal{M}\mathcal{M}_{\mathbf{A}}^{\vec{a}}(\lambda) \left( \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu_0) - \mathcal{P}[\hat{\beta}'](\lambda \rightarrow \nu_0) \right) = 0.$$

Take  $\mathbf{A}$  to be a finite length specialization into usual parameters  $(\alpha_1, \dots, \alpha_N)$  and multiply both sides by  $\Pi_q(\vec{a}; \mathbf{A})$  to get that

$$\sum_{\lambda \in \mathbb{GT}_N^+} P_\lambda(\vec{a}) Q_\lambda(\alpha_1, \dots, \alpha_N) \left( \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu_0) - \mathcal{P}[\hat{\beta}'](\lambda \rightarrow \nu_0) \right) = 0$$

for any  $\alpha_1, \dots, \alpha_N \geq 0$ , which contradicts the fact that  $Q_\lambda(\alpha_1, \dots, \alpha_N)$  are linearly independent as polynomials in  $\alpha_1, \dots, \alpha_N$ . □

It follows from (3.9) that both operators  $\mathcal{P}[\hat{\beta}]$  and  $\mathcal{P}[\alpha]$  are stochastic, i.e. for any  $\lambda \in \mathbb{GT}_N^+$

$$\sum_{\nu \in \mathbb{GT}_N^+} \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu) = \sum_{\nu \in \mathbb{GT}_N^+} \mathcal{P}[\alpha](\lambda \rightarrow \nu) = 1. \quad (4.6)$$

**Remark 4.2.** If  $N = 1$  in Proposition 4.1, then both dynamics  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  (living on  $\mathbb{Z}_{\geq 0} = \mathbb{GT}_1^+$ ) are rather simple. Namely, under both dynamics, at each discrete time step the only particle  $\lambda_1^{(1)} \in \mathbb{Z}_{\geq 0} = \mathbb{GT}_1^+$  jumps to the right according to

- (1) the  $q$ -geometric distribution with parameter  $\alpha a_1$ , i.e.,  $\mathbf{p}_{\alpha a_1}(n) := (\alpha a_1; q)_\infty \frac{(\alpha a_1)^n}{(q; q)_n}$ ,  $n = 0, 1, 2, \dots$ ,<sup>17</sup> in the case of dynamics  $\mathcal{P}[\alpha]$ , or

---

<sup>17</sup>The fact that this is indeed a probability distribution follows from the  $q$ -binomial theorem.

(2) the *Bernoulli distribution* with parameter  $\beta a_1$  in the case of dynamics  $\mathcal{P}[\hat{\beta}]$ : the particle jumps to the right by one with probability  $\beta a_1/(1 + \beta a_1)$ , and stays put with the complementary probability  $1/(1 + \beta a_1)$ .<sup>18</sup>

More generally, one can show that under the dynamics on  $\mathbb{GT}_N^+$ , the quantities  $|\lambda^{(N)}|$  evolve as follows. For  $\mathcal{P}[\alpha]$ , at each discrete time step  $|\lambda^{(N)}|$  is increased by the sum of  $N$  independent  $q$ -geometric random variables with parameters  $\alpha a_1, \dots, \alpha a_N$ . For  $\mathcal{P}[\hat{\beta}]$ , at each discrete time step  $|\lambda^{(N)}|$  is increased by the sum of  $N$  independent Bernoulli random variables with parameters  $\beta a_1, \dots, \beta a_N$ . To see this, use (4.6) to write

$$\begin{aligned} \sum_{\nu \in \mathbb{GT}_N^+} \frac{P_\nu(\vec{a})}{P_\lambda(\vec{a})} \phi_{\nu/\lambda} \alpha^{|\nu| - |\lambda|} &= \prod_{j=1}^N \frac{1}{(\alpha a_j; q)_\infty}, \\ \sum_{\nu \in \mathbb{GT}_N^+} \frac{P_\nu(\vec{a})}{P_\lambda(\vec{a})} \psi'_{\nu/\lambda} \beta^{|\nu| - |\lambda|} &= \prod_{j=1}^N (1 + \beta a_j) \end{aligned}$$

for any  $\lambda \in \mathbb{GT}_N^+$ . Substituting  $\alpha u$  instead of  $\alpha$  (or  $\beta u$  instead of  $\beta$ ) in these equalities leads to

$$\begin{aligned} \sum_{\nu \in \mathbb{GT}_N^+} \mathcal{P}[\alpha](\lambda \rightarrow \nu) u^{|\nu| - |\lambda|} &= \prod_{j=1}^N \frac{(\alpha a_j; q)_\infty}{(\alpha a_j u; q)_\infty}, \\ \sum_{\nu \in \mathbb{GT}_N^+} \mathcal{P}[\hat{\beta}](\lambda \rightarrow \nu) u^{|\nu| - |\lambda|} &= \prod_{j=1}^N \frac{1 + \beta a_j u}{1 + \beta a_j}. \end{aligned}$$

The observation follows, since both left hand sides are the probability generating functions of  $|\nu| - |\lambda|$  in formal variable  $u$ .

We will call the dynamics  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  the *univariate dynamics*, and the corresponding dynamics on interlacing arrays  $\mathcal{Q}[\alpha]$  and  $\mathcal{Q}[\hat{\beta}]$  (which we aim to construct) the *multivariate*

---

<sup>18</sup>This parametrization of Bernoulli random variables will be used throughout the paper.

*dynamics.* In a way, multivariate dynamics on arrays  $\boldsymbol{\lambda} = (\lambda^{(1)} \prec_{\mathfrak{h}} \dots \prec_{\mathfrak{h}} \lambda^{(N)})$  *stitch together* univariate dynamics on all levels  $\lambda^{(j)}$ ,  $j = 1, \dots, N$ : Namely, started from a  $q$ -Gibbs distribution, the multivariate evolution of the array  $\boldsymbol{\lambda}$  reduces to the corresponding univariate dynamics on each of the levels  $\lambda^{(j)}$ ,  $j = 1, \dots, N$ . See §4.2 below and also [14, §2], for more discussion.

Instead of the case of univariate dynamics (driven by identity (3.15)), the problem of constructing multivariate dynamics (involving identity (3.13)) has a *whole family of solutions*. This phenomenon was known in the Schur ( $q = 0$ ) case for some time, with the presence of the RSK-type (e.g., see [53], [54]) and the push-block [12] dynamics (see §4.4 below for more detail). A similar phenomenon was investigated in [14] for continuous time dynamics increasing the parameter  $\gamma$  in the  $q$ -Whittaker processes.

**Remark 4.3.** Since the  $q$ -Whittaker polynomials  $P_{\lambda}(\vec{a})$  entering (4.4) and (4.5) are not given by an especially nice formula, transition probabilities of the univariate dynamics are harder to analyze. On the other hand, RSK-type multivariate dynamics which we construct in the present paper turn out to have simpler transition probabilities. Note also that multivariate dynamics on  $q$ -Gibbs distributions can be used to “simulate” the univariate ones, cf. the above discussion about “stitching”.

When  $q = 0$ , we have  $\psi_{\lambda/\mu} = \phi_{\lambda/\mu} = \mathbf{1}_{\mu \prec_{\mathfrak{h}} \lambda}$  and  $\psi'_{\lambda/\mu} = \mathbf{1}_{\mu \prec_{\mathfrak{v}} \lambda}$ . Univariate discrete time dynamics on the first level  $\mathbb{GT}_1^+ = \mathbb{Z}_{\geq 0}$  look as in Remark 4.2 with the understanding that the  $q$ -geometric distribution in the case of  $\mathcal{P}[\alpha]$  has to be replaced by the usual geometric distribution  $\mathbf{p}_{\alpha a_1}(n)|_{q=0} = (1 - \alpha a_1)(\alpha a_1)^n$ ,  $n = 0, 1, 2, \dots$

**Remark 4.4.** The continuous time dynamics on  $\mathbb{GT}_1^+$  increasing the parameter  $\gamma$  of the specialization is the usual Poisson process which can be obtained from either of the discrete time dynamics  $\mathcal{P}[\alpha]$  or  $\mathcal{P}[\hat{\beta}]$  in a small  $\alpha$  or small  $\beta$  limit, respectively. In fact, this observation is also true in the general  $q > 0$  case.

The univariate dynamics  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  at any higher level  $\mathbb{GT}_N^+$ ,  $N = 2, 3, \dots$  (described in a  $q = 0$  version of Proposition 4.1), can be obtained from the  $N = 1$  dynamics via the *Doob's h-transform* procedure. Informally, to get the dynamics of  $N$  distinct particles  $(x_1 > \dots > x_N)$  on  $\mathbb{Z}_{\geq 0}$  (this state space is the same as  $\mathbb{GT}_N^+$  up to a shift  $x_i = \lambda_i + N - i$ ), one should consider the dynamics of  $N$  *independent* particles  $x_j$  each of which evolves according to the corresponding  $N = 1$  dynamics, and then impose the *condition* that the particles *never collide* and have relative asymptotic speeds  $a_1, \dots, a_N$ , respectively. This conditioning gives rise to the presence of the factors  $s_\nu(\vec{a})/s_\lambda(\vec{a})$  in transition probabilities (cf. Proposition 4.1). We refer to, e.g., [46], [45], [54], [61] for details on noncolliding dynamics.

It is worth noting that the Dyson's Brownian motion coming from  $N \times N$  GUE random matrices [27] arises via a similar procedure by considering noncolliding Brownian particles. One may thus think that the univariate dynamics  $\mathcal{P}[\alpha]$  and  $\mathcal{P}[\hat{\beta}]$  on  $\mathbb{GT}_N^+$  are certain discrete analogues of the Dyson's Brownian motion.

**4.2. Main equations.** Here we write down linear equations whose solutions correspond to multivariate discrete time Markov dynamics on  $q$ -Whittaker processes. We will be looking within the class of sequential update dynamics (2.21), which includes all four dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$ ,  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$ . For a sequential update dynamics it suffices to describe the evolution at any two consecutive levels  $j - 1$  and  $j$ .

**Theorem 4.5.** *A sequential update dynamics  $\mathcal{Q}$  defined via (2.18)–(2.19) preserves the class of  $q$ -Whittaker processes  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}$  and adds a new usual parameter  $\alpha$  to the specialization  $\mathbf{A}$  if and only if*

$$\sum_{\bar{\lambda} \in \mathbb{GT}_{j-1}^+} \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})(\alpha a_j)^{|\lambda| - |\nu| - (|\bar{\lambda}| - |\bar{\nu}|)} \psi_{\lambda/\bar{\lambda}} \phi_{\bar{\nu}/\bar{\lambda}} = (\alpha a_j; q)_\infty \psi_{\nu/\bar{\nu}} \phi_{\nu/\lambda} \quad (4.7)$$

for any  $j = 1, 2, \dots, N$  and any  $\lambda, \nu \in \mathbb{GT}_j^+$ ,  $\bar{\nu} \in \mathbb{GT}_{j-1}^+$ , such that the four signatures  $\bar{\lambda}, \bar{\nu}, \lambda, \nu$  are related as on Fig. 15, left (in particular, the above summation is taken only over  $\bar{\lambda}$  satisfying  $\bar{\lambda} \prec_{\mathfrak{h}} \bar{\nu}$ ,  $\bar{\lambda} \prec_{\mathfrak{h}} \lambda$ ). For  $j = 1$  we take  $\bar{\lambda} = \bar{\nu} = \emptyset$  in this equation and it becomes equivalent to  $\mathcal{U}_1 = \mathcal{P}[\alpha]$  at level  $\mathbb{GT}_1^+$  (as in Remark 4.2).

Similarly, a dynamics  $\mathcal{Q}$  preserves the class of  $q$ -Whittaker processes and adds a new dual parameter  $\beta$  to the specialization  $\mathbf{A}$  if and only if

$$\sum_{\bar{\lambda} \in \mathbb{GT}_{j-1}^+} \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})(\beta a_j)^{|\lambda| - |\nu| - (|\bar{\lambda}| - |\bar{\nu}|)} \psi_{\lambda/\bar{\lambda}} \psi'_{\bar{\nu}/\bar{\lambda}} = \frac{1}{1 + \beta a_j} \psi_{\nu/\bar{\nu}} \psi'_{\nu/\lambda} \quad (4.8)$$

for any  $j = 1, 2, \dots, N$  and any  $\lambda, \nu \in \mathbb{GT}_j^+$ ,  $\bar{\nu} \in \mathbb{GT}_{j-1}^+$ , such that the four signatures  $\bar{\lambda}, \bar{\nu}, \lambda, \nu$  are related as on Fig. 15, right (in particular, the above summation is taken only over  $\bar{\lambda}$  satisfying  $\bar{\lambda} \prec_{\mathfrak{v}} \bar{\nu}$ ,  $\bar{\lambda} \prec_{\mathfrak{h}} \lambda$ ). For  $j = 1$  we take  $\bar{\lambda} = \bar{\nu} = \emptyset$  in this equation and it becomes equivalent to  $\mathcal{U}_1 = \mathcal{P}[\hat{\beta}]$  at level  $\mathbb{GT}_1^+$  (as in Remark 4.2).



FIGURE 15. Squares of four signatures on two consecutive levels relevant to conditional transition  $\lambda \rightarrow \nu$  on the upper level given the transition  $\bar{\lambda} \rightarrow \bar{\nu}$  on the lower level, under dynamics  $\mathcal{Q}[\alpha]$  (left) and  $\mathcal{Q}[\hat{\beta}]$  (right). Note the similarity to blocks in Fomin’s growth diagrams (about the latter, see [30], [31], [32], [33]).

The proof of these equations was established in [14, §2.2] using a more general framework of Gibbs-like measures. However, for the sake of completeness, we reproduce it here in our particular setting of the  $q$ -Whittaker processes.

*Proof.* Let us consider only the case of adding  $(\alpha)$ , as the case of  $(\hat{\beta})$  is analogous.

The fact that a sequential update dynamics  $\mathcal{Q}$  defined via (2.18)–(2.19) preserves the class of  $q$ -Whittaker processes  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}$  and adds a new usual parameter  $\alpha$  to the specialization  $\mathbf{A}$  means that

$$\begin{aligned} & \sum_{\lambda} \frac{1}{\prod_q(\vec{a}; \mathbf{A})} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) Q_{\lambda^{(N)}}(\mathbf{A}) \\ & \mathcal{U}_1(\lambda^{(1)} \rightarrow \nu^{(1)}) \mathcal{U}_2(\lambda^{(2)} \rightarrow \nu^{(2)} \mid \lambda^{(1)} \rightarrow \nu^{(1)}) \dots \mathcal{U}_N(\lambda^{(N)} \rightarrow \nu^{(N)} \mid \lambda^{(N-1)} \rightarrow \nu^{(N-1)}) = \\ & = \frac{1}{\prod_q(\vec{a}; \mathbf{A} \cup (\alpha))} P_{\nu^{(1)}}(a_1) P_{\nu^{(2)}/\nu^{(1)}}(a_2) \dots P_{\nu^{(N)}/\nu^{(N-1)}}(a_N) Q_{\nu^{(N)}}(\mathbf{A} \cup (\alpha)) \quad \text{for every } \nu \text{ and } \mathbf{A}. \end{aligned} \tag{4.9}$$

Using (3.1), we can rewrite (4.9) as

$$\begin{aligned} & \sum_{\lambda} \left( \prod_{j=1}^N \mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) a_j^{(|\lambda^{(j)}| - |\nu^{(j)}| - (|\lambda^{(j-1)}| - |\nu^{(j-1)}|))} \psi_{\lambda^{(j)}/\lambda^{(j-1)}} \right) Q_{\lambda^{(N)}}(\mathbf{A}) = \\ & = \left( \prod_{j=1}^N (\alpha a_j; q)_{\infty} \psi_{\nu^{(j)}/\nu^{(j-1)}} \right) \sum_{\nu \in \mathbb{GT}_N^+} Q_{\nu}(\mathbf{A}) \alpha^{|\nu^{(N)}| - |\nu|} \phi_{\nu^{(N)}/\nu} \quad \text{for every } \nu \text{ and } \mathbf{A}. \end{aligned}$$

Since  $Q_{\lambda}(\mathbf{A})$  are linearly independent as polynomials in  $u_1, \dots, u_N$  for a finite length specialization  $\mathbf{A}$  into usual variables  $(u_1, \dots, u_N)$ , this is equivalent to saying that

$$\begin{aligned} & \sum_{\lambda: \lambda^{(N)} = \lambda} \prod_{j=1}^N \mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) (\alpha a_j)^{(|\lambda^{(j)}| - |\nu^{(j)}| - (|\lambda^{(j-1)}| - |\nu^{(j-1)}|))} \psi_{\lambda^{(j)}/\lambda^{(j-1)}} = \\ & = \phi_{\nu^{(N)}/\lambda^{(N)}} \prod_{j=1}^N (\alpha a_j; q)_{\infty} \psi_{\nu^{(j)}/\nu^{(j-1)}} \quad \text{for all } \nu \text{ and } \lambda. \end{aligned} \tag{4.10}$$

Suppose that  $\mathcal{U}_1 = \mathcal{P}[\alpha]$  at level  $\mathbb{GT}_1^+$ , and  $\mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)})$  satisfy (4.7) for  $2 \leq j \leq N$ . Then we can show by induction on  $k$ , that

$$\sum_{\lambda: \lambda^{(k)} = \lambda} \prod_{j=1}^k \mathcal{U}_j(\lambda^{(j)} \rightarrow \nu^{(j)} \mid \lambda^{(j-1)} \rightarrow \nu^{(j-1)}) (\alpha a_j)^{(|\lambda^{(j)}| - |\nu^{(j)}| - (|\lambda^{(j-1)}| - |\nu^{(j-1)}|))} \psi_{\lambda^{(j)}/\lambda^{(j-1)}} =$$

$$= \phi_{\nu^{(k)}/\lambda^{(k)}} \prod_{j=1}^k (\alpha a_j; q)_\infty \psi_{\nu^{(j)}/\nu^{(j-1)}} \text{ for all } 1 \leq k \leq N, \boldsymbol{\nu} = (\nu^{(1)} \prec_{\mathbf{h}} \nu^{(2)} \prec_{\mathbf{h}} \dots \prec_{\mathbf{h}} \nu^{(k)}), \lambda \in \mathbb{GT}_k^+. \quad (4.11)$$

Base for  $k = 1$  follows from the fact that  $\mathcal{U}_1 = \mathcal{P}[\alpha]$  at level  $\mathbb{GT}_1^+$ , while the inductive step follows from (4.7). So (4.10) holds.

For the other direction, suppose that (4.10) (and hence (4.9)) holds. For  $1 \leq k \leq N$  by summing (4.9) over  $\nu^{(k+1)}, \dots, \nu^{(N)}$  and applying (3.9) we get

$$\begin{aligned} & \sum_{\lambda} \frac{1}{\prod_q(a_1, \dots, a_k; \mathbf{A})} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \dots P_{\lambda^{(k)}/\lambda^{(k-1)}}(a_k) Q_{\lambda^{(k)}}(\mathbf{A}) \\ & \mathcal{U}_1(\lambda^{(1)} \rightarrow \nu^{(1)}) \mathcal{U}_2(\lambda^{(2)} \rightarrow \nu^{(2)} \mid \lambda^{(1)} \rightarrow \nu^{(1)}) \dots \mathcal{U}_k(\lambda^{(k)} \rightarrow \nu^{(k)} \mid \lambda^{(k-1)} \rightarrow \nu^{(k-1)}) = \\ & = \frac{1}{\prod_q(a_1, \dots, a_k; \mathbf{A} \cup (\alpha))} P_{\nu^{(1)}}(a_1) P_{\nu^{(2)}/\nu^{(1)}}(a_2) \dots P_{\nu^{(k)}/\nu^{(k-1)}}(a_k) Q_{\nu^{(k)}}(\mathbf{A} \cup (\alpha)) \end{aligned}$$

for every  $\boldsymbol{\nu}$  and  $\mathbf{A}$ , which implies (4.11). For  $k = 1$  it means that  $\mathcal{U}_1 = \mathcal{P}[\alpha]$  at level  $\mathbb{GT}_1^+$ , while for  $k \geq 2$  using (4.11) for both  $k$  and  $k - 1$  implies (4.7). □

In a continuous time setting, there also exist linear equations governing multivariate dynamics, cf. [14, §2.4]. In fact, the latter equations arise as small  $\alpha$  or small  $\beta$  limits of (4.7) or (4.8), respectively. Markov dynamics on  $q$ -Whittaker processes corresponding to solutions to these continuous time equations were constructed in [58], [14], [17].

**4.3. Discussion of main equations.** Let us make a number of general remarks about the main equations of Theorem 4.5.

4.3.1. The paper [14] contains a classification result in continuous time setting, which was achieved by further restricting the class of dynamics by imposing certain *nearest neighbor interaction* constraints. Under these constraints, putting together continuous time linear equations (which look similarly to (4.7) and (4.8)) with fixed  $\lambda$  and  $\bar{\nu}$  in a generic position,

at level  $j$  one arrives at a system of  $j$  linear equations with  $3j - 2$  variables. Solutions of such a system admit a reasonable classification.

It remains unclear how to impose (preferably, natural) constraints on solutions of discrete time equations (4.7) or (4.8) so that the family of all solutions would admit a reasonable description. Indeed, for example, in the case of a usual parameter (4.7), the number of variables is infinite while the number of available equations is finite. Therefore, in §5 and §6 below we devote our attention to constructing certain particular multivariate discrete time dynamics satisfying equations (4.8) and (4.7), respectively.

4.3.2. Note that summing (4.7) or (4.8) over  $\nu \in \mathbb{GT}_j^+$  leads to the skew Cauchy identity with both specializations being into one parameter (cf. (3.9)):

$$\sum_{\bar{\lambda} \in \mathbb{GT}^+} P_{\lambda/\bar{\lambda}}(a_j) Q_{\bar{\nu}/\bar{\lambda}}(\mathbf{B}) = \frac{1}{\Pi_q(a_j; \mathbf{B})} \sum_{\nu \in \mathbb{GT}^+} P_{\nu/\bar{\nu}}(a_j) Q_{\nu/\lambda}(\mathbf{B}), \quad \mathbf{B} = (\alpha) \text{ or } (\hat{\beta}). \quad (4.12)$$

Identity (4.12) may also be interpreted as a certain commutation relation between the univariate Markov operators  $\mathcal{P}[\alpha]$  or  $\mathcal{P}[\hat{\beta}]$  (of Proposition 4.1) and Markov projection operators (or *links*)<sup>19</sup>

$$\Lambda_{j-1}^j(\lambda, \bar{\lambda}) := \frac{P_{\bar{\lambda}}(a_1, \dots, a_{j-1})}{P_{\lambda}(a_1, \dots, a_j)} P_{\lambda/\bar{\lambda}}(a_j), \quad \lambda \in \mathbb{GT}_j^+, \quad \bar{\lambda} \in \mathbb{GT}_{j-1}^+,$$

in the sense that

$$\mathcal{P}[\alpha]^{(j)} \Lambda_{j-1}^j = \Lambda_{j-1}^j \mathcal{P}[\alpha]^{(j-1)}, \quad (4.13)$$

and similarly for  $\mathcal{P}[\hat{\beta}]$ . Indices  $j$  and  $j - 1$  in  $\mathcal{P}[\alpha]$  above mean the level of the interlacing array at which the transition operator of the univariate dynamics acts.

---

<sup>19</sup>These links in fact determine the  $q$ -Gibbs property (3.14); e.g., see [14, §2] for more detail.



One can thus say that each solution to the main equations (4.7) or (4.8) (and, therefore, each discrete time Markov dynamics on  $q$ -Whittaker processes) corresponds to a *refinement* of the skew Cauchy identity (4.12) (or of the commutation relation (4.13)).

4.3.3. The parameters  $a_1, \dots, a_{j-1}$  (but not  $a_j$ ) essentially do not contribute to the main equations (4.7), (4.8): they enter the equations only as a requirement that  $\bar{\nu} \in \mathbb{GT}_{j-1}^+$  and  $\lambda, \nu \in \mathbb{GT}_j^+$ . Thus, equations (4.7), (4.8) essentially depend on two specializations: a specialization into one usual parameter  $\mathbf{\Lambda} = (a_j)$  which corresponds to increasing the level number, and a specialization  $\mathbf{B} = (\alpha)$  or  $(\hat{\beta})$  which corresponds to time evolution. This allows to think of diagrams as on Fig. 15, as well as of main equations, for *any* specializations  $\mathbf{\Lambda}$  and  $\mathbf{B}$  (see Fig. 16). It suffices to consider three elementary cases for

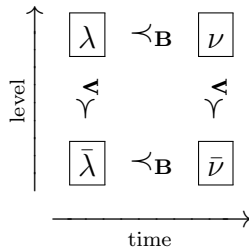


FIGURE 16. A square of four signatures corresponding to arbitrary specializations  $\mathbf{\Lambda}$  and  $\mathbf{B}$ . Notation  $\bar{\lambda} \prec_{\mathbf{\Lambda}} \lambda$  means that  $P_{\lambda/\bar{\lambda}}(\mathbf{\Lambda}) > 0$ , and similarly for  $\prec_{\mathbf{B}}$ . When the specialization  $\mathbf{\Lambda}$  is into a single usual or dual parameter,  $\prec_{\mathbf{\Lambda}}$  reduces to  $\prec_{\mathbf{h}}$  or  $\prec_{\mathbf{v}}$ , respectively.

$\mathbf{\Lambda}$  and  $\mathbf{B}$  as in (4.2). This yields 9 possible systems of equations for dynamics. If one of the specializations is pure Plancherel (case (3) in (4.2)), then the corresponding Markov dynamics on  $q$ -Whittaker processes were essentially constructed in [14], [17]. This leaves four systems of equations in which both  $\mathbf{\Lambda}$  and  $\mathbf{B}$  are specializations into a single usual or dual parameter. In this paper we address two of these four cases corresponding to  $\mathbf{\Lambda} = (a_j)$ , which in particular give rise to two new discrete time  $q$ -PushTASEPs (as marginally Markovian projections of dynamics on interlacing arrays, see §5.2 and §6.3).

4.3.4. In fact, one can define the quantities  $\psi_{\lambda/\mu}(q, t)$ ,  $\phi_{\lambda/\mu}(q, t)$ ,  $\psi'_{\lambda/\mu}(q, t)$  for the general Macdonald parameters  $(q, t)$  (see [51, Ch. VI]), and thus write down the corresponding main linear equations for any specializations  $\mathbf{A}$  and  $\mathbf{B}$ . (In particular, for  $t \neq 0$  the right-hand side of the identity (3.6) defining a specialization should be multiplied by  $\prod_{i=1}^{\infty} (t\alpha_i u; q)_{\infty}$ .) It is not known whether there exist other solutions to the main equations for general  $(q, t)$  yielding honest Markov dynamics (i.e., having *nonnegative* transition probabilities) except the *push-block solution* (see §4.4 below for the definition). We do not address this question in the present paper.

There is a rather simple transformation of the main equations for general  $(q, t)$  (related to transposition of Young diagrams) which interchanges  $q \leftrightarrow t$  and swaps usual and dual parameters in both specializations  $\mathbf{A}$  and  $\mathbf{B}$  [17]. This transformation relates the  $q$ -Whittaker ( $t = 0$ ) and the Hall–Littlewood ( $q = 0$ ) settings.

The remaining two cases of the ( $q$ -Whittaker) main equations mentioned above (corresponding to  $\mathbf{A} = (\hat{b})$ , a specialization into a dual parameter) should thus be thought of as discrete time versions of the continuous time equations of [17] (relevant to the Hall–Littlewood setting). As such, (conjectural) solutions to the former equations leading to discrete time dynamics on interlacing arrays are unlikely to produce new marginally Markovian TASEP-like particle systems in one space dimension (see also discussion in [14, §8.3]). In the present paper, we do not address these two remaining cases corresponding to the Hall–Littlewood setting.

**4.4. Push-block dynamics.** There is a rather straightforward general construction (dating back to an idea of [25]) leading to certain particular multivariate dynamics. Namely, assume that the conditional probabilities  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  entering the main equations (Theorem 4.5) *do not depend* on  $\bar{\lambda}$ . Then each equation (corresponding to fixed  $\lambda, \nu \in \mathbb{GT}_j^+$ , and  $\bar{\nu} \in \mathbb{GT}_{j-1}^+$ ) contains only one unknown  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu})$ . Thus, the main equations admit a unique solution. Let us consider the case of a usual parameter  $\alpha$  (4.7).

Observe that the left-hand side of (4.7) takes the following form (where signatures satisfy conditions on Fig. 15, left):

$$\begin{aligned}
\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) &= \sum_{\bar{\lambda} \in \mathbb{GT}_{j-1}^+} (\alpha a_j)^{|\lambda| - |\nu| - (|\bar{\lambda}| - |\bar{\nu}|)} \psi_{\lambda/\bar{\lambda}} \phi_{\bar{\nu}/\bar{\lambda}} \\
&= \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) \alpha^{|\lambda| - |\nu|} a_j^{-|\nu| + |\bar{\nu}|} \sum_{\bar{\lambda} \in \mathbb{GT}_{j-1}^+} P_{\lambda/\bar{\lambda}}(a_j) Q_{\bar{\nu}/\bar{\lambda}}(\alpha) \\
&= \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) (\alpha a_j; q)_\infty \sum_{\varkappa \in \mathbb{GT}_j^+} (\alpha a_j)^{|\varkappa| - |\nu|} \psi_{\varkappa/\bar{\nu}} \phi_{\varkappa/\lambda},
\end{aligned}$$

where we have used the skew Cauchy identity (4.12). Then (4.7) yields the solution

$$\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) = \frac{(\alpha a_j)^{|\nu|} \psi_{\nu/\bar{\nu}} \phi_{\nu/\lambda}}{\sum_{\varkappa \in \mathbb{GT}_j^+} (\alpha a_j)^{|\varkappa|} \psi_{\varkappa/\bar{\nu}} \phi_{\varkappa/\lambda}}. \quad (4.14)$$

In (4.14) as well as in the above computation, it should be  $\lambda \prec_{\mathfrak{h}} \nu$ ,  $\bar{\nu} \prec_{\mathfrak{h}} \nu$  and  $\lambda \prec_{\mathfrak{h}} \varkappa$ ,  $\bar{\nu} \prec_{\mathfrak{h}} \varkappa$ , see Fig. 15, left.

Similarly, the solution of (4.8) not depending on  $\bar{\lambda}$  looks as

$$\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) = \frac{(\beta a_j)^{|\nu|} \psi_{\nu/\bar{\nu}} \psi'_{\nu/\lambda}}{\sum_{\varkappa \in \mathbb{GT}_j^+} (\beta a_j)^{|\varkappa|} \psi_{\varkappa/\bar{\nu}} \psi'_{\varkappa/\lambda}}. \quad (4.15)$$

The signatures have to be related as on Fig. 15, right, i.e.,  $\lambda \prec_{\mathfrak{v}} \nu$ ,  $\bar{\nu} \prec_{\mathfrak{h}} \nu$ , and  $\lambda \prec_{\mathfrak{v}} \varkappa$ ,  $\bar{\nu} \prec_{\mathfrak{h}} \varkappa$ .

**Definition 4.6.** We will call the multivariate dynamics defined by (4.14) or (4.15) the (*discrete time*) *push-block dynamics* on  $q$ -Whittaker processes adding a specialization  $(\alpha)$  or  $(\hat{\beta})$ , respectively. We denote these dynamics by  $\mathcal{Q}_{\text{pb}}^q[\alpha]$  and  $\mathcal{Q}_{\text{pb}}^q[\hat{\beta}]$ .

The construction of push-block dynamics can be equivalently described as follows. Recall the commutation relation between the univariate dynamics  $\mathcal{P}$  and the stochastic links  $\Lambda_{j-1}^j$  (4.13). Then one can say that the multivariate dynamics chooses  $\nu$  at random according

to the distribution of the middle signature in a chain of Markov operators

$$\lambda \xrightarrow{\mathfrak{P}^{(j)}} \nu \xrightarrow{\Lambda_{j-1}^j} \bar{\nu},$$

conditioned on the first signature  $\lambda$  and the last signature  $\bar{\nu}$ . Denominators in formulas (4.14) and (4.15) reflect this conditioning.

Setting  $q = 0$  greatly simplifies formulas (4.14) and (4.15) thus leading to nice push-block multivariate dynamics on interlacing arrays. They were introduced and studied in [12]. For the analogous dynamics in the case of Dyson's Brownian motions see [73].

Due to the sequential nature of multivariate dynamics (2.18), we will consider evolution at consecutive levels  $j - 1$  and  $j$ . Assuming that the movement  $\bar{\lambda} \rightarrow \bar{\nu}$  at level  $j - 1$  and the old configuration  $\lambda$  at level  $j$  are given, we will describe the probability distribution of  $\nu \in \mathbb{GT}_j^+$  corresponding to  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$ .

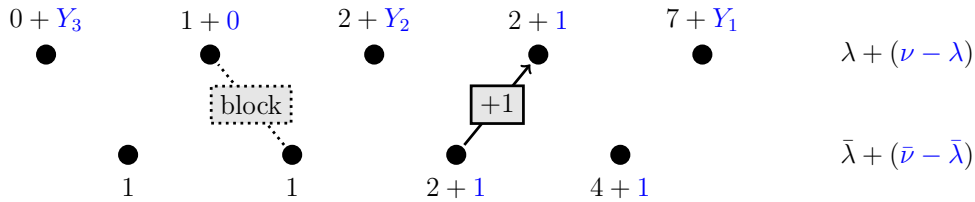


FIGURE 17. An example of a step of  $\mathcal{Q}_{\text{pb}}^{q=0}[\hat{\beta}]$  at levels 4 and 5. Here  $\lambda = (7, 2, 2, 1, 0)$ ,  $\bar{\lambda} = (4, 2, 1, 1)$ , and  $\bar{\nu} = (5, 3, 1, 1)$ . The move  $\lambda_2 = 2 \rightarrow \nu_2 = 2 + 1$  on the upper level is dictated by the corresponding move  $\bar{\lambda}_2 = 2 \rightarrow \bar{\nu}_2 = 2 + 1$  on the lower level (due to the short-range pushing mechanism), so no further move of  $\nu_2$  is possible. The particle  $\lambda_4 = 1$  cannot move because it is blocked by  $\bar{\nu}_3 = \lambda_4$ . All other particles are free to move (including  $\lambda_3$  which was blocked before the movement at the lower level), and their jumps  $Y_1, Y_2, Y_3$  are independent identically distributed Bernoulli random variables with  $P(Y_1 = 0) = 1/(1 + \beta a_j)$ .

Let us first focus on the case of  $\mathcal{Q}_{\text{pb}}^{q=0}[\hat{\beta}]$  (see Fig. 17).<sup>20</sup> In this case (4.15) simplifies to

$$\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\nu}) = \frac{(\beta a_j)^{|\nu|} \mathbf{1}_{\bar{\nu} \prec_{\text{h}} \nu} \mathbf{1}_{\lambda \prec_{\text{v}} \nu}}{\sum_{\varkappa \in \mathbb{GT}_j^+} (\beta a_j)^{|\varkappa|} \mathbf{1}_{\bar{\nu} \prec_{\text{h}} \varkappa} \mathbf{1}_{\lambda \prec_{\text{v}} \varkappa}},$$

i.e. for any  $\nu', \nu'' \in \mathbb{GT}_j^+$ , such that  $\bar{\nu} \prec_{\text{h}} \nu', \lambda \prec_{\text{v}} \nu'$  and  $\bar{\nu} \prec_{\text{h}} \nu'', \lambda \prec_{\text{v}} \nu''$ ,

$$\frac{\mathcal{U}_j(\lambda \rightarrow \nu' \mid \bar{\nu})}{\mathcal{U}_j(\lambda \rightarrow \nu'' \mid \bar{\nu})} = (\beta a_j)^{|\nu'| - |\nu''|}.$$

It is clear that the only dynamics with such property fits the following description. During one step of the dynamics, each particle  $\lambda_i$ ,  $1 \leq i \leq j$ , can either stay, or jump to the right by one, according to the rules:

- (1) (*short-range pushing*) If  $\bar{\nu}_i = \lambda_i + 1$ , then the move  $\lambda_i \rightarrow \nu_i = \lambda_i + 1$  is mandatory to restore the interlacing (which was broken by the move  $\bar{\lambda} \rightarrow \bar{\nu}$ ) during the same step of the discrete time.
- (2) (*blocking*) If  $\lambda_i = \bar{\nu}_{i-1}$ , then the particle  $\lambda_i$  is blocked and must stay, i.e.,  $\nu_i$  is forced to be equal to  $\lambda_i$ .
- (3) (*independent jumps*) All other particles  $\lambda_i$  which are neither pushed nor blocked, jump to the right by 0 or 1 according to an independent Bernoulli random variable with probability of staying  $1/(1 + \beta a_j)$ .

By the same explanation the dynamics  $\mathcal{Q}_{\text{pb}}^{q=0}[\alpha]$  at two consecutive levels looks as follows (see Fig. 18). Each particle  $\lambda_i$ ,  $1 \leq i \leq j$ , independently jumps to the right by a random distance which has the geometric distribution with parameter  $\alpha a_j$  conditioned to stay in

---

<sup>20</sup>To simplify pictures, here and below we will display interlacing arrays of integers (cf. Fig. 2), but will still speak about particles jumping to the right.

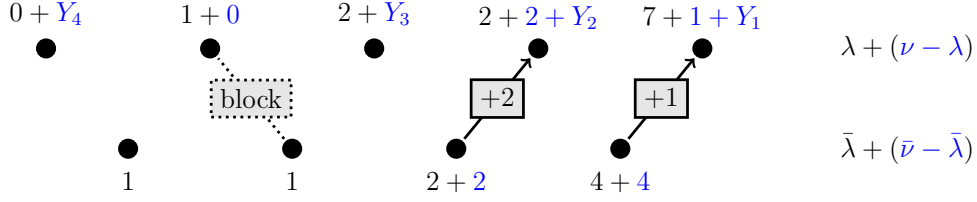


FIGURE 18. An example of a step of  $Q_{\text{pb}}^{q=0}[\alpha]$  at levels 4 and 5. The move  $\bar{\lambda}_1 = 4 \rightarrow \bar{\nu}_1 = 4 + 4$  forces  $\lambda_1$  to move to the right by 1, and similarly the move  $\bar{\lambda}_2 = 2 \rightarrow \bar{\nu}_2 = 2 + 2$  forces  $\lambda_2$  to move to the right by 2 (short-range pushing); note that these forced moves do not exhaust all possible distance traveled by  $\lambda_1$  or  $\lambda_2$ . The particle  $\lambda_4 = 1$  is blocked by  $\bar{\nu}_3 = \lambda_4$  and thus cannot move. All other parts of the movement  $\lambda \rightarrow \nu$  are determined by independent identically distributed geometric random variables  $Y_i$ ,  $1 \leq i \leq 4$  with parameter  $\alpha a_j$ , where each variable is conditioned to stay in the maximal interval not breaking the interlacing:  $Y_1 \geq 0$  (i.e., no conditioning),  $0 \leq Y_2 \leq 4$ ,  $0 \leq Y_3 \leq 2$ ,  $0 \leq Y_4 \leq 1$ .

the interval from  $(\bar{\nu}_i - \lambda_i)_+ := \max\{0, \bar{\nu}_i - \lambda_i\}$  to  $\bar{\nu}_{i-1} - \lambda_i$  (with the agreement that  $\lambda_0 = +\infty$ ).<sup>21</sup> This conditioning corresponds to the denominator in (4.14).

**4.5. RSK-type dynamics.** Let us now define an important subclass of multivariate dynamics which is central to the present paper.

**Definition 4.7.** A multivariate sequential update dynamics  $\mathcal{Q}$  (which corresponds to conditional probabilities  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  satisfying (2.19) and the main equations (4.7) or (4.8)) is called *RSK-type* if

$$\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) = 0 \quad \text{unless } |\nu| - |\lambda| \geq |\bar{\nu}| - |\bar{\lambda}|, \quad \text{for all } \lambda, \nu \in \mathbb{GT}_j^+, \bar{\lambda}, \bar{\nu} \in \mathbb{GT}_{j-1}^+.$$

In the above definition,  $|\bar{\nu}| - |\bar{\lambda}|$  is the total distance traveled by particles at level  $j - 1$ , and similarly  $|\nu| - |\lambda|$  is the total distance traveled by particles at level  $j$ . Informally, under an RSK-type dynamics all movement at level  $j - 1$  *must propagate further* to level  $j$  (and, consequently, to all upper levels of the array).

<sup>21</sup>Due to the memorylessness of the geometric distribution, this description is equivalent to what is illustrated on Fig. 18.

By Remark 4.2, under an RSK-type dynamics the quantity  $|\lambda^{(j)}| - |\lambda^{(j-1)}|$  (for any  $j = 1, \dots, N$ ) at each step of the discrete time is increased by adding a  $q$ -geometric random variable with parameter  $\alpha a_j$  (in the case of  $\mathcal{Q}[\alpha]$ ), or a Bernoulli random variable with parameter  $\beta a_j$  (in the case of  $\mathcal{Q}[\hat{\beta}]$ ).

**Remark 4.8.** This feature of RSK-type dynamics separates them from the push-block dynamics of §4.4. Indeed, under a push-block dynamics movements at level  $j-1$  generically *do not propagate upwards* because the quantities  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  do not depend on  $\bar{\lambda}$ . More precisely, the only steps at level  $j-1$  that can propagate to level  $j$  correspond to the situation  $\bar{\nu} \not\prec_{\mathfrak{h}} \lambda$ . Then a part of the movement  $\lambda \rightarrow \nu$  is *mandatory*, as it is dictated by the need to immediately (i.e., during the same time step of the multivariate dynamics) restore the interlacing between the levels  $j-1$  and  $j$ .

RSK-type dynamics on  $q$ -Whittaker processes that we construct in §5 and §6 give rise to discrete time  $q$ -TASEPs and  $q$ -PushTASEPs as their Markovian marginals. On the other hand, discrete time push-block dynamics do not seem to produce any TASEP-like processes.<sup>22</sup> Note also that in general the denominator in (4.14) or (4.15) does not seem to be given by an explicit formula, so the discrete time push-block dynamics are not easy to work with (cf. Remark 4.3). This provides an additional motivation for constructing and studying RSK-type dynamics.

---

<sup>22</sup>The continuous time push-block dynamics on  $q$ -Whittaker processes has lead to the discovery of the continuous time  $q$ -TASEP in [8]. A continuous time RSK-type dynamics on  $q$ -Whittaker processes was later employed in [14] to discover the continuous time  $q$ -PushTASEP, a close relative of the  $q$ -TASEP (see also §5.6 below). In fact,  $q$ -PushTASEP and  $q$ -TASEP can be unified to produce another nice particle system on  $\mathbb{Z}$ , namely, the  $q$ -PushASEP, which also extends to a certain dynamics on interlacing arrays [22].

## 5. RSK-TYPE DYNAMICS $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ AND $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ ADDING A DUAL PARAMETER

In this section we explain the construction of two RSK-type dynamics on  $q$ -Whittaker processes adding a dual parameter  $\beta$  to the specialization (in the sense of (4.1)). For  $q = 0$ , these dynamics degenerate to  $(\hat{\beta})$  dynamics on Schur processes arising from row and column RSK insertion. We also discover that for  $0 < q < 1$ , the row and column dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  are related by a certain transformation (we call it *complementation*). Moreover, in a small  $\beta$  limit the complementation provides a direct connection between continuous time RSK-type dynamics on  $q$ -Whittaker processes introduced in [58] (column version) and [14] (row version).

**5.1. Row insertion dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ .** Let us now describe one time step  $\lambda \rightarrow \nu$  of the multivariate Markov dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  on  $q$ -Whittaker processes of depth  $N$ . A part of randomness during this step comes from independent Bernoulli random variables  $V_1, \dots, V_N \in \{0, 1\}$  with parameters  $\beta a_1, \dots, \beta a_N$ , respectively (these random variables are resampled during each time step).

The bottommost particle of the interlacing array is updated as  $\nu_1^{(1)} = \lambda_1^{(1)} + V_1$  (as it should be, cf. Remark 4.2). Next, sequentially for each  $j = 2, \dots, N$ , given the movement  $\bar{\lambda} \rightarrow \bar{\nu}$  at level  $j - 1$ , we will randomly update  $\lambda \rightarrow \nu$  at level  $j$ . To describe this update, write

$$\bar{\nu} - \bar{\lambda} = \sum_{i=1}^{j-1} c_i \bar{e}_i, \quad c_i \in \{0, 1\}, \quad \bar{e}_i \text{ are basis vectors of length } j - 1,$$

and say that numbers  $(k, m)$ , where  $1 \leq k \leq m \leq j - 1$ , form *island* $(k, m)$  if

$$c_{k-1} = 0 \quad (\text{or } k = 1), \quad c_k = c_{k+1} = \dots = c_m = 1, \quad \text{and} \quad c_{m+1} = 0 \quad (\text{or } m = j - 1).$$



That is, all particles that have moved at level  $j - 1$  split into several disjoint islands. Also denote for any  $i = 1, \dots, j - 1$ :

$$\mathbf{f}_i = \mathbf{f}_i(\bar{\nu}, \lambda) := \frac{1 - q^{\lambda_i - \bar{\nu}_i + 1}}{1 - q^{\bar{\nu}_{i-1} - \bar{\nu}_i + 1}}, \quad \mathbf{g}_i = \mathbf{g}_i(\bar{\nu}, \lambda) := 1 - q^{\lambda_i - \bar{\nu}_i + 1} \quad (5.1)$$

(by agreement, let  $\bar{\nu}_0 := +\infty$ ). Note that all these quantities are between 0 and 1.

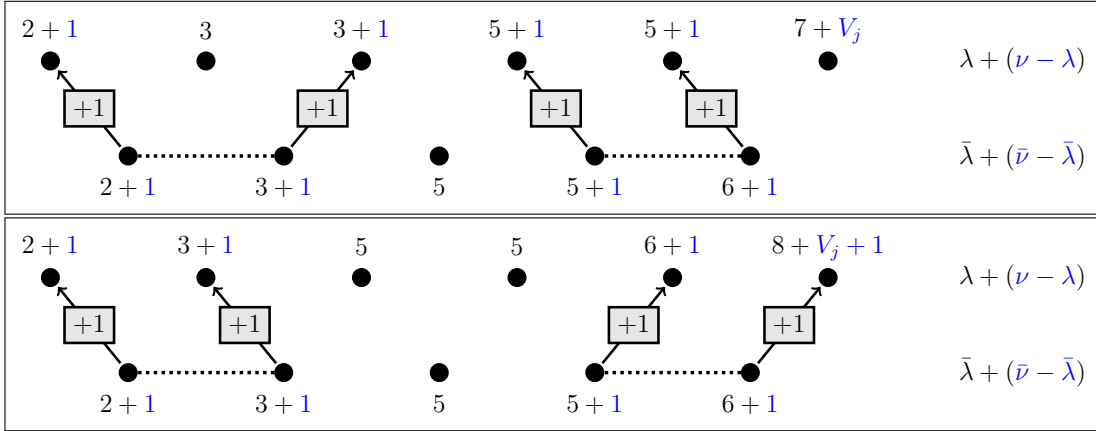


FIGURE 19. An example of a step of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  at levels 5 and 6. There are two islands,  $(1, 2)$  and  $(4, 5)$ , moving at level  $j - 1$ . Above:  $V_j = 1$ , and the probability of the displayed transition is  $1 \cdot (1 - \mathbf{f}_4)\mathbf{g}_5 = 1 - q$  (note that here the particle  $\lambda_4 = 3$  cannot be chosen not to move because  $\mathbf{f}_4 = 0$ ). Below:  $V_j = 0$ , and the probability of the displayed transition is  $(1 - \mathbf{f}_1)(1 - \mathbf{g}_2) \cdot \mathbf{f}_4 = q^3$  (note that here the particle  $\lambda_4 = 5$  must be chosen not to move because  $\mathbf{f}_4 = 1$ ).

The update  $\lambda \rightarrow \nu$  at level  $j$  goes as follows (see Fig. 19). First, the rightmost particle jumps to the right by  $V_j$ , i.e.,  $\nu_1 = \lambda_1 + V_j e_1$ . Then, independently for every island  $(k, m)$  of particles that have moved at level  $j - 1$ , perform the following updates:

- (1) If  $V_j = 1$  and  $k = 1$  (i.e., the particle  $\lambda_1$  has already moved, and the island contains the first particle at level  $j - 1$ ), then move the particles  $\lambda_2, \dots, \lambda_{m+1}$  at level  $j$  to the right by one with probability 1.
- (2) If  $V_j = 1$  and  $k > 1$ , or  $V_j = 0$  (i.e., island  $(k, m)$  does not interfere with the movement of  $\lambda_1$  coming from  $V_j$ , or there is no independent movement of  $\lambda_1$ ), then

island( $k, m$ ) triggers the movement (to the right by one) of all particles  $\lambda_k, \dots, \lambda_{m+1}$  except one. The particle which does not move is chosen at random:

- $\lambda_k$  is chosen not to move with probability

$$\mathbf{f}_k = \frac{1 - q^{\lambda_k - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}; \quad (5.2)$$

- each  $\lambda_s, k + 1 \leq s \leq m$ , is chosen not to move with probability

$$(1 - \mathbf{f}_k)(1 - \mathbf{g}_{k+1}) \dots (1 - \mathbf{g}_{s-1})\mathbf{g}_s = \frac{q^{\lambda_k - \bar{\nu}_k + 1} - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}} q^{\sum_{i=k+1}^{s-1} (\lambda_i - \bar{\nu}_i + 1)} (1 - q^{\lambda_s - \bar{\nu}_s + 1}); \quad (5.3)$$

- $\lambda_{m+1}$  is chosen not to move with probability

$$(1 - \mathbf{f}_k)(1 - \mathbf{g}_{k+1}) \dots (1 - \mathbf{g}_{m-1})(1 - \mathbf{g}_m) = \frac{q^{\lambda_k - \bar{\nu}_k + 1} - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}} q^{\sum_{i=k+1}^m (\lambda_i - \bar{\nu}_i + 1)}. \quad (5.4)$$

Probabilities (5.2), (5.3), and (5.4) are nonnegative, and their sum telescopes to 1. This completes the description of the  $(\hat{\beta})$  row insertion RSK-type dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ . Clearly, thus defined conditional probabilities  $\mathcal{U}_j, j = 1, \dots, N$ , for this dynamics satisfy (2.19).

**Remark 5.1.** The  $q$ -deformed probabilities (5.2), (5.3), and (5.4) ensure that mandatory pushing and blocking mechanisms (built into Definitions 2.6 and 2.7) work automatically:

- If  $\lambda_s = \bar{\nu}_s - 1$  for any  $k \leq s \leq m$ , then the particle  $\lambda_s$  cannot be chosen not to move. This agrees with the mandatory pushing of  $\lambda_s$  by the move of  $\bar{\lambda}_s = \lambda_s$  which is necessary to restore the interlacing.
- If  $\lambda_k = \bar{\nu}_{k-1}$  (i.e.,  $\lambda_k$  is blocked), then  $\mathbf{f}_k = 1$ , so  $\lambda_k$  must be chosen not to move.

This means that in this dynamics no move donations ever arise (cf Remark 2.8).

**Theorem 5.2.** *The dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  defined above satisfies the main equations (4.8), and hence preserves the class of  $q$ -Whittaker processes and adds a new dual parameter  $\beta$  to the specialization  $\mathbf{A}$  as in (4.1).*

*Proof.* We need to prove (4.8) for any fixed  $j = 2, \dots, N$  and  $\lambda, \nu \in \mathbb{GT}_j^+$ ,  $\bar{\nu} \in \mathbb{GT}_{j-1}^+$ , where  $\lambda \prec_{\nu} \bar{\nu} \succ_{\mathbf{h}} \bar{\nu}$  (cf. Fig. 15, right). For a subset  $I \subseteq \{1, 2, \dots, j-1\}$ , set

$$U_I := (1 + \beta a_j) \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \frac{\psi_{\lambda/\bar{\lambda}} \psi'_{\bar{\nu}/\bar{\lambda}}}{\psi_{\nu/\bar{\nu}} \psi'_{\nu/\lambda}},$$

where  $\bar{\lambda} = \bar{\nu} - \sum_{i \in I} \bar{e}_i$ , i.e.,  $\bar{\lambda} \in \mathbb{GT}_{j-1}^+$  is obtained from  $\bar{\nu}$  by shifting back (by one) all particles with indices belonging to  $I$ . By agreement, if  $I$  is such that  $\bar{\lambda}$  does not satisfy  $\lambda \succ_{\mathbf{h}} \bar{\lambda} \prec_{\nu} \bar{\nu}$  (cf. Fig. 15, right), then  $U_I = 0$ . With this notation, the desired identity (4.8) turns into

$$\sum_{I \subseteq \{1, 2, \dots, j-1\}} U_I (\beta a_j)^{|\lambda| - |\nu| - (|\bar{\lambda}| - |\bar{\nu}|)} = 1. \quad (5.5)$$

Note that the denominator  $(1 + \beta a_j)$  coming from the Bernoulli distribution of  $V_j$  will always cancel the corresponding factor in all  $U_I$ 's.

First, let us consider a particular case when  $\nu = \lambda + \sum_{i=k}^m e_i$ , i.e., the movement  $\lambda \rightarrow \nu$  involves a consecutive group of particles from  $k$  to  $m$ , where  $1 \leq k \leq m \leq j-1$ . There are four subcases:

1. If  $k > 1$  and  $m < j$ , then necessarily  $V_j = 0$ , and (5.5) becomes

$$U_{[k-1, m-1]} + \sum_{s=k}^{m-1} U_{[k-1, s-1] \cup [s+1, m]} + U_{[k, m]} = 1 \quad (5.6)$$

(here and below by  $[k-1, m-1]$ , etc., we mean the corresponding interval of indices). See Fig. 20. Using (3.3), (3.7), we have (as before, here and below in the proof we agree that

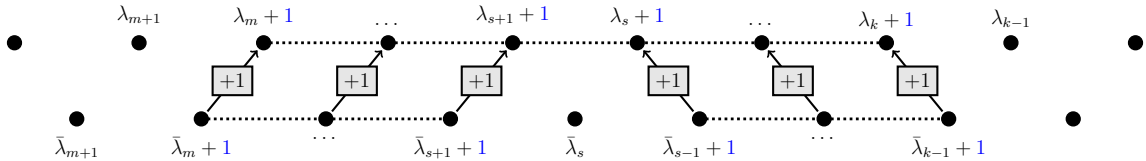


FIGURE 20. Situation corresponding to the  $s$ -th term in (5.6).

$$\bar{\nu}_0 = +\infty)$$

$$\begin{aligned}
U_{[k-1, m-1]} &= \underbrace{f_{k-1}(\bar{\nu}, \lambda)}_{u_j} \cdot \underbrace{\frac{\binom{\lambda_m - \lambda_{m+1}}{\lambda_m - \bar{\nu}_m}_q \binom{\lambda_{k-1} - \lambda_k}{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}_q}{\binom{\lambda_{m+1} - \lambda_{m+1}}{\lambda_{m+1} - \bar{\nu}_m}_q \binom{\lambda_{k-1} - \lambda_{k-1}}{\lambda_{k-1} - \bar{\nu}_{k-1}}_q}}_{\psi_{\lambda/\bar{\lambda}}/\psi_{\nu/\bar{\nu}}} \cdot \underbrace{\frac{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}}}_{\psi'_{\bar{\nu}/\bar{\lambda}}/\psi'_{\nu/\lambda}} \\
&= \frac{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}}{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}} \frac{1 - q^{\lambda_m - \bar{\nu}_m + 1}}{1 - q^{\lambda_m - \lambda_{m+1} + 1}} \frac{1 - q^{\lambda_{k-1} - \lambda_k}}{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}} \frac{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}} \\
&= \frac{1 - q^{\lambda_m - \bar{\nu}_m + 1}}{1 - q^{\lambda_m - \lambda_{m+1} + 1}}.
\end{aligned}$$

Also for any  $k \leq s \leq m-1$ ,

$$\begin{aligned}
&U_{[k-1, s-1] \cup [s+1, m]} \\
&= \underbrace{f_{k-1}(\bar{\nu}, \lambda) \cdot (1 - f_{s+1}(\bar{\nu}, \lambda))(1 - g_{s+2}(\bar{\nu}, \lambda)) \cdots (1 - g_m(\bar{\nu}, \lambda))}_{u_j} \\
&\quad \times \underbrace{\frac{\binom{\lambda_m - \lambda_{m+1}}{\lambda_m - \bar{\nu}_m + 1}_q \binom{\lambda_s - \lambda_{s+1}}{\lambda_s - \bar{\nu}_s}_q \binom{\lambda_{k-1} - \lambda_k}{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}_q}{\binom{\lambda_{m+1} - \lambda_{m+1}}{\lambda_{m+1} - \bar{\nu}_m}_q \binom{\lambda_{s+1} - \lambda_{s+1}}{\lambda_{s+1} - \bar{\nu}_s}_q \binom{\lambda_{k-1} - \lambda_{k-1}}{\lambda_{k-1} - \bar{\nu}_{k-1}}_q}}_{\psi_{\lambda/\bar{\lambda}}/\psi_{\nu/\bar{\nu}}} \cdot \underbrace{\frac{(1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1})(1 - q^{\bar{\nu}_s - \bar{\nu}_{s+1} + 1})}{1 - q^{\lambda_{k-1} - \lambda_k}}}_{\psi'_{\bar{\nu}/\bar{\lambda}}/\psi'_{\nu/\lambda}} \\
&= \frac{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}}{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}} \frac{q^{\lambda_{s+1} - \bar{\nu}_{s+1} + 1} - q^{\bar{\nu}_s - \bar{\nu}_{s+1} + 1}}{1 - q^{\bar{\nu}_s - \bar{\nu}_{s+1} + 1}} q^{\sum_{i=s+2}^m (\lambda_i - \bar{\nu}_i + 1)} \\
&\quad \times \frac{1 - q^{\bar{\nu}_m - \lambda_{m+1}}}{1 - q^{\lambda_{m+1} - \lambda_{m+1}}} \frac{1 - q^{\lambda_s + 1 - \bar{\nu}_s}}{1 - q^{\bar{\nu}_s - \lambda_{s+1}}} \frac{1 - q^{\lambda_{k-1} - \lambda_k}}{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}} \frac{(1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1})(1 - q^{\bar{\nu}_s - \bar{\nu}_{s+1} + 1})}{1 - q^{\lambda_{k-1} - \lambda_k}} \\
&= \frac{(1 - q^{\lambda_s + 1 - \bar{\nu}_s})(1 - q^{\bar{\nu}_m - \lambda_{m+1}})}{1 - q^{\lambda_{m+1} - \lambda_{m+1}}} q^{\sum_{i=s+1}^m (\lambda_i - \bar{\nu}_i + 1)},
\end{aligned}$$

and

$$\begin{aligned}
U_{[k, m]} &= \underbrace{(1 - f_k(\bar{\nu}, \lambda))(1 - g_{k+1}(\bar{\nu}, \lambda)) \cdots (1 - g_m(\bar{\nu}, \lambda))}_{u_j} \\
&\quad \times \underbrace{\frac{\binom{\lambda_m - \lambda_{m+1}}{\lambda_m - \bar{\nu}_m + 1}_q \binom{\lambda_{k-1} - \lambda_k}{\lambda_{k-1} - \bar{\nu}_{k-1}}_q}{\binom{\lambda_{m+1} - \lambda_{m+1}}{\lambda_{m+1} - \bar{\nu}_m}_q \binom{\lambda_{k-1} - \lambda_{k-1}}{\lambda_{k-1} - \bar{\nu}_{k-1}}_q}}_{\psi_{\lambda/\bar{\lambda}}/\psi_{\nu/\bar{\nu}}} \cdot \underbrace{\frac{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}}}_{\psi'_{\bar{\nu}/\bar{\lambda}}/\psi'_{\nu/\lambda}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{q^{\lambda_k - \bar{\nu}_k + 1} - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}} q^{\sum_{i=k+1}^m (\lambda_i - \bar{\nu}_i + 1)} \frac{1 - q^{\bar{\nu}_m - \lambda_{m+1}}}{1 - q^{\lambda_{m+1} - \lambda_{m+1}}} \frac{1 - q^{\lambda_{k-1} - \lambda_k}}{1 - q^{\bar{\nu}_{k-1} - \lambda_k}} \frac{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}} \\
&= \frac{1 - q^{\bar{\nu}_m - \lambda_{m+1}}}{1 - q^{\lambda_{m+1} - \lambda_{m+1}}} q^{\sum_{i=k}^m (\lambda_i - \bar{\nu}_i + 1)}.
\end{aligned}$$

The summation in (5.6) thus telescopes and gives 1 as desired (similarly to the sum of expressions (5.2), (5.3), and (5.4)).

**2.** If  $k > 1$  and  $m = j$ , then also necessarily  $V_j = 0$ , and there is only one  $I$ , namely,  $[k-1, j-1]$ , contributing to (5.5). We have

$$\begin{aligned}
U_{[k-1, j-1]} &= f_{k-1}(\bar{\nu}, \lambda) \cdot \frac{\binom{\lambda_{k-1} - \lambda_k}{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}_q}{\binom{\lambda_{k-1} - \lambda_k - 1}{\lambda_{k-1} - \bar{\nu}_{k-1}}_q} \cdot \frac{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}} \\
&= \frac{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}}{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}} \frac{1 - q^{\lambda_{k-1} - \lambda_k}}{1 - q^{\lambda_{k-1} - \bar{\nu}_{k-1} + 1}} \frac{1 - q^{\bar{\nu}_{k-2} - \bar{\nu}_{k-1} + 1}}{1 - q^{\lambda_{k-1} - \lambda_k}} = 1,
\end{aligned}$$

so we see that (5.5) holds.

**3.** If  $k = 1$  and  $m < j$ , then  $V_j$  can be either 0 or 1, and (5.5) now looks as

$$U_{[1, m]} + (a_j \beta)^{-1} \sum_{s=1}^{m-1} U_{[1, s-1] \cup [s+1, m]} + (a_j \beta)^{-1} U_{[1, m-1]} = 1.$$

This identity is established similarly to the subcase 1. Namely, one readily sees that

$$\begin{aligned}
U_{[1, m]} &= \frac{1 - q^{\bar{\nu}_m - \lambda_{m+1}}}{1 - q^{\lambda_m - \lambda_{m+1} + 1}} q^{\sum_{i=1}^m (\lambda_i - \bar{\nu}_i + 1)}; \\
U_{[1, s-1] \cup [s+1, m]} &= (a_j \beta) \frac{(1 - q^{\bar{\nu}_m - \lambda_{m+1}})(1 - q^{\lambda_s - \bar{\nu}_s + 1})}{1 - q^{\lambda_m - \lambda_{m+1} + 1}} q^{\sum_{i=s+1}^m (\lambda_i - \bar{\nu}_i + 1)}; \\
U_{[1, m-1]} &= (a_j \beta) \frac{1 - q^{\lambda_m - \bar{\nu}_m + 1}}{1 - q^{\lambda_m - \lambda_{m+1} + 1}},
\end{aligned}$$

and the sum of these quantities telescopes and gives 1.

**4.** If  $k = 1$  and  $m = j$ , this means that necessarily  $V_j = 1$ , and the only term that enters (5.5) is  $U_{[1, j-1]} = \beta a_j$ , so the desired identity also holds.

We have now established the desired identity in the particular case  $\nu = \lambda + \sum_{i=k}^m e_i$ . In the general case there could be several consecutive groups of particles forming the move  $\lambda \rightarrow \nu$  at level  $j$ . Let there be gaps of at least two not moving particles between neighboring moving groups. Then, by the product nature of the quantities  $\psi$  and  $\psi'$  (3.3), (3.7), as well as by the independence of propagation for different islands at level  $j - 1$ , cf. Fig. 19, the sum in the left-hand side of (5.5) can clearly be represented as a product of sums corresponding to individual groups of moving particles. Each such individual sum is the same as in one of the subcases 1–4 above, and therefore is equal to 1. This implies (5.5) in the case when moving groups at level  $j$  are sufficiently far apart.

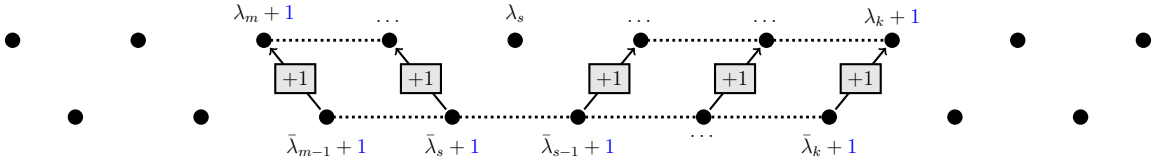


FIGURE 21. Two islands at level  $j$  corresponding to a single island at level  $j - 1$ .

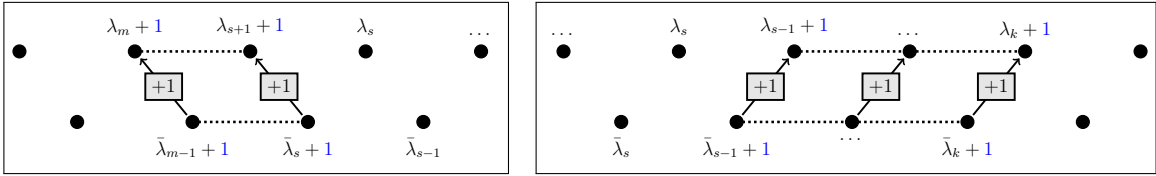


FIGURE 22. Two configurations giving the same contribution as the one on Fig. 21.

Finally, it remains to check (5.5) in the case when there could be moving groups at level  $j$  separated by one not moving particle. Consider two such neighboring groups. The only configuration of moves at level  $j - 1$  (corresponding to these two groups at level  $j$ ) that could prevent the sum in (5.5) to be of product form is given on Fig. 21. However, one readily sees that the contribution of this configuration is the same as the product of contributions of two configurations on Fig. 22. Indeed, factors involving the quantities  $\psi$  are already in

a product form, and the remaining factors (coming from  $\mathcal{U}_j$  and the quantities  $\psi'$ ) are

$$\begin{aligned} & \underbrace{\frac{q^{\lambda_k - \bar{\lambda}_k} - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}}{1 - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}} q^{\sum_{i=k+1}^{s-1} (\lambda_i - \bar{\lambda}_i)} (1 - q^{\lambda_s - \bar{\lambda}_s})}_{(1 - f_k)(1 - g_{k+1}) \dots (1 - g_{s-1})g_s \text{ on Fig. 21}} \cdot \frac{1 - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}}{(1 - q^{\lambda_{k-1} - \lambda_k})(1 - q^{\lambda_s - \lambda_{s+1}})} \\ &= \underbrace{\frac{1 - q^{\lambda_s - \bar{\lambda}_s}}{1 - q^{\bar{\lambda}_{s-1} - \bar{\lambda}_s}}}_{f_s \text{ on Fig. 22, left}} \cdot \frac{1 - q^{\bar{\lambda}_{s-1} - \bar{\lambda}_s}}{1 - q^{\lambda_s - \lambda_{s+1}}} \times \underbrace{\frac{q^{\lambda_k - \bar{\lambda}_k} - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}}{1 - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}} q^{\sum_{i=k+1}^{s-1} (\lambda_i - \bar{\lambda}_i)}}_{(1 - f_k)(1 - g_{k+1}) \dots (1 - g_{s-1}) \text{ on Fig. 22, right}} \cdot \frac{1 - q^{\bar{\lambda}_{k-1} - \bar{\lambda}_k}}{1 - q^{\lambda_{k-1} - \lambda_k}}. \end{aligned}$$

Note that we have expressed everything in terms of signatures  $\lambda$  and  $\bar{\lambda}$  because the signatures  $\bar{\nu}$  differ on Fig. 21 and Fig. 22.

Therefore, in the last remaining case we can still rewrite (5.5) in a product form. This completes the proof of the theorem.  $\square$

**Remark 5.3** (Schur degeneration). If one sets  $q = 0$ , then in a generic situation (when particles at levels  $j - 1$  and  $j$  are sufficiently far apart from each other) all quantities  $f_i$  and  $g_i$  become equal to one, see (5.1). One readily sees that the dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  reduces to the dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\hat{\beta}]$  on Schur processes. The latter dynamics is based on the classical Robinson–Schensted–Knuth row insertion (§2.6).

**5.2. Bernoulli  $q$ -PushTASEP.** One can readily check that under the dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  we have just constructed, the rightmost  $N$  particles  $\lambda_1^{(j)}$  of the interlacing array evolve in a *marginally Markovian manner* (i.e., their evolution does not depend on the dynamics of the rest of the interlacing array). Namely, at each discrete time step  $t \rightarrow t + 1$  the bottommost particle is updated as  $\lambda_1^{(1)}(t + 1) = \lambda_1^{(1)}(t) + V_1$ , and for any  $j = 2, \dots, N$ :

- If  $\lambda_1^{(j-1)}$  has not moved, then the rightmost particle at level  $j$  is updated as

$$\lambda_1^{(j)}(t + 1) = \lambda_1^{(j)}(t) + V_j;$$

- If  $\lambda_1^{(j-1)}$  has moved to the right by one, then the same particle is updated as

$$\lambda_1^{(j)}(t+1) = \lambda_1^{(j)}(t) + V_j + (1 - V_j) \cdot \mathbf{1}_{\text{pushing by } \lambda_1^{(j-1)}},$$

where pushing by  $\lambda_1^{(j-1)}$  happens with probability  $1 - f_1 = q^{\lambda_1^{(j)}(t) - \lambda_1^{(j-1)}(t)}$  which depends only on the rightmost particles of the array.

(Recall that the  $V_i$ 's are independent Bernoulli random variables which are independently resampled each step of the discrete time.) This evolution of the rightmost particles  $\lambda_1^{(j)}$ ,  $1 \leq j \leq N$ , leads to a new interacting particle system on  $\mathbb{Z}$  which we call the (*discrete time*) *Bernoulli  $q$ -PushTASEP*. We discuss this process in detail in §7 below.

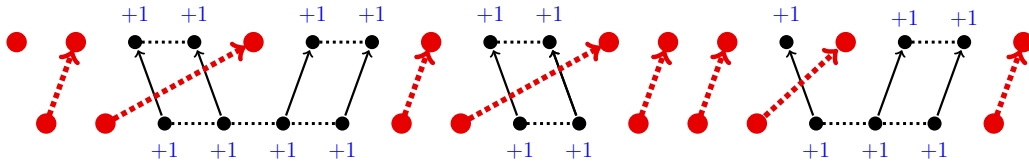


FIGURE 23. Complementation of propagation rules turning the dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  (with move propagation given by thin solid arrows) into  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  (corresponding to thick dashed arrows).

**5.3. Complementation.** Let us take another look at propagation rules employed in the definition of the row insertion dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  on  $q$ -Whittaker processes (see the beginning of §5.1). These rules state that, generically, an island of moving particles at level  $j-1$  splits (at random) into two moving islands at level  $j$  separated by exactly one staying particle (either of two moving islands at level  $j$  is allowed to be empty). Now consider the *pattern of staying particles* at levels  $j-1$  and  $j$ . We see that an island  $(k, m)$  (where  $k \leq m$ ) of staying particles at level  $j-1$  always gives rise to an island  $(k+1, m)$  of staying particles at level  $j$ , plus one more staying particle somewhere to the right of  $k$  (but to the left of the next staying particle at level  $j$ ). The latter staying particle (whose index is chosen at random) is precisely the one separating the two moving islands at level  $j$ . See Fig. 23.



The transformation of propagation rules of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  that we just described informally in fact leads to a new RSK-type multivariate dynamics on  $q$ -Whittaker processes. Let us work in a more general setting:

**Definition 5.4** (Complementation of a dynamics). Assume that  $\mathcal{Q}$  is a multivariate sequential update dynamics on  $q$ -Whittaker processes adding a specialization  $(\hat{\beta})$ . For  $j = 2, \dots, N$  and signatures  $\lambda, \nu \in \mathbb{GT}_j^+$ ,  $\bar{\lambda}, \bar{\nu} \in \mathbb{GT}_{j-1}^+$  satisfying conditions on Fig. 15, right, let  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  be the corresponding conditional probabilities. Assume that the dynamics is *translation invariant*, i.e., that the values  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  do not change if one adds the same number to all coordinates of all four signatures.

For  $S$  a sufficiently large positive integer, define the *complement conditional probabilities* as

$$\mathcal{U}'_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) := (a_j \beta)^{-2(|\lambda| - |\nu| - |\bar{\lambda}| + |\bar{\nu}|) - 1} \mathcal{U}_j\left([S - \lambda] \rightarrow [S + 1 - \nu] \mid [S - \bar{\lambda}] \rightarrow [S + 1 - \bar{\nu}]\right),$$

where

$$[S - \lambda] := (S - \lambda_j \geq S - \lambda_{j-1} \geq \dots \geq S - \lambda_1)$$

is the complement of the Young diagram  $\lambda$  in the  $j \times S$  rectangle, and similarly for  $[S + 1 - \nu]$ ,  $[S - \bar{\lambda}]$ , and  $[S + 1 - \bar{\nu}]$  (hence the name “complementation”). Note that these four new signatures also satisfy conditions on Fig. 15, right.

Let us denote by  $\mathcal{Q}'$  the dynamics on interlacing arrays corresponding to  $\mathcal{U}'_j$ ,  $j = 2, \dots, N$ . Note that due to translation invariance, the complement dynamics  $\mathcal{Q}'$  does not depend on  $S$  provided that  $S$  is large enough.

**Lemma 5.5.** *Let  $S$  be sufficiently large. For  $\bar{\mu} \in \mathbb{GT}_{k-1}^+$ ,  $\mu \in \mathbb{GT}_k^+$  such that  $\bar{\mu} \prec_{\mathfrak{h}} \mu$ , we have*

$$\psi_{[S-\mu]/[S-\bar{\mu}]} = \psi_{\mu/\bar{\mu}}.$$

For  $\mu, \varkappa \in \mathbb{GT}_k^+$  such that  $\mu \prec_{\mathfrak{v}} \varkappa$ , we have

$$\psi'_{[S+1-\varkappa]/[S-\mu]} = \psi'_{\varkappa/\mu}.$$

*Proof.* A straightforward verification using definitions (3.3), (3.7).  $\square$

**Proposition 5.6.** *If  $\mathcal{Q}$  is a multivariate sequential update dynamics adding a specialization  $(\hat{\beta})$ , then so is the complement dynamics  $\mathcal{Q}'$ .*

*Proof.* One can show that the complement dynamics  $\mathcal{Q}'$  satisfies the same main equations (4.8) as the original dynamics  $\mathcal{Q}$ . Indeed, Lemma 5.5 ensures that the coefficients  $\psi_{\lambda/\bar{\lambda}}\psi'_{\bar{\nu}/\bar{\lambda}}$  and  $\psi_{\nu/\bar{\nu}}\psi'_{\nu/\lambda}$  do not change under complementation, and powers of  $(a_j\beta)$  also transform as they should:

$$\begin{aligned} & \mathcal{U}'_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})(a_j\beta)^{|\lambda|-|\nu|-|\bar{\lambda}|+|\bar{\nu}|} \\ &= (a_j\beta)^{-|\lambda|+|\nu|+|\bar{\lambda}|-|\bar{\nu}|-1} \mathcal{U}_j\left([S-\lambda] \rightarrow [S+1-\nu] \mid [S-\bar{\lambda}] \rightarrow [S+1-\bar{\nu}]\right) \\ &= (a_j\beta)^{|[S-\lambda]|-|[S+1-\nu]|-|[S-\bar{\lambda}]|+|[S+1-\bar{\nu}]|} \mathcal{U}_j\left([S-\lambda] \rightarrow [S+1-\nu] \mid [S-\bar{\lambda}] \rightarrow [S+1-\bar{\nu}]\right). \end{aligned}$$

This establishes the main equations for the complement dynamics.  $\square$

**5.4. Column insertion dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ .** Clearly, the row insertion dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  on  $q$ -Whittaker processes is translation invariant (in the sense of Definition 5.4), so one can define the complement dynamics. Denote it by  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ . Let us describe (in an explicit way) the evolution of the interlacing array under this new dynamics during one step of the discrete time. See Fig. 24 for an example.

As before, a part of randomness comes from independent Bernoulli random variables  $V_j \in \{0, 1\}$  with  $P(V_j = 0) = 1/(1 + \beta a_j)$ ,  $j = 1, \dots, N$ . The bottommost particle of the interlacing array is updated as  $\nu_1^{(1)} = \lambda_1^{(1)} + V_1$ . Sequentially for each  $j = 2, \dots, N$ , given the movement  $\bar{\lambda} \rightarrow \bar{\nu}$  at level  $j - 1$ , we will randomly update  $\lambda \rightarrow \nu$  at level  $j$ . Let us denote (as usual,  $\bar{\nu}_0 = +\infty$ )

$$f'_k = f'_k(\bar{\nu}, \lambda) := \frac{1 - q^{\bar{\nu}_{k-1} - \lambda_k}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}, \quad g'_s = g'_s(\bar{\nu}, \lambda) := 1 - q^{\bar{\nu}_{s-1} - \lambda_s}. \quad (5.7)$$

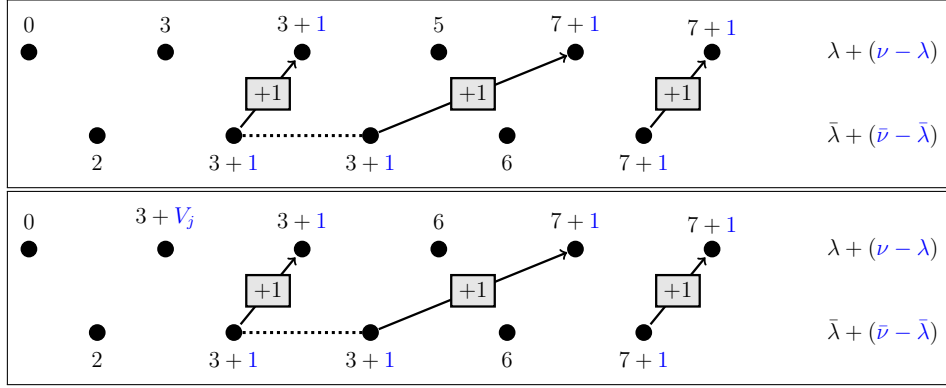


FIGURE 24. An example of a step of  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  at levels 5 and 6. Above:  $V_j = 0$ , and the probability of the displayed transition is  $1 - f'_3 = (q + q^2)/(1 + q + q^2)$ . Below:  $V_j = 1$ , and the probability of the displayed transition is  $(1 - g'_6)(1 - f'_3) = q^2$ . Note that in the latter case the particle  $\lambda_3 = 6$  cannot be chosen to move because it is blocked by  $\bar{\lambda}_2 = \lambda_3$  which is not moving; this agrees with  $f'_3 = 0$ .

The update  $\lambda \rightarrow \nu$  looks as follows:

- (1) Consider a pair of moved particles  $(\bar{\lambda}_r, \bar{\lambda}_k)$  at level  $j - 1$ , where  $0 \leq r < k \leq j - 1$ , such that the particles  $\bar{\lambda}_{r+1}, \dots, \bar{\lambda}_{k-1}$  in between did not move (by agreement,  $r = 0$  corresponds to  $\bar{\lambda}_k$  being the rightmost moved particle at level  $j - 1$ ). Regardless of the value of  $V_j$ , each such pair of moved particles at level  $j - 1$  triggers the move (to the right by one) of exactly one particle  $\lambda_s$ ,  $r + 1 \leq s \leq k$ , between them at level  $j$ . If  $r + 1 = k$ , then there is only one choice  $s = k$ , so  $\lambda_k$  must move. Otherwise,

the moving particle  $\lambda_s$  is chosen at random (independently of everything else) with the following probabilities:

- If  $s = k$ , then  $\lambda_s$  is chosen to move with probability

$$\mathbf{f}'_k = \frac{1 - q^{\bar{\nu}_{k-1} - \lambda_k}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}; \quad (5.8)$$

- If  $r + 1 < s < k$ , then  $\lambda_s$  is chosen to move with probability

$$(1 - \mathbf{f}'_k)(1 - \mathbf{g}'_{k-1}) \cdots (1 - \mathbf{g}'_{s+1})\mathbf{g}'_s = \frac{q^{\bar{\nu}_{k-1} - \lambda_k} - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}} q^{\sum_{i=s}^{k-2} (\bar{\nu}_i - \lambda_{i+1})} (1 - q^{\bar{\nu}_{s-1} - \lambda_s}); \quad (5.9)$$

- If  $s = r + 1$ , then  $\lambda_s$  is chosen to move with probability

$$(1 - \mathbf{f}'_k)(1 - \mathbf{g}'_{k-1}) \cdots (1 - \mathbf{g}'_{r+3})(1 - \mathbf{g}'_{r+2}) = \frac{q^{\bar{\nu}_{k-1} - \lambda_k} - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}}{1 - q^{\bar{\nu}_{k-1} - \bar{\nu}_k + 1}} q^{\sum_{i=r+1}^{k-2} (\bar{\nu}_i - \lambda_{i+1})}. \quad (5.10)$$

Clearly, these probabilities are nonnegative, and their sum telescopes to 1.

- (2) If  $V_j = 1$ , then in addition to the moves described above, exactly one more particle at level  $j$  is chosen to move (to the right by one). Namely, let  $\bar{\lambda}_m$  be the leftmost moved particle at level  $j - 1$ . If  $m = j - 1$ , then the additional moving particle at level  $j$  is  $\lambda_j$ , the leftmost particle. If  $m < j - 1$ , then one of the particles  $\lambda_s$  with  $m + 1 \leq s \leq j$  is randomly chosen to move (independently of everything else) with the following probabilities:

- If  $s = j$ , then  $\lambda_s$  is chosen to move with probability

$$\mathbf{g}'_j = 1 - q^{\bar{\nu}_{j-1} - \lambda_j}; \quad (5.11)$$

- If  $m + 1 < s < j$ , then  $\lambda_s$  is chosen to move with probability

$$(1 - \mathbf{g}'_j)(1 - \mathbf{g}'_{j-1}) \cdots (1 - \mathbf{g}'_{s+1})\mathbf{g}'_s = (1 - q^{\bar{\nu}_{s-1} - \lambda_s}) q^{\sum_{i=s}^{j-1} (\bar{\nu}_i - \lambda_{i+1})}; \quad (5.12)$$

- If  $s = m + 1$ , then  $\lambda_s$  is chosen to move with probability

$$(1 - \mathbf{g}'_j)(1 - \mathbf{g}'_{j-1}) \cdots (1 - \mathbf{g}'_{m+3})(1 - \mathbf{g}'_{m+2}) = q^{\sum_{i=m+1}^{j-1} (\bar{\nu}_i - \lambda_{i+1})}. \quad (5.13)$$

The sum of these probabilities also telescopes to 1.

This completes the description of the  $(\hat{\beta})$  column insertion RSK-type dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ .

**Theorem 5.7.** *The dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  defined above satisfies the main equations (4.8), and hence preserves the class of  $q$ -Whittaker processes and adds a new dual parameter  $\beta$  to the specialization  $\mathbf{A}$  as in (4.1).*

*Proof.* One can readily check that  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  is the complement of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ . Then the desired statement follows from Theorem 5.2 and Proposition 5.6.  $\square$

**Remark 5.8.** Similarly to  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  (cf. Remark 5.1), probabilities (5.8)–(5.13) employed in the definition of  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  ensure the mandatory pushing, blocking, and move donation mechanisms (described in Definitions 2.6 and 2.7 and Remark 2.8). Namely, observe that

- If  $\bar{\lambda}_k = \lambda_k$  for some  $k$  and  $\bar{\lambda}_k$  has moved at level  $j - 1$ , then  $\mathbf{f}'_k = 1$ , which means that  $\lambda_k$  is chosen to move with probability 1.
- If  $\lambda_s = \bar{\lambda}_{s-1}$ , and  $\bar{\lambda}_{s-1}$  has not moved, then  $\mathbf{g}'_s = \mathbf{f}'_s = 0$ , so according to (5.8), (5.9) the particle  $\lambda_s$  at level  $j$  cannot be chosen to move. If, moreover,  $\bar{\lambda}_s$  has moved at level  $j - 1$ , then this move will trigger some other particle to the right of  $\lambda_s$  at level  $j$  to move. In other words, the moving impulse coming from  $\bar{\lambda}_s \rightarrow \bar{\nu}_s = \bar{\lambda}_s + 1$  will be donated further to the right of  $\lambda_s$ .

**Remark 5.9** (Schur degeneration). When  $q = 0$ , one readily sees from (5.7) that generically (i.e., when particles at levels  $j - 1$  and  $j$  are sufficiently far apart) we have  $\mathbf{f}'_k = \mathbf{g}'_s = 1$ . This implies that the dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  degenerates to the multivariate dynamics  $\mathcal{Q}_{\text{col}}^{q=0}[\hat{\beta}]$  on

Schur processes. The latter is based on the classical Robinson–Schensted–Knuth column insertion (§2.6).

**5.5. Bernoulli  $q$ -TASEP.** Under the dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ , the leftmost  $N$  particles  $\lambda_j^{(j)}$  of the interlacing array evolve in a *marginally Markovian manner*. Indeed, one can readily check that at each discrete time step  $t \rightarrow t + 1$  the bottommost particle is updated as  $\lambda_1^{(1)}(t + 1) = \lambda_1^{(1)}(t) + V_1$ , and for any  $j = 2, \dots, N$ :

- If  $\lambda_{j-1}^{(j-1)}$  has moved, then the leftmost particle at level  $j$  is updated as

$$\lambda_j^{(j)}(t + 1) = \lambda_j^{(j)}(t) + V_j;$$

- If  $\lambda_{j-1}^{(j-1)}$  has not moved, then the same particle is updated as

$$\lambda_j^{(j)}(t + 1) = \lambda_j^{(j)}(t) + V_j \cdot \mathbf{1}_{\lambda_j^{(j)} \text{ is chosen to move}},$$

where  $\lambda_j^{(j)}$  is chosen to move with probability  $\mathbf{g}'_j = 1 - q^{\lambda_{j-1}^{(j-1)}(t) - \lambda_j^{(j)}(t)}$  which depends only on the leftmost particles of the array.

This evolution of the leftmost particles  $\lambda_j^{(j)}$ ,  $1 \leq j \leq N$ , is the (*discrete time*) *Bernoulli  $q$ -TASEP* which was introduced and studied in [7].

**5.6. Small  $\beta$  continuous time limit.** If one sends the parameter  $\beta$  to zero and simultaneously rescales time from discrete to continuous, then both dynamics  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  turn into certain continuous time Markov dynamics on  $q$ -Whittaker processes. At the level  $j = 1$  (cf. Remark 4.2), this limit transition coincides with the one bringing the (one-sided) discrete time random walk to the continuous time Poisson process. In continuous time setting, at most one particle can move at each level  $j = 1, \dots, N$  during an instance of continuous time.

The continuous time limit  $\mathcal{Q}_{\text{Plancherel, row}}^q[\gamma]$  of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  looks as follows. Each rightmost particle  $\lambda_1^{(j)}$  of the interlacing array has an independent exponential clock with rate  $a_j$ . When the clock rings, the particle jumps to the right by one.

There is also a jump propagation mechanism present: If at level  $j - 1$  some particle  $\lambda_m^{(j-1)}$  has moved (to the right by one), then this move instantaneously triggers the move of the upper left neighbor  $\lambda_{m+1}^{(j)}$  with probability  $f_m = \frac{1 - q^{\lambda_m^{(j)} - \lambda_m^{(j-1)}}}{1 - q^{\lambda_{m-1}^{(j-1)} - \lambda_m^{(j-1)}}}$ ,<sup>23</sup> or the move of the upper right neighbor  $\lambda_m^{(j)}$  with the complementary probability  $1 - f_m$ . This dynamics was introduced in [14] (Dynamics 8). Under it, the rightmost particles of the array also evolve in a marginally Markovian manner. This leads to the *continuous time  $q$ -PushTASEP* on  $\mathbb{Z}$  [14, §8.3], [22].

The continuous time limit  $\mathcal{Q}_{\text{Plancherel, col}}^q[\gamma]$  of  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  looks as follows. Each particle  $\lambda_k$ ,  $1 \leq k \leq j$ , at level  $j$  has an independent exponential clock with rate

$$\begin{cases} a_j \mathbf{g}'_j, & k = j; \\ a_j (1 - \mathbf{g}'_j)(1 - \mathbf{g}'_{j-1}) \dots (1 - \mathbf{g}'_{k+1}) \mathbf{g}'_k, & 1 < k < j; \\ a_j (1 - \mathbf{g}'_j)(1 - \mathbf{g}'_{j-1}) \dots (1 - \mathbf{g}'_3)(1 - \mathbf{g}'_2), & k = 1. \end{cases}$$

These quantities correspond to (5.11)–(5.13) with  $\bar{\nu} = \bar{\lambda}$  (because if an independent jump occurs at level  $j$  then at level  $j - 1$  there could be no movement). When the clock of  $\lambda_k$  rings, this particle jumps to the right by one. Note that the move donation mechanism described in Remark 2.8 follows from the above probabilities.

There is also a jump propagation mechanism: If a particle  $\bar{\lambda}_k$  has moved at level  $j - 1$ , then it triggers the move (to the right by one) of exactly one particle  $\lambda_s$ ,  $1 \leq s \leq k$ , at

---

<sup>23</sup>Note that this formula is written using particle coordinates before the move at level  $j - 1$ , cf. (5.1).

level  $j$ , where  $s$  is chosen at random with probabilities

$$\begin{cases} \mathbf{f}'_k, & s = k; \\ (1 - \mathbf{f}'_k)(1 - \mathbf{g}'_{k-1}) \cdots (1 - \mathbf{g}'_{s+1})\mathbf{g}'_s, & 1 < s < k; \\ (1 - \mathbf{f}'_k)(1 - \mathbf{g}'_{k-1}) \cdots (1 - \mathbf{g}'_3)(1 - \mathbf{g}'_2), & s = 1. \end{cases}$$

The above probabilities are given by (5.8)–(5.10) where  $\bar{\nu}$  differs from  $\bar{\lambda}$  as  $\bar{\nu} = \bar{\lambda} + \bar{e}_k$ . This dynamics on  $q$ -Whittaker processes was introduced in [58]. Under it, the leftmost particles of the interlacing array evolve in a marginally Markovian manner as a  $q$ -TASEP. This continuous time particle system was introduced in [8]. See also, e.g., [11], [9], [29] for further results on the  $q$ -TASEP.

Thus, the two continuous time dynamics on  $q$ -Whittaker processes (or, in other words,  $q$ -randomized Robinson–Schensted insertion tableau dynamics) introduced in [58] and [14] are the  $\beta \rightarrow 0$  degenerations of  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ , respectively. On the other hand, complementation (§5.3) provides a straightforward link between the two latter discrete time dynamics.

## 6. RSK-TYPE DYNAMICS $\mathcal{Q}_{\text{row}}^q[\alpha]$ AND $\mathcal{Q}_{\text{col}}^q[\alpha]$ ADDING A USUAL PARAMETER

In this section we explain the construction of two RSK-type dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$  on  $q$ -Whittaker processes adding a usual parameter  $\alpha$  to the specialization (as in (4.1)). For  $q = 0$ , these dynamics degenerate to  $(\alpha)$  dynamics on Schur processes arising from row and column RSK insertion. As in the case of  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$  and  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  dynamics, in a small  $\alpha$  limit the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$  degenerate to continuous time RSK-type dynamics from [58] (column version) and [14] (row version).

**6.1. The  $q$ -deformed Beta-binomial distribution.** We will use the following quantities:



**Definition 6.1.** Let  $y \in \{0, 1, 2, \dots\} \cup \{+\infty\}$ , and  $s \in \{0, 1, \dots, y\}$ . Recall the  $q$ -notation from (3.2). Let

$$\varphi_{q,\xi,\eta}(s | y) := \xi^s \frac{(\eta/\xi; q)_s (\xi; q)_{y-s}}{(\eta; q)_y} \frac{(q; q)_y}{(q; q)_s (q; q)_{y-s}}. \quad (6.1)$$

If  $y = +\infty$ , the limits of the above quantities are

$$\varphi_{q,\xi,\eta}(s | +\infty) = \xi^s \frac{(\eta/\xi; q)_s (\xi; q)_\infty}{(q; q)_s (\eta; q)_\infty}. \quad (6.2)$$

An important property of the quantities (6.1) and (6.2) is that for all  $y \in \{0, 1, 2, \dots\} \cup \{+\infty\}$ , we have

$$\sum_{s=0}^y \varphi_{q,\xi,\eta}(s | y) = 1. \quad (6.3)$$

This statement may be rewritten as the  $q$ -Chu-Vandermonde identity for the basic hypergeometric series  ${}_2\phi_1$ . For the proof and more details see [35], [20]. Recall that in general the unilateral basic hypergeometric series  ${}_j\phi_k$  is defined via

$${}_j\phi_k \left[ \begin{matrix} a_1 & \dots & a_j \\ b_1 & \dots & b_k \end{matrix} ; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_j; q)_n}{(b_1, \dots, b_k; q)_n} \left( (-1)^n q^{\binom{n}{2}} \right)^{1+k-j} z^n, \quad (6.4)$$

where  $(c_1, \dots, c_m; q)_n = \prod_{i=1}^m (c_i; q)_n$ . Later on in this section to prove some identities we will need to apply transformation formulas for certain hypergeometric series.

Therefore, for all values of the parameters  $(q, \xi, \eta)$  for which  $\varphi_{q,\xi,\eta}(s | y)$  is well-defined and nonnegative for every  $0 \leq y \leq s$ , (6.1) defines a probability distribution on  $\{0, 1, \dots, y\}$ . One such family of parameters is  $0 \leq q < 1$ ,  $0 \leq \eta \leq \xi < 1$ , cf. [65], [20]. Another choice of parameters leading to a probability distribution which we will use is  $\varphi_{q^{-1}, q^a, q^b}(\cdot | c)$ , where  $a \leq b$ ,  $c \leq b$  are nonnegative integers.

The distribution  $\varphi_{q,\xi,\eta}$  appears (under a simple change of parameters) as the orthogonality weight of the classical  $q$ -Hahn orthogonal polynomials [44], and is also related to a

very natural  $q$ -deformation of the Polya urn scheme [39]. As such,  $\varphi_{q,\xi,\eta}$  may be regarded as a  $q$ -deformed *Beta-binomial distribution*, since the latter is the orthogonality weight for the Hahn orthogonal polynomials, and also arises from the ordinary Polya urn scheme. We can also directly see by taking  $q = e^{-\epsilon}$ ,  $\xi = e^{-\alpha\epsilon}$ ,  $\eta = e^{-(\alpha+\beta)\epsilon}$  and letting  $\epsilon \rightarrow 0+$ , that  $\varphi_{q,\xi,\eta}(s | y)$  converges to

$$\frac{\Gamma(\alpha + y - s)\Gamma(\beta + s)\Gamma(\alpha + \beta)\Gamma(y + 1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + y)\Gamma(s + 1)\Gamma(y - s + 1)},$$

which is the probability of  $s$  under the beta-binomial distribution with parameters  $y, \alpha, \beta$ .

Let us now record two straightforward observations which we will be using below. First,

$$\binom{n}{k}_{q^{-1}} = q^{-k(n-k)} \binom{n}{k}_q. \quad (6.5)$$

Second, if  $a \leq b$ ,  $c \leq b$  are nonnegative integers ( $b$  might also be  $+\infty$ ), then for any  $s \in \{0, 1, \dots, c\}$  one has

$$\lim_{q \searrow 0} \varphi_{q^{-1}, q^a, q^b}(s | c) = \mathbf{1}_{s=\max\{c-a, 0\}}. \quad (6.6)$$

Indeed, in this case

$$\varphi_{q^{-1}, q^a, q^b}(s | c) = q^{s(a-c+s)} \frac{(q^a; q^{-1})_{c-s} (q^{b-a}; q^{-1})_s}{(q^b; q^{-1})_c} \binom{s}{c}_q.$$

If  $a > c$ , as  $q \rightarrow 0$  this converges to 1 for  $s = 0$  and to 0 for  $s > 0$ . If  $a \leq c$ , as  $q \rightarrow 0$  this converges to 0 for  $0 \leq s < c - a$ , since  $(q^a; q^{-1})_{c-s}$  vanishes, to 0 for  $s > c - a$ , since a positive power of  $q$  tends to 0, and to 1 for  $s = c - a$ .

**6.2. Row insertion dynamics**  $\mathcal{Q}_{\text{row}}^q[\alpha]$ . Let us now describe one time step  $\boldsymbol{\lambda} \rightarrow \boldsymbol{\nu}$  of the multivariate Markov dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  on  $q$ -Whittaker processes of depth  $N$ . A part of randomness a time step comes from independent  $q$ -geometric random variables  $V_1, \dots, V_N \in$

$\mathbb{Z}_{\geq 0}$  with parameters  $\alpha a_1, \dots, \alpha a_N$ , respectively (these random variables are resampled during each time step).

The bottommost particle of the interlacing array is updated as  $\nu_1^{(1)} = \lambda_1^{(1)} + V_1$ . Next, sequentially for each  $j = 2, \dots, N$ , given the movement  $\bar{\lambda} = \lambda^{(j-1)} \rightarrow \bar{\nu} = \nu^{(j-1)}$  at level  $j - 1$ , we will randomly update  $\lambda = \lambda^{(j)} \rightarrow \nu = \nu^{(j)}$  at level  $j$ . To describe this update, write

$$\bar{\nu} - \bar{\lambda} = \sum_{i=1}^{j-1} c_i \bar{e}_i, \quad c_i \in \mathbb{Z}_{\geq 0}, \quad \bar{e}_i \text{ are basis vectors of length } j - 1.$$

Note that by interlacing, it must be that  $c_i \leq \bar{\lambda}_{i-1} - \bar{\lambda}_i$ .

Sample independent random variables  $W_1, \dots, W_{j-1}$ , such that each  $W_i \in \{0, 1, \dots, c_i\}$  is distributed according to

$$\varphi_{q^{-1}, \xi_i, \eta_i}(\cdot \mid c_i), \quad \text{where } \xi_i := q^{\lambda_i - \bar{\lambda}_i} \text{ and } \eta_i := q^{\bar{\lambda}_{i-1} - \bar{\lambda}_i} \quad (6.7)$$

(this is a probability distribution because  $\lambda_i - \bar{\lambda}_i \leq \bar{\lambda}_{i-1} - \bar{\lambda}_i$  and  $c_i \leq \bar{\lambda}_{i-1} - \bar{\lambda}_i$ , cf. §6.1).

We will use the conventions  $\bar{\lambda}_0 = +\infty$  and  $\eta_1 = 0$ . Define a sequence of signatures

$$\lambda = \mu(0), \mu(1), \dots, \mu(j - 1)$$

via

$$\mu(i) := \mu(i - 1) + W_{j-i} e_{j-i} + (c_{j-i} - W_{j-i}) e_{j-i+1} \quad \text{for } 1 \leq i \leq j - 1$$

(where  $e_i$  are basis vectors of length  $j$ ). Finally, define  $\nu := \mu(j - 1) + V_j e_1$ , this is our new signature at level  $j$ .

In words, each  $i$ th particle on the  $(j - 1)$ -st level which has moved by  $c_i$ , must trigger a total of  $c_i$  moves (to the right by one) at level  $j$  (this the RSK-type property, see Definition 4.7). Each such particle at level  $j - 1$  independently from the others, *in parallel*, splits

contribution from its jump between its nearest neighbors on the level  $j$ , according to the distribution (6.7). After this pushing, the rightmost particle on the  $j$ -th level additionally performs an independent jump according to the  $q$ -geometric distribution with parameter  $\alpha a_j$ . Clearly, thus defined conditional probabilities  $\mathcal{U}_j$ ,  $j = 1, \dots, N$ , for this dynamics are nonnegative and satisfy (2.19). See Fig. 25.

One must verify that the interlacing properties (as on Fig. 15, left) are preserved by this dynamics:

**Lemma 6.2.** *If  $\bar{\lambda} \prec_h \bar{\nu}$ ,  $\bar{\lambda} \prec_h \lambda$  and  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) > 0$ , then  $\bar{\nu} \prec_h \nu$  and  $\lambda \prec_h \nu$ .*

*Proof.* Observe that for  $a \leq b$  and  $c \leq b$

$$\varphi_{q^{-1}, q^a, q^b}(s \mid c) = 0, \quad \text{if } s > b - a \text{ or } c - s > a. \quad (6.8)$$

Apply this for  $a = \lambda_i - \bar{\lambda}_i$ ,  $b = \bar{\lambda}_{i-1} - \bar{\lambda}_i$ ,  $c = c_i$  to get  $c_i - \lambda_i + \bar{\lambda}_i \leq W_i \leq \bar{\lambda}_{i-1} - \lambda_i$ .

Since  $\nu_i = \lambda_i + W_i + c_{i-1} - W_{i-1}$ , we have

$$\begin{aligned} \nu_i &\leq \lambda_i + \bar{\lambda}_{i-1} - \lambda_i + c_{i-1} - W_{i-1} = \bar{\lambda}_{i-1} + c_{i-1} - W_{i-1} = \bar{\nu}_{i-1} - W_{i-1} \leq \bar{\nu}_{i-1}, \\ \bar{\nu}_i &= \bar{\lambda}_i + c_i \leq \lambda_i + W_i \leq \nu_i, \end{aligned}$$

so  $\nu \succ_h \bar{\nu}$ . Moreover, we can also write

$$\nu_i \leq \lambda_i + \bar{\lambda}_{i-1} - \lambda_i + \lambda_{i-1} - \bar{\lambda}_{i-1} = \lambda_{i-1}, \quad \lambda_i \leq \nu_i$$

which implies that  $\nu \succ_h \lambda$ . □

This verification completes the description of the  $(\alpha)$  row insertion RSK-type dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$ .

**Remark 6.3.** (Schur degeneration). If one sets  $q = 0$ , then the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  reduces to the dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  on Schur processes based on the classical Robinson–Schensted–Knuth row insertion (§2.6). To see this, observe that (6.6) implies

$$\varphi_{q^{-1}, \xi_i, \eta_i}(s \mid c_i) \rightarrow \mathbf{1}_{s=\max\{c_i - \lambda_i + \bar{\lambda}_i, 0\}} \quad \text{as } q \rightarrow 0,$$

that is, each  $W_i$  becomes equal to  $\max\{c_i - \lambda_i + \bar{\lambda}_i, 0\}$  in the  $q \searrow 0$  limit. Therefore, the update  $\lambda \rightarrow \nu$  is reduced to applying  $c_i$  operations pull at positions  $i$  from  $j - 1$  to 1, plus an additional independent jump of the rightmost particle according to the geometric distribution with parameter  $\alpha a_j$ .

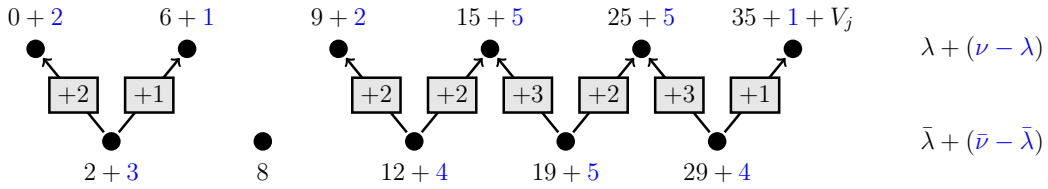


FIGURE 25. An example of a step of  $\mathcal{Q}_{\text{row}}^q[\alpha]$  at levels 5 and 6, with  $V_j = 3$ . The probability of this update is equal to  $\varphi_{q^{-1}, q^4, q^6}(1 \mid 3) \varphi_{q^{-1}, q^3, q^7}(2 \mid 4) \varphi_{q^{-1}, q^6, q^{10}}(2 \mid 5) \varphi_{q^{-1}, q^6, 0}(1 \mid 4) (\alpha a_6; q)_\infty \frac{(\alpha a_6)^3}{(q; q)_3}$ . Note that, e.g.,  $\varphi_{q^{-1}, q^3, q^7}(0 \mid 4) = 0$ , which ensures the mandatory pushing (by at least 1) of  $\lambda_3$  by the move of  $\bar{\lambda}_3$ .

**Theorem 6.4.** *The dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  defined above satisfies the main equations (4.7), and hence preserves the class of  $q$ -Whittaker processes and adds a new usual parameter  $\alpha$  to the specialization  $\mathbf{A}$  as in (4.1).*

*Proof.* We will prove (4.7) by induction on  $j$ . Case  $j = 1$  is straightforward because  $\bar{\lambda}$  is empty (cf. Remark 4.2).

Assume now that (4.7) holds for signatures  $\lambda, \nu$  having length  $j - 1$ , and let us prove this identity for  $\lambda, \nu$  or length  $j$ . The idea is to expand each term in the sum in the left-hand side of (4.7) with respect to what happens to the leftmost particle on the  $(j - 1)$ -st level

(and its neighborhood), and then use the inductive assumption and the fact that the  $\varphi$ 's sum to 1.

For a signature  $\mu = (\mu_1 \geq \dots \geq \mu_m)$  we denote by  $\mu^-$  the signature  $(\mu_1 \geq \dots \geq \mu_{m-1})$  obtained by deleting the smallest part of  $\mu$ , and by  $\mu + [s]_{\text{lm}}$  the signature  $(\mu_1 \geq \dots \geq \mu_{m-1} \geq \mu_m + s)$  obtained by adding  $s$  to the smallest part of  $\mu$  (for  $s \leq \mu_{m-1} - \mu_m$ ). To simplify certain notations below, also denote

$$\mathcal{V}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) := \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \frac{(\alpha a_j)^{|\lambda| - |\bar{\lambda}| - |\nu| + |\bar{\nu}|}}{(\alpha a_j; q)_\infty}. \quad (6.9)$$

Temporarily let  $t$  stand for  $c_{j-1} = \bar{\nu}_{j-1} - \bar{\lambda}_{j-1}$  which is the move of the leftmost particle on the  $(j-1)$ -st level. In order to have at least one nonzero summand in the left-hand side of (4.7), we need to have (see Fig. 26):

- $t \geq \nu_j - \lambda_j$ , since the jump of the leftmost particle on the  $j$ -th level happens due to contribution of a part of the jump of the leftmost particle on the  $(j-1)$ -st level.
- $t \geq \nu_j - \lambda_j + \bar{\nu}_{j-1} - \lambda_{j-1}$ , since  $\varphi_{q^{-1}, \xi_{j-1}, \eta_{j-1}}(t - \nu_j + \lambda_j \mid t) > 0$  implies by (6.8) that  $\nu_j - \lambda_j \leq \lambda_{j-1} - \bar{\nu}_{j-1} + t$ .
- $t \leq \bar{\nu}_{j-1} - \lambda_j$ , since we must have  $\bar{\lambda}_{j-1} \geq \lambda_j$ .
- $t \leq \nu_{j-1} - \lambda_{j-1} + \nu_j - \lambda_j$ , since the contribution from the jump of the leftmost particle on the  $(j-1)$ -st level is split between particles  $\lambda_j$  and  $\lambda_{j-1}$  at the level  $j$ .

Denote the interval of  $t$  satisfying the above inequalities by  $I$ . We also must have

- $t \leq \bar{\lambda}_{j-2} - \lambda_{j-1} + \nu_j - \lambda_j$ , since  $\varphi_{q^{-1}, \xi_{j-1}, \eta_{j-1}}(t - \nu_j + \lambda_j \mid t) > 0$  implies by (6.8) that  $t - \nu_j + \lambda_j \leq \bar{\lambda}_{j-2} - \lambda_{j-1}$ . For  $j = 2$  this last inequality should be omitted.

We will use the notation  $\tilde{\lambda} := \bar{\lambda}^-$ . Denote by  $J(t)$  the set of signatures  $\tilde{\lambda}$  of length  $j-2$ , such that  $\tilde{\lambda} \prec_{\text{h}} \bar{\nu}$ ,  $\tilde{\lambda} \prec_{\text{h}} \lambda^-$ , and  $\tilde{\lambda}_{j-2} \geq t + \lambda_{j-1} - \nu_j + \lambda_j$ . For  $j = 2$  this set consists of just the empty signature. If  $\bar{\lambda}$  is such that  $\bar{\lambda} \prec_{\text{h}} \bar{\nu}$ ,  $\bar{\lambda} \prec_{\text{h}} \lambda$  and  $\mathcal{V}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \neq 0$ , then  $\bar{\lambda}^- \in \bigsqcup_{t \in I} J(t)$ .

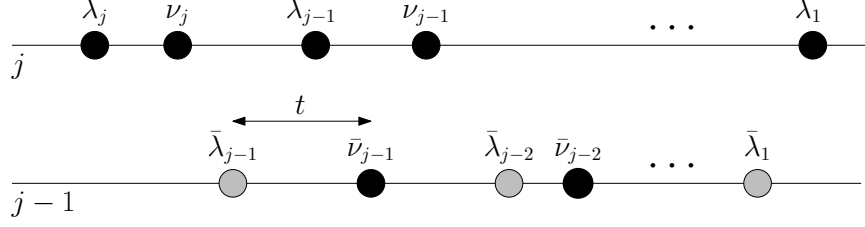


FIGURE 26. We expand sum with respect to the jump  $t = c_{j-1} = \bar{\nu}_{j-1} - \bar{\lambda}_{j-1}$  of the leftmost particle on the  $(j - 1)$ -st level. Note that the signatures  $\nu, \lambda, \bar{\nu}$  are fixed, while the positions of particles  $\bar{\lambda}_i$  vary in the sum.

The left-hand side of (4.7) divided by the right-hand side of the same equation is equal to

$$\begin{aligned}
& \sum_{\bar{\lambda}} \mathcal{V}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \frac{\psi_{\lambda/\bar{\lambda}} \phi_{\bar{\nu}/\bar{\lambda}}}{\psi_{\nu/\bar{\nu}} \phi_{\nu/\lambda}} \\
&= \sum_{t \in I} \sum_{\bar{\lambda} \in J(t)} \underbrace{\binom{t}{\nu_j - \lambda_j}_{q^{-1}} \frac{(q^{\lambda_{j-1} - \bar{\nu}_{j-1} + t}; q^{-1})_{\nu_j - \lambda_j} (q^{\bar{\lambda}_{j-2} - \lambda_{j-1}}; q^{-1})_{t - \nu_j + \lambda_j}}{(q^{\bar{\lambda}_{j-2} - \bar{\nu}_{j-1} + t}; q^{-1})_t} q^{(\lambda_{j-1} - \bar{\nu}_{j-1} + t)(t - \nu_j + \lambda_j)}}_{\varphi_{q^{-1}, q^{\bar{\lambda}_{j-2} - \lambda_{j-1}}, q^{\bar{\lambda}_{j-2} - \bar{\nu}_{j-1} + t}(t - \nu_j + \lambda_j | t)}} \\
& \quad \times \mathcal{V}_{j-1}(\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}} \rightarrow \nu^- \mid \tilde{\lambda} \rightarrow \bar{\nu}^-) \cdot \frac{\psi_{\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}/\tilde{\lambda}} \phi_{\bar{\nu}^-/\tilde{\lambda}}}{\psi_{\nu^-/\bar{\nu}^-} \phi_{\nu^-/\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}}} \\
& \quad \times \underbrace{\frac{\binom{\lambda_{j-1} - \lambda_j}{\bar{\nu}_{j-1} - t - \lambda_j}_q \cdot \binom{\bar{\lambda}_{j-2} - \bar{\nu}_{j-1} + t}{t}_q \cdot \frac{(q^{\nu_{j-1} - \lambda_{j-1}}; q^{-1})_{t - \nu_j + \lambda_j}}{(q^{\bar{\lambda}_{j-2} - \lambda_{j-1}}; q^{-1})_{t - \nu_j + \lambda_j}}}{\frac{\psi_{\lambda/\bar{\lambda}} \psi_{\nu^-/\bar{\nu}^-}}{\psi_{\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}/\tilde{\lambda}} \psi_{\nu/\bar{\nu}}}} \cdot \frac{\phi_{\bar{\nu}^-/\tilde{\lambda}} \psi_{\nu^-/\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}}}{\phi_{\nu/\lambda} \phi_{\bar{\nu}^-/\tilde{\lambda}}}} \\
&= \sum_{t \in I} \left( \varphi_{q^{-1}, q^{\nu_{j-1} - \bar{\nu}_{j-1}}, q^{\nu_{j-1} - \nu_j}(t - \nu_j + \lambda_j \mid \nu_{j-1} - \lambda_{j-1})} \right. \\
& \quad \times \left. \sum_{\tilde{\lambda} \in J(t)} \mathcal{V}_{j-1}(\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}} \rightarrow \nu^- \mid \tilde{\lambda} \rightarrow \bar{\nu}^-) \cdot \frac{\psi_{\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}/\tilde{\lambda}} \phi_{\bar{\nu}^-/\tilde{\lambda}}}{\psi_{\nu^-/\bar{\nu}^-} \phi_{\nu^-/\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}}}} \right) \\
&= \sum_{t \in I} \varphi_{q^{-1}, q^{\nu_{j-1} - \bar{\nu}_{j-1}}, q^{\nu_{j-1} - \nu_j}(t - \nu_j + \lambda_j \mid \nu_{j-1} - \lambda_{j-1}) = 1.
\end{aligned}$$

Above  $\mathcal{V}_{j-1}$  and  $\mathcal{V}_j$  have the same value of the parameter  $a = a_j$ . We have also used the fact that

$$\begin{aligned} |\nu| - |\lambda| - |\bar{\nu}| + |\bar{\lambda}| &= |\nu^-| - |\lambda^-| + \nu_j - \lambda_j - |\bar{\nu}^-| + |\bar{\lambda}^-| - \bar{\nu}_{j-1} + \bar{\lambda}_{j-1} \\ &= |\nu^-| - |\lambda^-| + [t - \nu_j + \lambda_j]_{\text{lm}} - |\bar{\nu}^-| + |\bar{\lambda}^-|, \end{aligned}$$

hence  $\mathcal{V}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  involves the same power of  $\alpha a_j$  as  $\mathcal{V}_{j-1}(\lambda^- + [t - \nu_j + \lambda_j]_{\text{lm}} \rightarrow \nu^- \mid \bar{\lambda}^- \rightarrow \bar{\nu}^-)$ . Also, (6.8) implies that  $\varphi_{q^{-1}, q^{\nu_{j-1} - \bar{\nu}_{j-1}}, q^{\nu_j - \nu_j}}(t - \nu_j + \lambda_j \mid \nu_{j-1} - \lambda_{j-1})$  is nonzero only for  $t \in I$ , hence one gets 1 after summing these quantities over  $t \in I$ .

This concludes the proof, and also establishes Theorem 1.1 from Introduction.  $\square$

**6.3. Geometric  $q$ -PushTASEP.** Under the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  we have just constructed, the rightmost  $N$  particles  $\lambda_1^{(j)}$  of the interlacing array evolve in a *marginally Markovian manner* (i.e., their evolution does not depend on the dynamics of the rest of the interlacing array). Namely, at each discrete time step  $t \rightarrow t + 1$  the bottommost particle is updated as  $\lambda_1^{(1)}(t + 1) = \lambda_1^{(1)}(t) + V_1$ , and for any  $j = 2, \dots, N$  if we let  $\text{gap}_j(t) = \lambda_1^{(j)}(t) - \lambda_1^{(j-1)}(t)$  be the gap between the rightmost particles on the  $(j - 1)$ -st and the  $j$ -th levels at time  $t$ , then

$$\lambda_1^{(j)}(t + 1) = \lambda_1^{(j)}(t) + V_j + W_{j,t}$$

for an independent random variable  $W_{j,t}$  distributed according to

$$\varphi_{q^{-1}, q^{\text{gap}_j(t)}, 0}(\cdot \mid \lambda_1^{(j-1)}(t + 1) - \lambda_1^{(j-1)}(t)).$$

The random variable  $V_j$  (recall that it has the  $q$ -geometric distribution with parameter  $\alpha a_j$  which is resampled during each time step) represents an independent jump of  $\lambda_1^{(j)}$ . The variable  $W_{j,t}$  represents the pushing of  $\lambda_1^{(j)}$  by the move of  $\lambda_1^{(j-1)}$ .



This evolution of the rightmost particles  $\lambda_1^{(j)}$ ,  $1 \leq j \leq N$ , leads to a new interacting particle system on  $\mathbb{Z}$  which we call the (*discrete time*) *geometric  $q$ -PushTASEP*.

**6.4. Column insertion dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$ . Description and discussion.** Let us now describe one time step  $\lambda \rightarrow \nu$  of the multivariate Markov dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  on  $q$ -Whittaker processes of depth  $N$ . As in the previous case, the bottommost particle of the interlacing array is updated as  $\nu_1^{(1)} = \lambda_1^{(1)} + X$  for a  $q$ -geometric random variable  $X$  with parameter  $\alpha a_1$ . Next, sequentially for each  $j = 2, \dots, N$ , given the movement  $\bar{\lambda} \rightarrow \bar{\nu}$  at level  $j - 1$ , we will randomly update  $\lambda \rightarrow \nu$  at level  $j$ . To describe this update we write, as usual,

$$\bar{\nu} - \bar{\lambda} = \sum_{i=1}^{j-1} c_i \bar{e}_i, \quad c_i \in \mathbb{Z}_{\geq 0}.$$

All randomness during this update comes from a collection of  $3j$  dependent random variables  $X_1, \dots, X_j, Y_1, \dots, Y_j, Z_1, \dots, Z_j$  (they are resampled during each time step), and

$$\nu_{j-i+1} - \lambda_{j-i+1} = \underbrace{X_i}_{\text{voluntary jump}} + \underbrace{Y_i}_{\text{push from } \bar{\lambda}_{j-i+1}} + \underbrace{Z_i}_{\text{push from the "stabilization fund"}}, \quad i = 1, \dots, j.$$

(It will be convenient to let  $i$  represent the position of the particle counted from the left.) Observe that  $Y_1$  must be identically zero. The “stabilization fund” means the leftover push from the first  $i - 2$  particles from the left at level  $j - 1$  (i.e., from  $\bar{\lambda}_{j-1}, \dots, \bar{\lambda}_{j-i+2}$ ) (in particular,  $Z_1$  and  $Z_2$  are identically zero).

Let us first formally define the distribution of all the parts of the jumps:

- (1) Set  $\theta_1 := 1$ . For  $i$  from 1 to  $j$  sample  $X_i$  according to

$$X_i \sim \varphi_{q, \alpha a_j \theta_i, 0}(\cdot \mid \bar{\lambda}_{j-i} - \lambda_{j-i+1}) \tag{6.10}$$

and set

$$\theta_{i+1} := \theta_i q^{\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i}.$$

$X_i$  comes from the input  $V_j$ , see Remark (6.7). Here the convention  $\bar{\lambda}_0 = +\infty$  applies when  $i = j$ .

(2) Set  $Y_1 := 0$ . For  $i$  from 2 to  $j - 1$  take  $Y_i = y$  with probability

$$Y_i \sim \varphi_{q^{-1}, q^{c_{j-i+1}}, q^{\bar{\lambda}_{j-i} - \bar{\lambda}_{j-i+1}}}(\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - y \mid \bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i). \quad (6.11)$$

Finally, set  $Y_j := c_1$ .

(3) Set  $r_1 = r_2 = 1$  and  $Z_1 = Z_2 := 0$ . Set  $r_3 := r_2 q^{c_{j-1} - Y_2}$ . For  $i$  from 3 to  $j - 1$  take  $Z_i = z$  with probability

$$Z_i \sim \varphi_{q^{-1}, r_i, 0}(\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - Y_i - z \mid \bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - Y_i) \quad (6.12)$$

and set

$$r_{i+1} := r_i q^{c_{j-i+1} - Y_i - Z_i}.$$

Finally, let  $Z_j := \log_q r_j$ .

**Remark 6.5.** For fixed  $s, u, d \geq 0$  (possibly  $u = \infty$ ) and  $D \rightarrow \infty$  observe that

$$\varphi_{q^{-1}, q^s, q^{u+D}}(D - d \mid D) = q^{(s-d)(D-d)} (q^s; q^{-1})_d \binom{D}{d}_q \frac{(q^{u+D-s}; q^{-1})_{D-d}}{(q^{u+D}; q^{-1})_D} \rightarrow \mathbf{1}_{d=s}.$$

Therefore, the definitions of  $Z_j = \log_q r_j$  and  $Y_j = c_1$  are consistent with the definitions of  $Z_i$  and  $Y_i$  ( $i < j$ ), respectively. In words, the consistency for  $Z_j$  means that the stabilization fund is depleted for the push of the rightmost particle on the  $j$ -th level. The consistency for  $Y_j$  means that the whole value of the jump of the rightmost particle on the  $(j - 1)$ -st level is transferred to the rightmost particle on the  $j$ -th level via immediate pushing.

**Lemma 6.6.** *If  $\bar{\lambda} \prec_h \bar{\nu}$ ,  $\bar{\lambda} \prec_h \lambda$  and  $\mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) > 0$ , then  $\bar{\nu} \prec_h \nu$  and  $\lambda \prec_h \nu$ .*

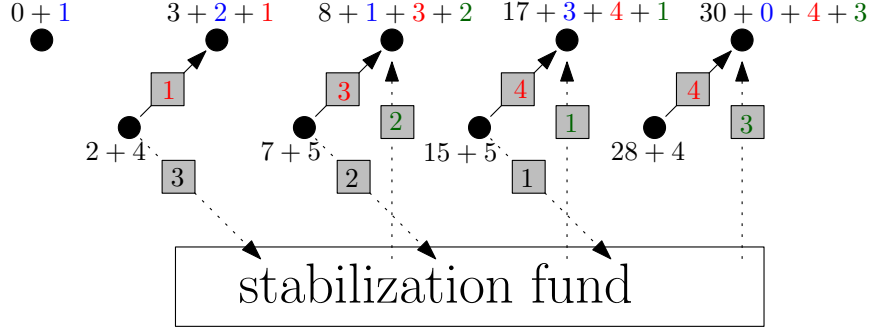


FIGURE 27. An example of a step of  $\mathcal{Q}_{\text{col}}^q[\alpha]$  at levels 4 and 5.

*Proof.* It is straightforward from the definition of the dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  that  $\nu_{j-i+1} \leq \bar{\lambda}_{j-i} \leq \min(\bar{\nu}_{j-i}, \lambda_{j-i})$ . Also for  $2 \leq i \leq j$  (6.11) together with (6.8) implies that  $\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - Y_i \leq \bar{\lambda}_{j-i} - \bar{\lambda}_{j-i+1} - c_{j-i+1}$ , hence  $\nu_{j-i+1} \geq \lambda_{j-i+1} + X_i + Y_i \geq \bar{\nu}_{j-i+1}$ . It follows that the interlacing properties are preserved.  $\square$

In the rest of this subsection we will describe the column insertion dynamics in words, and also discuss its various properties. The (rather involved) proof that this dynamics acts on  $q$ -Whittaker processes in a desired way is postponed to the next subsection.

There are two stages of the update of particle positions  $\lambda_j, \lambda_{j-1}, \dots, \lambda_1$ , performed in order *from left to right*, which we will describe below.

During the first stage of the update, the particles at level  $j$  level make voluntary jumps in order from left to right. The value  $X_i$  of the voluntary jump of  $\lambda_{j+1-i}$  depends on the previous jump  $X_{i-1}$ , where  $2 \leq i \leq j$ . Indeed, this dependence comes from the parameters  $\theta_i$  (note that they are nonincreasing in  $i$ ), see (6.10). Note that unlike the  $\mathcal{Q}_{\text{row}}^q[\alpha]$  case, in which all random movements not coming from pushing are restricted to the right edge, in the case of  $\mathcal{Q}_{\text{col}}^q[\alpha]$  any particle might make a voluntary jump.

**Remark 6.7.** The random variable  $X_1 + \dots + X_j$  has the  $q$ -geometric distribution with parameter  $\alpha a_j$ , as it should be by Remark 4.2 and the discussion of §4.5. This is seen by applying inductively the following lemma:

**Lemma 6.8.** *Let  $A$  and  $B$  be random variables such that  $A$  is distributed according to  $\varphi_{q,\alpha,0}(\cdot | a)$ , and  $B$  given  $A$  is distributed according to  $\varphi_{q,\alpha q^{a-A},0}(\cdot | b)$  (where  $b$  might be  $+\infty$ ). Then  $A + B$  is distributed according to  $\varphi_{q,\alpha,0}(\cdot | a + b)$ .*

*Proof.* Indeed, we have

$$\begin{aligned}
\text{Prob}(A + B = y) &= \sum_{s=0}^y \text{Prob}(A = s) \text{Prob}(B = y - s | A = s) \\
&= \sum_{s=0}^y \alpha^s (\alpha; q)_{a-s} \binom{a}{s}_q (\alpha q^{a-s})^{y-s} (\alpha q^{a-s}; q)_{b-y+s} \binom{b}{y-s}_q \\
&= \alpha^y (\alpha; q)_{a+b-y} q^{a(y-s)} \frac{(q^a; q^{-1})_s (q^b; q^{-1})_{y-s}}{(q; q)_y} q^{-s(y-s)} \binom{y}{s}_q \\
&= \alpha^y (\alpha; q)_{a+b-y} \binom{a+b}{y}_q \cdot \sum_{s=0}^y \varphi_{q^{-1}, q^a, q^{a+b}}(y - s | y) \\
&= \varphi_{q,\alpha,0}(y | a + b),
\end{aligned}$$

which establishes the desired statement. □

The second stage of the update consists of pushing, in order from left to right. We start an initially empty stabilization fund, which will collect impulses not immediately used for pushing, and will be a source of the pushes  $Z_i$ . The value of the stabilization fund just before the movement of  $\lambda_{j+1-i}$  is  $\log_q r_i$  (by agreement,  $r_1 = r_2 = 1$  always). For each  $i$

ranging from 2 to  $j$ , the following three steps happen:

- (1) The particle  $\lambda_{j+1-i}$  gets a push  $Y_i$  from its lower left neighbor  $\bar{\lambda}_{j+1-i}$ . The size of this push (distributed according to (6.11)) is at most  $c_{j-i+1}$ .
- (2) Then  $\lambda_{j+1-i}$  gets a push from the stabilization fund (if it is not empty) of size not exceeding the current value of the stabilization fund. This push is (6.13) distributed according to  $Z_i$  (6.12).
- (3) Finally, the amount of pushing not used in (1) above, i.e.,  $c_{j-i+1} - Y_i$ , is added to the stabilization fund.

One can also think that the above two update stages are performed together for each particle  $\lambda_j, \lambda_{j-1}, \dots, \lambda_1$ .

**Proposition 6.9.** *One can switch the order of the lower left neighbor pushing and stabilization fund pushing (i.e., steps (1) and (2) in (6.13)) without changing the dynamics.*

24

*Proof.* Fix  $k = 2, \dots, j$ . Suppose that after the voluntary displacement stage the distance from the  $k$ -th particle from the left at level  $j$  (denote this particle by  $P$ ) to  $\bar{\lambda}_{j+1-k}$  is  $h := \bar{\lambda}_{j-k} - \lambda_{j-k+1} - X_{j-k+1}$ . Also set  $\ell := \bar{\nu}_{j-k+1} - \bar{\lambda}_{j-k+1}$ ,  $b := \bar{\lambda}_{j-k} - \bar{\lambda}_{j-k+1}$ , and let the current size of the stabilization fund be  $R$ .

If the steps (1) and (2) in (6.13) are not interchanged, then the probability that  $P$  jumps by  $s \geq 0$  is

$$\sum_{y=0}^s \varphi_{q^{-1}, q^\ell, q^b}(h - y \mid h) \varphi_{q^{-1}, q^R, 0}(h - s \mid h - y).$$

---

<sup>24</sup>Here for the version with interchanged steps (1) and (2) we would have  $Z_i \sim \varphi_{q^{-1}, r_i, 0}(\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - \cdot \mid \bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i)$  and  $Y_i \sim \varphi_{q^{-1}, q^{c_{j-i+1}}, q^{\bar{\lambda}_{j-i} - \bar{\lambda}_{j-i+1}}}(\bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - Z_i - \cdot \mid \bar{\lambda}_{j-i} - \lambda_{j-i+1} - X_i - Z_i)$ .

If the steps (1) and (2) in (6.13) are interchanged, then the same probability is given by

$$\sum_{y=0}^s \varphi_{q^{-1}, q^R, 0}(h-s+y | h) \varphi_{q^{-1}, q^\ell, q^b}(h-s | h-s+y).$$

After dividing each of these two expressions by  $\frac{q^{(R+\ell)(h-s)}(q^{b-\ell}; q^{-1})_{h-s}}{(q^b; q^{-1})_{h-s}} \binom{h}{s}_{q^{-1}}$  we arrive to the following identity we need to verify

$$\begin{aligned} \sum_{y=0}^s \binom{s}{y}_{q^{-1}} q^{\ell(s-y)} (q^\ell; q^{-1})_y (q^R; q^{-1})_{s-y} (q^{b-\ell-h+s}; q^{-1})_{s-y} \\ = \sum_{y=0}^s \binom{s}{y}_{q^{-1}} q^{Ry} (q^\ell; q^{-1})_y (q^R; q^{-1})_{s-y} (q^{b-h+1}; q)_{s-y}. \end{aligned} \quad (6.14)$$

We are very grateful to Christian Krattenthaler for providing us with a proof of the  $q$ -binomial identity (6.14), which we reproduce below.

First, use a transformation formula for  ${}_3\phi_2$  series [35, (III.12)]:

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q \right] = \frac{(e/c; q)_n}{(e; q)_n} c^n {}_3\phi_2 \left[ \begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; q, \frac{bq}{e} \right]$$

Sending  $b \rightarrow 0$  we obtain

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, 0, c \\ d, e \end{matrix}; q, q \right] = \frac{(e/c; q)_n}{(e; q)_n} c^n {}_2\phi_2 \left[ \begin{matrix} q^{-n}, c \\ d, cq^{1-n}/e \end{matrix}; q, \frac{dq}{e} \right] \quad (6.15)$$

Multiply both sides of (6.15) by  $c^{-n}(d; q)_n(e; q)_n$  to obtain

$$\begin{aligned} c^{-n} \sum_{y=0}^n \frac{(q^{-n}; q)_y}{(q; q)_y} (c; q)_y (dq^{n-1}; q^{-1})_{n-y} (eq^{n-1}; q^{-1})_{n-y} q^y = \\ \sum_{y=0}^n \frac{(q^{-n}; q)_y (e/c; q)_n}{(q; q)_y (cq^{1-n}/e; q)_y} (c; q)_y (dq^{n-1}; q^{-1})_{n-y} (-1)^y q^{y(y-1)/2} (dq/e)^y. \end{aligned}$$

This equality can be rewritten as

$$\begin{aligned} \sum_{y=0}^n \binom{n}{y}_{q^{-1}} c^{y-n} (c^{-1}; q^{-1})_y (dq^{n-1}; q^{-1})_{n-y} (eq^{n-1}; q^{-1})_{n-y} = \\ \sum_{y=0}^n \binom{n}{y}_{q^{-1}} (e/c; q)_{n-y} (c^{-1}; q^{-1})_y (dq^{n-1}; q^{-1})_{n-y} (dq^{n-1})^y. \end{aligned}$$

Now make the substitution  $n := s$ ,  $d := q^{1+R-s}$ ,  $c := q^{-\ell}$ ,  $e := q^{1+b-h-\ell}$  to arrive to (6.14).  $\square$

**Remark 6.10.** (Schur degeneration) If one sets  $q = 0$ , then the dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  reduces to the dynamics  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  on Schur processes based on the classical Robinson–Schensted–Knuth column insertion (§2.6). Indeed, observe that

$$\lim_{q \rightarrow 0} \varphi_{q, uq^t, 0}(s \mid g) = \mathbf{1}_{s=0} \quad \text{for } t > 0, \quad \text{and} \quad \lim_{q \rightarrow 0} \varphi_{q, u, 0}(s \mid g) = (1 - u + u\mathbf{1}_{s=g})u^s.$$

Thus, the first update stage (voluntary movements) reduces to the propagation of the impulse the leftmost particle receives (which has geometric distribution with parameter  $\alpha a_j$ ). The lower left neighbor pushing due to (6.6) and the stabilization fund pushing together degenerate to performing  $c_{j-1} + \dots + c_1$  operations **push** (Definition 2.7) in order from left to right.

### 6.5. Column insertion dynamics $\mathcal{Q}_{\text{col}}^q[\alpha]$ . **Proof.**

**Theorem 6.11.** *The dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  defined above satisfies the main equations (4.7), and hence preserves the class of  $q$ -Whittaker processes and adds a new usual parameter  $\alpha$  to the specialization  $\mathbf{A}$  as in (4.1).*

*Proof.* We aim to prove the desired statement by induction on  $j$ . To apply this induction, we will need a more general statement. To describe it, introduce the following notation. For a nonnegative integer  $h$ , use  $\mathcal{U}_j^h(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$  to denote the probability that transition

$\bar{\lambda} \rightarrow \bar{\nu}$  on the  $(j-1)$ -st level spurs a transition  $\lambda \rightarrow \nu$  on the  $j$ -th level according to the rules of  $\mathcal{Q}_{\text{col}}^q[\alpha]$  specified above, but modified so that  $Z_2 = z$  with probability

$$\varphi_{q^{-1}, r_2, 0}(\bar{\lambda}_{j-2} - \lambda_{j-1} - X_2 - Y_2 - z \mid \bar{\lambda}_{j-2} - \lambda_{j-1} - X_2 - Y_2), \quad r_2 := q^h.$$

Note that the original dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  has  $r_2 = 1$ . In other words, the modification  $\mathcal{U}_j^h$  means that we introduce an additional impulse of size  $h$  which is distributed among particles at level  $j$  (except for  $\lambda_j$ ), as if coming from (nonexistent) particles preceding the leftmost particle on the  $(j-1)$ -st level.

Let  $\sigma := |\nu^-| - |\lambda^-| - |\bar{\nu}| + |\bar{\lambda}|$  (recall that the notation  $\mu^-$  means  $\mu$  without the last coordinate). Under the modified probabilities  $\mathcal{U}_j^h$  as above,  $\sigma - h$  is a sum of voluntary movements of particles on the  $j$ -th level except for the leftmost one. Note also that  $\mathcal{U}_j^h(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) = 0$  for  $h > \sigma$ .

To further simplify the notation, let (see Fig. 28)

$$\begin{aligned} a &:= \lambda_j, & k &:= \nu_j - \lambda_j, & b &:= \bar{\nu}_{j-1}, & t &:= \bar{\nu}_{j-1} - \bar{\lambda}_{j-1}, \\ c &:= \lambda_{j-1}, & d &:= \bar{\nu}_{j-2}, & s &:= \bar{\nu}_{j-2} - \bar{\lambda}_{j-2}, \\ \ell &:= \nu_{j-1} - \lambda_{j-1}, & x &:= X_2 & y &:= Y_2. \end{aligned}$$

For a nonnegative integer  $H$  define

$$\tilde{\mathcal{U}}_j^H(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) := \sum_{h=0}^H \binom{H}{h}_{q^{-1}} q^{(H-h)\sigma + h(b-t-a-k)} \mathcal{U}_j^h(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}). \quad (6.16)$$

In particular,  $\tilde{\mathcal{U}}_j^0(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) = \mathcal{U}_j(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu})$ . In general, the quantities  $\tilde{\mathcal{U}}_j^H$  are not probability distributions in  $\nu$ . Their only meaning is that they come up in the inductive proof below.



With all the above notation we are now able to describe and prove the generalized statement which we will prove by induction:

$$\sum_{\bar{\lambda} \in \text{GT}_{j-1}^+} \tilde{\mathcal{V}}_j^H(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \frac{\psi_{\lambda/\bar{\lambda}} \phi_{\bar{\nu}/\bar{\lambda}}}{\psi_{\nu/\bar{\nu}} \phi_{\nu/\lambda}} = 1 \quad \text{for any } H \geq 0. \quad (6.17)$$

Here and below  $\tilde{\mathcal{V}}_j^H$  is related to  $\tilde{\mathcal{U}}_j^H$  as in (6.9). For  $H = 0$  this statement gives us (4.7).

For  $j = 1$  we have  $\sigma = 0$ , so only the term  $h = 0$  contributes to (6.16). Therefore, checking this induction base is the same as in the proof for  $\Omega_{\text{row}}^q[\alpha]$  dynamics.

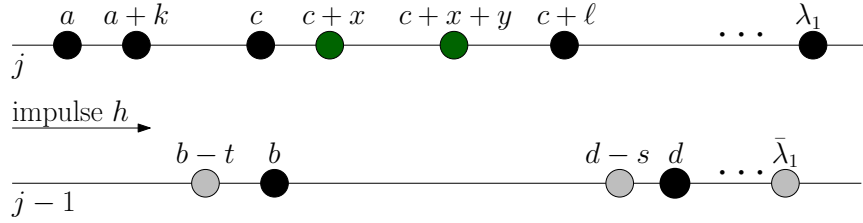


FIGURE 28. We expand sum with respect to jump  $t = c_{j-1} = \bar{\nu}_{j-1} - \bar{\lambda}_{j-1}$  of the leftmost particle on the  $(j-1)$ -st level, voluntary movement  $x$  of the second leftmost particle on the  $j$ -th level and push  $y$  from the leftmost particle on the  $(j-1)$ -st level.

Let us now perform the inductive step. Denote by  $I$  the range of  $(t, x, y, h)$ , such that

$$t, x, y, h \geq 0, \quad x + y \leq \ell, \quad h + t + x - \ell \geq 0, \quad t \leq b - a - k.$$

Then we may write (see Fig. 28)

$$\begin{aligned} & \sum_{\bar{\lambda} \in \text{GT}_{j-1}^+} \tilde{\mathcal{V}}_j^H(\lambda \rightarrow \nu \mid \bar{\lambda} \rightarrow \bar{\nu}) \frac{\psi_{\lambda/\bar{\lambda}} \phi_{\bar{\nu}/\bar{\lambda}}}{\psi_{\nu/\bar{\nu}} \phi_{\nu/\lambda}} \\ &= \sum_{(t,x,y,h) \in I} \frac{(q; q)_H}{(q; q)_h (q; q)_{H-h}} \frac{(q; q)_{c-a}}{(q; q)_{b-t-a} (q; q)_{c-b+t}} \frac{(q; q)_{b-a-k} (q; q)_{c+l-b}}{(q; q)_{c+l-a-k}} \frac{(q; q)_{d-s-b+t}}{(q; q)_t (q; q)_{d-s-b}} \\ & \quad \times \frac{(q; q)_k (q; q)_{c-a-k}}{(q; q)_{c-a}} \frac{(q; q)_{b-t-a}}{(q; q)_k (q; q)_{b-t-a-k}} \frac{(q; q)_{d-s-c}}{(q; q)_x (q; q)_{d-s-c-x}} \frac{(q; q)_{d-s-c-x}}{(q; q)_y (q; q)_{d-s-c-x-y}} \\ & \quad \times \frac{(q^t; q^{-1})_y (q^{d-s-b}; q^{-1})_{d-s-c-x-y}}{(q^{d-s-b+t}; q^{-1})_{d-s-c-x}} \frac{(q; q)_{d-s-c-x-y}}{(q; q)_{\ell-x-y} (q; q)_{d-s-c-\ell}} (q^h; q^{-1})_{\ell-x-y} \end{aligned}$$

$$\begin{aligned}
& \times q^{-h(H-h)-y(d-s-c-x-y)-(d-s-c-\ell)(\ell-x-y)+t(d-s-c-x-y)} \\
& \times q^{h(d-s-c-\ell)+(H-h)\sigma+h(b-t-a-k)+(b-t-a-k)x+(b-t-a-k+\ell-x)(\sigma-x-h)} \\
& \times \sum_{\tilde{\lambda} \in \text{GT}_{j-2}^+} \mathcal{V}_j^{h+t+x-\ell}(\lambda^- \rightarrow \nu^- \mid \tilde{\lambda} \rightarrow \bar{\nu}^-) \frac{\psi_{\lambda^-/\tilde{\lambda}} \phi_{\bar{\nu}^-/\tilde{\lambda}}}{\psi_{\nu^-/\bar{\nu}^-} \phi_{\nu^-/\lambda^-}} \\
& = \sum_{r=0}^{H+B} \binom{H+B}{r} q^{(H+B-r)(\sigma-\ell+t)+r(d-s-c-\ell)} \\
& \times \sum_{\tilde{\lambda} \in \text{GT}_{j-2}^+} \mathcal{V}_j^r(\lambda^- \rightarrow \nu^- \mid \tilde{\lambda} \rightarrow \bar{\nu}^-) \frac{\psi_{\lambda^-/\tilde{\lambda}} \phi_{\bar{\nu}^-/\tilde{\lambda}}}{\psi_{\nu^-/\bar{\nu}^-} \phi_{\nu^-/\lambda^-}} \\
& = \sum_{\tilde{\lambda} \in \text{GT}_{j-2}^+} \tilde{\mathcal{V}}_j^{H+B}(\lambda^- \rightarrow \nu^- \mid \tilde{\lambda} \rightarrow \bar{\nu}^-) \frac{\psi_{\lambda^-/\tilde{\lambda}} \phi_{\bar{\nu}^-/\tilde{\lambda}}}{\psi_{\nu^-/\bar{\nu}^-} \phi_{\nu^-/\lambda^-}} \\
& = 1.
\end{aligned}$$

Here we have applied Proposition 6.12 (see below) with  $A = H$ ,  $B = b - a - k$ ,  $C = c - b + \ell$ , where  $r := h + t - \ell + x$  is the value of the stabilization fund just before the push of the third leftmost particle on the  $j$ -th level plus the value of the additional impulse in the inductive assumption. This completes the inductive step in proving (6.17), and thus implies the theorem.  $\square$

**Proposition 6.12.** *For  $A, B, C, \ell, r \geq 0$ , such that  $A + B \geq r$  and  $B + C \geq \ell$ , one has*

$$\begin{aligned}
& \sum_{t=0}^B \sum_{x=0}^{\ell} \sum_{y=0}^{\ell-x} \left[ \binom{\ell}{x, y}_{q^{-1}} \binom{B}{t}_{q^{-1}} (q^t; q^{-1})_y (q^{r+\ell-x}; q^{-1})_t (q^{r+\ell-t-x}; q^{-1})_{\ell-x-y} \frac{(q; q)_r}{(q; q)_{r+\ell-x}} \right. \\
& \times \frac{(q; q)_A}{(q; q)_{A+B}} \frac{(q^{A+B-r}; q^{-1})_{B-t+\ell-x} (q^{C+t}; q^{-1})_{\ell} (q^C; q^{-1})_{\ell-x-y}}{(q^{B+C}; q^{-1})_{\ell} (q^{C+t}; q^{-1})_{\ell-x}} \\
& \left. \times q^{t(\ell-x-y)+(r+\ell-x)(B-t)+(A+B-r)x} \right] = 1.
\end{aligned}$$

Here and thereafter we use  $q$ -multinomial notation  $\binom{n}{m, k}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_m (q; q)_{n-m-k}}$ .

We are extremely grateful to Christian Krattenthaler for providing us with a proof of this proposition. We reproduce the proof below.

*Proof.* The left hand side of the equality can be expressed as a power series in  $q^A, q^r, q^C$ , hence we can set  $\alpha = q^A, \beta = q^r, \gamma = q^C$  and prove a more general equality:

$$\sum_{t=0}^B \sum_{x=0}^{\ell} \sum_{y=0}^{\ell-x} \left[ \binom{\ell}{x, y}_{q^{-1}} \binom{B}{t}_{q^{-1}} (q^t; q^{-1})_y \frac{(\beta q^{\ell-x}; q^{-1})_{t+\ell-x-y}}{(\beta q^{\ell-x}; q^{-1})_{\ell-x}} \right. \\ \left. \times \frac{(\alpha q^B / \beta; q^{-1})_{B-t+\ell-x} (\gamma q^t; q^{-1})_{\ell} (\gamma; q^{-1})_{\ell-x-y}}{(\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (\gamma q^t; q^{-1})_{\ell-x}} \times \alpha^x \beta^{B-t-x} q^{-ty+Bl} \right] = 1.$$

By first summing over  $y$ , the left hand side can be written as

$$\sum_{t=0}^B \sum_{x=0}^{\ell} \left[ {}_3\phi_2 \left[ \begin{matrix} q^{-t}, 0, q^{-\ell+x} \\ \beta q^{1-t}, \gamma q^{1-\ell+x} \end{matrix}; q, q \right] \binom{\ell}{x}_{q^{-1}} \binom{B}{t}_{q^{-1}} \frac{(\beta q^{\ell-x}; q^{-1})_{t+\ell-x}}{(\beta q^{\ell-x}; q^{-1})_{\ell-x}} \right. \\ \left. \times \frac{(\alpha q^B / \beta; q^{-1})_{B-t+\ell-x} (\gamma q^t; q^{-1})_{\ell} (\gamma; q^{-1})_{\ell-x}}{(\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (\gamma q^t; q^{-1})_{\ell-x}} \times \alpha^x \beta^{B-t-x} q^{Bl} \right].$$

We now apply transformation formula (6.15) to rewrite this as

$$= \sum_{t=0}^B \sum_{x=0}^{\ell} \left[ {}_2\phi_2 \left[ \begin{matrix} q^{-t}, q^{-\ell+x} \\ \beta q^{1-t}, q^{-t}/\gamma \end{matrix}; q, \beta q^{1+\ell-t-x}/\gamma \right] \binom{\ell}{x}_{q^{-1}} \binom{B}{t}_{q^{-1}} \frac{(\beta q^{\ell-x}; q^{-1})_{t+\ell-x}}{(\beta q^{\ell-x}; q^{-1})_{\ell-x}} \frac{(\gamma q; q)_t}{(\gamma q^{1-\ell+x}; q)_t} \right. \\ \left. \times \frac{(\alpha q^B / \beta; q^{-1})_{B-t+\ell-x} (\gamma q^t; q^{-1})_{\ell} (\gamma; q^{-1})_{\ell-x}}{(\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (\gamma q^t; q^{-1})_{\ell-x}} \times \alpha^x \beta^{B-t-x} q^{Bl-\ell t+tx} \right] \\ = \sum_{t=0}^B \sum_{x=0}^{\ell} \sum_{y=0}^{\min\{t, \ell-x\}} \left[ (-1)^y \frac{(q^{-t}; q)_y (q^{-\ell+x}; q)_y}{(q^{-t}/\gamma; q)_y (\beta q^{1-t}; q)_y (q; q)_y} \binom{\ell}{x}_{q^{-1}} \binom{B}{t}_{q^{-1}} \frac{(\gamma q; q)_t}{(\gamma q^{1-\ell+x}; q)_t} \right. \\ \left. \times \frac{(\beta; q^{-1})_t (\alpha q^B / \beta; q^{-1})_{B-t+\ell-x} (\gamma q^t; q^{-1})_{\ell} (\gamma; q^{-1})_{\ell-x}}{(\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (\gamma q^t; q^{-1})_{\ell-x}} \right. \\ \left. \times \alpha^x \beta^{B-t-x+y} \gamma^{-y} q^{Bl+\ell y-\ell t+tx+y^2/2+y/2-ty-xy} \right]$$

$$\begin{aligned}
&= \sum_{t=0}^B \sum_{y=0}^{\ell} \left[ {}_1\phi_1 \left[ \begin{matrix} q^{-\ell+y} \\ \alpha q^{1-\ell+t}/\beta \end{matrix}; q, \alpha q^{1+t-y}/\beta \right] (-1)^y \frac{(q^{-\ell}; q)_y (q^{-t}; q)_y}{(q^{-t}/\gamma; q)_y (\beta q^{1-t}; q)_y (q; q)_y} \right. \\
&\quad \left. \times \binom{B}{t}_{q^{-1}} \frac{(\beta; q^{-1})_t (\alpha q^B/\beta; q^{-1})_{B-t+\ell} (\gamma q^t; q^{-1})_{\ell}}{(\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell}} \times \beta^{B-t+y} \gamma^{-y} q^{B\ell - \ell t + \ell y + y^2/2 + y/2 - ty} \right].
\end{aligned}$$

The last equality is obtained by summing over  $x$ . We now use the summation formula [35, (II.5)]:

$${}_1\phi_1 \left[ \begin{matrix} a \\ c \end{matrix}; q, c/a \right] = \frac{(c/a; q)_{\infty}}{(c; q)_{\infty}},$$

and by summing over  $y$  rewrite our expression as

$$\begin{aligned}
&= \sum_{t=0}^B \left[ {}_3\phi_2 \left[ \begin{matrix} q^{-\ell}, \beta q^{-t}/\alpha, q^{-t} \\ \beta q^{1-t}, q^{-t}/\gamma \end{matrix}; q, \alpha q^{1+\ell}/\gamma \right] \right. \\
&\quad \left. \times \binom{B}{t}_{q^{-1}} \frac{(\beta; q^{-1})_t (\alpha q^B/\beta; q^{-1})_{B-t+\ell} (\gamma q^t; q^{-1})_{\ell}}{(\alpha q^t/\beta; q^{-1})_{\ell} (\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell}} \times \beta^{B-t} q^{B\ell - \ell t} \right].
\end{aligned}$$

We now aim to use the transformation formula [35, (III.13)]:

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, deq^n/bc \right] = \frac{(e/c; q)_n}{(e; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; q, q \right]. \quad (6.18)$$

Applying it, we can rewrite our expression as

$$\begin{aligned}
&= \sum_{t=0}^B \left[ {}_3\phi_2 \left[ \begin{matrix} q^{-\ell}, q^{-t}, \alpha q \\ \beta q^{1-t}, \gamma q^{1-\ell} \end{matrix}; q, q \right] \right. \\
&\quad \left. \times \binom{B}{t}_{q^{-1}} \frac{(1/\gamma; q)_{\ell} (\beta; q^{-1})_t (\alpha q^B/\beta; q^{-1})_{B-t+\ell} (\gamma q^t; q^{-1})_{\ell}}{(\alpha q^t/\beta; q^{-1})_{\ell} (\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (q^{-t}/\gamma; q)_{\ell} (q; q)_y} \times \beta^{B-t} q^{B\ell - \ell t} \right] \\
&= \sum_{t=0}^B \sum_{y=0}^{\ell} \left[ \binom{B}{t}_{q^{-1}} \frac{(q^{-\ell}; q)_y (q^{-t}; q)_y (\alpha q; q)_y (\gamma; q^{-1})_{\ell-y} (\beta q^y; q^{-1})_t (\alpha q^B/\beta; q^{-1})_{B-t+\ell}}{(\alpha q^t/\beta; q^{-1})_{\ell} (\alpha q^B; q^{-1})_B (\gamma q^B; q^{-1})_{\ell} (\beta q^y; q^{-1})_y (q; q)_y} \times \beta^{B-t} q^{B\ell + y} \right]
\end{aligned}$$

$$= \sum_{y=0}^{\ell} \sum_{t=y}^B \left[ \binom{B-y}{t-y}_{q^{-1}} \frac{\beta^{B-t}(\beta; q^{-1})_{t-y}(\alpha q^B/\beta; q^{-1})_{B-t}}{(\alpha q^B; q^{-1})_{B-y}} \times \binom{\ell}{y}_{q^{-1}} \frac{q^{B(\ell-y)}(\gamma; q^{-1})_{\ell-y}(q^B; q^{-1})_y}{(\gamma q^B; q^{-1})_{\ell}} \right].$$

The fact that this expression is equal to 1 now follows by applying (6.3) twice.  $\square$

**6.6. Geometric  $q$ -TASEP.** Under the dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$ , the leftmost  $N$  particles  $\lambda_j^{(j)}$  of the interlacing array evolve in a *marginally Markovian manner*.

Namely, let  $\text{gap}_j(t) := \lambda_{j-1}^{(j-1)}(t) - \lambda_j^{(j)}(t)$  be the gap between the consecutive leftmost particles at time  $t$ . We assume  $\text{gap}_1(t) = +\infty$ . Then at each discrete time step  $t \rightarrow t+1$  the leftmost particle on the  $j$ -th level is updated as

$$\lambda_1^{(j)}(t+1) = \lambda_1^{(j)}(t) + W_{j,t}$$

for an independent random variable  $W_{j,t}$  distributed according to  $\varphi_{q, \alpha a_j, 0}(\cdot \mid \text{gap}_j(t))$ .

This evolution of  $\lambda_j^{(j)}$ ,  $1 \leq j \leq N$ , is the (*discrete time*) *geometric  $q$ -TASEP* which was introduced and studied in [7].

**6.7. Small  $\alpha$  continuous time limit.** Let us send the parameter  $\alpha$  to zero and simultaneously rescale time from discrete to continuous. Namely, set  $\alpha := (1-q)\Delta$ , and let each discrete time step correspond to continuous time  $\Delta$ . In the limit  $\Delta \rightarrow 0$ , both dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$  turn into the same continuous time Markov dynamics on  $q$ -Whittaker processes as in §5.6 above. That is, the limit of  $\mathcal{Q}_{\text{row}}^q[\alpha]$  is the dynamics introduced in [14], same as for  $\mathcal{Q}_{\text{row}}^q[\hat{\beta}]$ . The limit of  $\mathcal{Q}_{\text{col}}^q[\alpha]$  is the dynamics introduced in [58], same as for  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$ .

## 7. MOMENTS

In this section we briefly discuss moment formulas for the Bernoulli  $q$ -PushTASEP started from the step initial configuration (corresponding to  $\lambda_1^{(j)}(0) = 0$ ,  $j = 1, \dots, N$ ).

7.1. **Bernoulli  $q$ -PushTASEP on the line.** In this section it will be convenient to work in the shifted coordinates

$$x_i := -\lambda_1^{(i)} - i, \quad i = 1, \dots, N,$$

so that  $x_1 > \dots > x_N$ . We will think that the  $x_j$ 's encode positions of particles on the line  $\mathbb{Z}$  which jump to the *left*. Let us reformulate the definition of the Bernoulli  $q$ -PushTASEP (§5.2) in these terms.

**Definition 7.1.** Each discrete time step  $t \rightarrow t + 1$  of the Bernoulli  $q$ -PushTASEP consists of the following sequential updates (see Fig. 29):

- (1) The first particle  $x_1$  jumps to the left by one with probability  $\frac{a_1\beta}{1+a_1\beta}$ , and stays put with the complementary probability  $\frac{1}{1+a_1\beta}$ .
- (2) Sequentially for  $j = 2, \dots, N$ :
  - (a) If the particle  $x_{j-1}$  has not jumped, then  $x_j$  jumps to the left by one with probability  $\frac{a_j\beta}{1+a_j\beta}$ , and stays put with the complementary probability  $\frac{1}{1+a_j\beta}$ .
  - (b) If the particle  $x_{j-1}$  has jumped (to the left by one), then  $x_j$  jumps to the left by one with probability  $\frac{a_j\beta + q^{\text{gap}_j(t)}}{1+a_j\beta}$ , and stays put with the complementary probability  $\frac{1 - q^{\text{gap}_j(t)}}{1+a_j\beta}$ , where  $\text{gap}_j(t) := x_{j-1}(t) - x_j(t) - 1$  is the distance between the particles before the jump of  $x_{j-1}$ .<sup>25</sup>

We will assume that the Bernoulli  $q$ -PushTASEP starts from the step initial configuration  $x_i(0) = -i$ ,  $i = 1, \dots, N$ .

---

<sup>25</sup>Note that if  $x_{j-1}$  has jumped and  $x_j(t) = x_{j-1}(t) - 1$ , then the probability that  $x_j$  jumps is equal to one, as it should be.

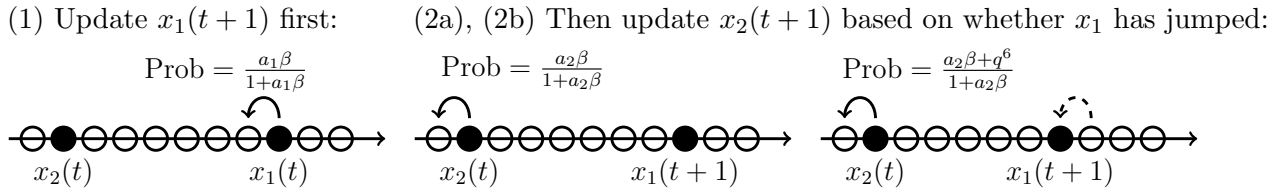


FIGURE 29. Bernoulli  $q$ -PushTASEP (on this picture,  $\text{gap}_2(t) = 6$ ).

**7.2. Connection to the Bernoulli  $q$ -TASEP.** The Bernoulli  $q$ -PushTASEP looks quite similar to the Bernoulli  $q$ -TASEP introduced in [7] (see also §5.5 above for an explanation of how the latter process arises from the dynamics  $\mathcal{Q}_{\text{col}}^q[\hat{\beta}]$  on  $q$ -Whittaker processes).

Moreover, there exists a *direct coupling* between the two processes which we now explain. Recall that under the Bernoulli  $q$ -TASEP (we will denote its particles with tildes:  $\tilde{x}_1(t) > \dots > \tilde{x}_N(t)$ ) particles jump to the *right* by one according to the rules on Fig. 30. Let this process also start from the step initial configuration  $\tilde{x}_i(0) = -i$ ,  $i = 1, \dots, N$ .

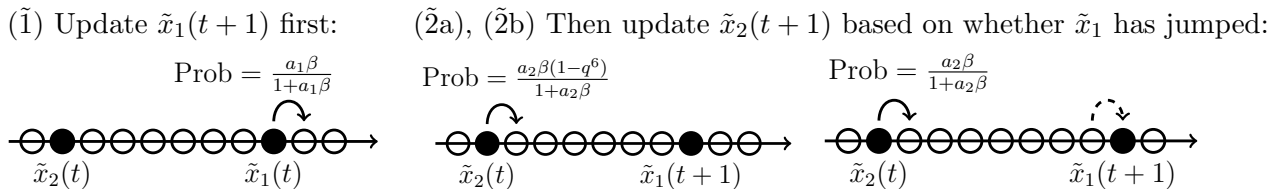


FIGURE 30. Bernoulli  $q$ -TASEP (on this picture,  $\text{gap}_2(t) = 6$ ).

**Proposition 7.2.** *Let  $\{x_i(t)\}_{t=0,1,\dots}$  be the Bernoulli  $q$ -PushTASEP started from the step initial configuration and depending on parameters  $\{a_i\}$  and  $\beta$ .*

*Then the evolution of the process  $\{t + x_i(t)\}_{t=0,1,\dots}$  coincides with the Bernoulli  $q$ -TASEP  $\{\tilde{x}_i(t)\}_{t=0,1,\dots}$  started from the step initial configuration and depending on the parameters  $\{a_i^{-1}\}$  and  $\beta^{-1}$ .*

*Proof.* The process  $\{t + x_i(t)\}$  jumps to the right, and, moreover, each of its particles makes a jump precisely when the corresponding  $q$ -PushTASEP particle  $x_i(t)$  stays put. In

particular, the first particle  $x_1(t)$  stays put with probability  $1/(1 + a_1\beta) = (a_1^{-1}\beta^{-1})/(1 + a_1^{-1}\beta^{-1})$ . Next, if  $x_1(t)$  stayed put, then  $x_2(t)$  stays put with probability  $1/(1 + a_2\beta) = (a_2^{-1}\beta^{-1})/(1 + a_2^{-1}\beta^{-1})$ . Otherwise, if  $x_1(t)$  jumped to the left, then  $x_2(t)$  stays put with probability

$$1 - \frac{a_2\beta + q^{\text{gap}_2(t)}}{1 + a_2\beta} = \frac{1 - q^{\text{gap}_2(t)}}{1 + a_2\beta} = \frac{a_2^{-1}\beta^{-1}(1 - q^{\text{gap}_2(t)})}{1 + a_2^{-1}\beta^{-1}}.$$

We see that the particles  $\{t + x_i(t)\}$  indeed perform the Bernoulli  $q$ -TASEP evolution with the desired parameters.  $\square$

One can think that this coupling between the two particle systems on  $\mathbb{Z}$  comes from the complementation procedure (§5.3) relating the corresponding two-dimensional dynamics.

**7.3. Nested contour integral formulas for  $q$ -moments.** The above coupling between the Bernoulli  $q$ -PushTASEP and the Bernoulli  $q$ -TASEP allows to readily write down moment formulas for the former process:

**Theorem 7.3.** *Let  $\{x_i(t)\}_{t=0,1,\dots}$  be the Bernoulli  $q$ -PushTASEP jumping to the left, started from the step initial configuration. Fix  $k \geq 1$ . For all  $t = 0, 1, 2, \dots$  and all integers  $N \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 0$ ,*

$$\begin{aligned} & \mathbb{E}^{\text{step}} \left( \prod_{i=1}^k q^{x_{n_i}(t) + n_i} \right) \\ &= \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi\mathbf{i})^k} \oint \dots \oint \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^k \left( \prod_{i=1}^{n_j} \frac{1}{1 - a_i z_j} \right) \left( \frac{1 + q^{-1}\beta z_j^{-1}}{1 + \beta z_j^{-1}} \right)^t \frac{dz_j}{z_j}, \end{aligned} \tag{7.1}$$

where the contour of integration for each  $z_A$  contains  $a_1^{-1}, \dots, a_N^{-1}$ , and the contours  $\{qz_B\}_{B>A}$ , but not poles 0 or  $(-\beta)$ .

*Proof.* Immediately follows from Proposition 7.2 and [7, Theorem 2.1.(3)].  $\square$



**Remark 7.4.** Since  $x_i(t) + i \geq -t$  for any  $i = 1, \dots, N$  and any  $t \geq 0$ , the  $q$ -moments in (7.1) admit an a priori bound. Therefore, for a fixed  $t \geq 0$  they determine the distribution of the random variables  $(x_1(t), \dots, x_N(t))$ .

**Remark 7.5.** One can also establish the nested contour integral formula (7.1) directly, similarly to [7] (see also [23]). Indeed, denote

$$I_t(\vec{y}) := q^{t(y_1+y_2+\dots+y_N)} \mathbb{E}^{\text{step}} \left( \prod_{i=0}^N q^{y_i(x_i(t)+i)} \right),$$

where  $(y_0, \dots, y_N) \in \mathbb{Z}_{\geq 0}^N$  and, by agreement, the product is zero if  $y_0 > 0$ .<sup>26</sup> One can directly show that these quantities satisfy certain linear equations in the  $y_j$ 's. For each  $i = 1, \dots, N$ , consider the following difference operators acting on functions in  $\vec{y}$ :

$$[\mathcal{H}^{q,\xi}]_i f(\vec{y}) := \sum_{s_i=0}^{y_i} \varphi_{q, a_i^{-1} \xi, 0}(s_i | y_i) f(y_0, y_1, \dots, y_{i-2}, y_{i-1} + s_i, y_i - s_i, y_{i+1}, \dots, y_N). \quad (7.2)$$

Here the quantities  $\varphi$  are defined in (6.1).

Also, denote by  $\mathcal{H}^{q,\xi}$  the operator which acts as  $[\mathcal{H}^{q,\xi}]_i$  in each variable  $y_i$ :

$$\mathcal{H}^{q,\xi} := [\mathcal{H}^{q,\xi}]_N [\mathcal{H}^{q,\xi}]_{N-1} \dots [\mathcal{H}^{q,\xi}]_1. \quad (7.3)$$

Applying operators  $[\mathcal{H}^{q,\xi}]_i$  in this order corresponds to first changing  $y_1$  (by decreasing it by  $s_1$ ), then  $y_2$  (by sending  $s_2$  to  $y_1 - s_1$ ), etc., up to  $y_N$ . In other words, these changes (encoded by  $s_1, \dots, s_N$ ) happen in parallel, simultaneously with each of  $y_1, y_2, \dots, y_N$ .

One can then show that for any  $t = 0, 1, 2, \dots$  and any  $\vec{y} = (y_0, y_1, \dots, y_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ , the quantities  $I_t(\vec{y})$  satisfy

$$\mathcal{H}^{q, -\beta^{-1}} I_{t+1}(\vec{y}) = \mathcal{H}^{q, -q\beta^{-1}} I_t(\vec{y}).$$

---

<sup>26</sup>One should think that the variables  $y_j$  encode the  $n_i$ 's in (7.1): each  $y_j$  denotes the number of  $n_i$ 's which are equal to  $j$ .

These linear equations can then be solved by the coordinate Bethe ansatz technique, because the action of each of the operators  $\mathcal{H}^{q,-\beta^{-1}}$  and  $\mathcal{H}^{q,-q\beta^{-1}}$  reduces to the action of a free operator (i.e., which acts on each of the variables  $n_i$  separately; note the identification of the  $y_j$ 's and  $n_i$ 's in the previous footnote) plus two-body boundary conditions. This immediately leads to the desired nested contour integral formula.

**7.4. Remark. Geometric  $q$ -PushTASEP formulas.** There are also nested contour integral formulas for  $q$ -moments of the geometric  $q$ -PushTASEP (§6.3). They can be obtained directly using the definition of the dynamics, similarly to the approach outlined in Remark 7.5. The moment formulas (for the geometric  $q$ -PushTASEP jumping to the left) will have the same form as in (7.1), with the following replacement of factors:

$$\prod_{j=1}^k \left( \frac{1 + q^{-1}\beta z_j^{-1}}{1 + \beta z_j^{-1}} \right)^t \longrightarrow \prod_{j=1}^k \frac{1}{(1 - \alpha q^{-1} z_j^{-1})^t}.$$

However, because particles in the geometric  $q$ -PushTASEP can jump arbitrarily far to the left (at least as far as by independent  $q$ -geometric jumps), only a finite number of  $q$ -moments of the form  $\mathbb{E}^{\text{step}} \left( \prod_{i=1}^k q^{x_{n_i}(t)+n_i} \right)$  exists. Therefore, these  $q$ -moments do not determine the distribution of the geometric  $q$ -PushTASEP.

## 8. POLYMER LIMITS OF $(\alpha)$ DYNAMICS ON $q$ -WHITTAKER PROCESSES

In this section we explain how the two  $(\alpha)$  dynamics on  $q$ -Whittaker processes behave in the limit as  $q \nearrow 1$ . This leads to discrete time stochastic processes related to geometric RSK correspondences and directed random polymers.

**8.1. Polymer partition functions.** Let us first describe the polymer models we will be dealing with. They are based on inverse-Gamma random variables:

**Definition 8.1.** A positive random variable  $X$  has *Gamma distribution* with shape parameter  $\theta > 0$  if it has probability density

$$P(X \in dx) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} dx.$$

We abbreviate this by  $X \sim \text{Gamma}(\theta)$ . Then  $X^{-1}$  has probability density

$$P(X^{-1} \in dx) = \frac{1}{\Gamma(\theta)} x^{-\theta-1} e^{-1/x} dx,$$

which is called *inverse-Gamma distribution* and denoted by  $\text{Gamma}^{-1}(\theta)$ .

We recall partition functions of two models of log-Gamma polymers in  $1 + 1$  dimensions studied previously in [69], [10], [21], [59], [57], [24] (see also [55] for a continuous time version). Both models are defined on the lattice strip  $\{(t, j) \mid t \in \{0, 1, 2, \dots\}, j \in \{1, 2, \dots, n\}\}$ . One should think of  $t$  as time. Suppose we have two collections of real numbers  $\theta_j$  for  $j \in \{1, 2, \dots, n\}$  and  $\hat{\theta}_t$  for  $t \in \{0, 1, 2, \dots\}$ , such that  $\theta_j + \hat{\theta}_t > 0$  for all  $j$  and  $t$ .

**Definition 8.2** (Log-Gamma polymer [69]; Fig. 31, left). Each vertex  $(t, j)$  in the strip is equipped with a random weight  $d_{t,j}$ . These weights are independent, and  $d_{t,j}$  is distributed according to  $\text{Gamma}^{-1}(\theta_j + \hat{\theta}_t)$ . The *log-Gamma polymer partition function* with parameters  $\theta_j, \hat{\theta}_t$  is given by

$$R_1^j(t) := \sum_{\pi: (1,1) \rightarrow (t,j)} \prod_{(s,i) \in \pi} d_{s,i}, \quad (8.1)$$

where the sum is over directed up/right lattice paths  $\pi$  from  $(1, 1)$  to  $(t, j)$ , which are made of horizontal edges  $(s, i) \rightarrow (s + 1, i)$  and vertical edges  $(s, i) \rightarrow (s, i + 1)$ . Extend this definition to denote by  $R_k^j(t)$  for  $t \geq k$  the weighted sum over all  $k$ -tuples of *nonintersecting* up/right lattice paths starting from  $(1, 1), (1, 2), \dots, (1, k)$  and going respectively to  $(t, j - k + 1), (t, j - k + 2), \dots, (t, j)$ . The weight of a tuple of paths is defined by taking

a product of weights of vertices of these paths. The inequality  $t \geq k$  ensures that  $R_k^j(t)$  is positive.

**Definition 8.3** (Strict-weak polymer [24], [57];<sup>27</sup> Fig. 31, right). Each horizontal edge  $e$  in the strip is equipped with a random weight  $d_e$ . These weights are independent, and  $d_{(t-1,j) \rightarrow (t,j)}$  is distributed according to  $\text{Gamma}(\theta_j + \hat{\theta}_t)$ . The *strict-weak polymer partition function* with parameters  $\theta_j, \hat{\theta}_t$  is given by

$$L_1^j(t) := \sum_{\pi: (0,1) \rightarrow (t,j)} \prod_{e \in \pi} d_e, \quad (8.2)$$

where the sum is over directed lattice paths from  $(0, 1)$  to  $(t, j)$  which are made of horizontal edges  $(s, i) \rightarrow (s+1, i)$  and diagonal moves  $(s, i) \rightarrow (s+1, i+1)$ . The product is taken only over horizontal edges of the path. Extend this definition to denote by  $L_k^j(t)$  for  $t \geq j - k$  the weighted sum over all  $k$ -tuples of the corresponding *nonintersecting* lattice paths starting from  $(0, 1), (0, 2), \dots, (0, k)$  and going respectively to  $(t, j - k + 1), (t, j - k + 2), \dots, (t, j)$ . The weight of a tuple of paths is defined by taking a product of weights of horizontal edges of these paths. The inequality  $t \geq j - k$  ensures that  $L_k^j(t)$  is positive.

Distributions of ratios of the polymer partition functions defined above are sometimes called *Whittaker processes* (or, to be more precise,  $\alpha$ -*Whittaker processes*), cf. [8]. They arise as limits (as  $q$ , the  $a_j$ 's and the  $\alpha_t$ 's simultaneously go to 1) of suitably rescaled particle positions in an interlacing integer array distributed according to the  $q$ -Whittaker process  $\mathcal{M}_{\mathbf{A}}^{\vec{a}}$  (§3.2), where

$$\mathbf{A} = (\alpha_1, \dots, \alpha_t), \quad \vec{a} = (a_1, \dots, a_n).$$

---

<sup>27</sup>These two papers independently introduce essentially the same model. We will be using the notation of [24].

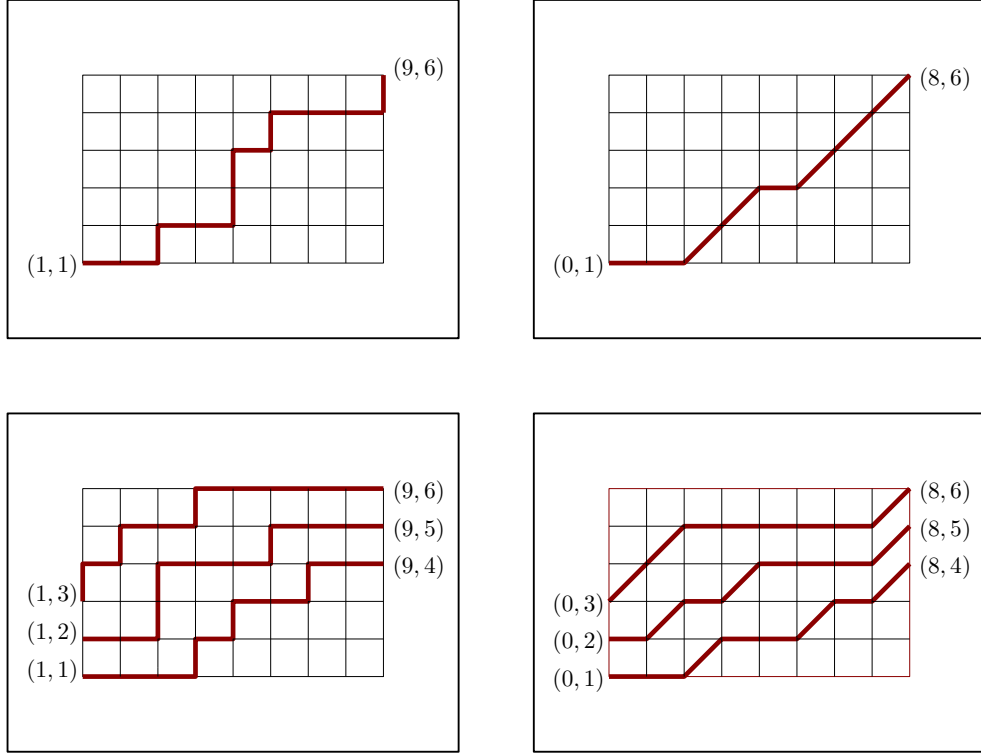


FIGURE 31. Paths and tuples of paths that contribute to the polymer partition functions:  $R_1^5(9)$  (top left),  $R_3^5(9)$  (bottom left),  $L_1^6(8)$  (top right),  $L_3^6(8)$  (bottom right).

The convergence of  $q$ -Whittaker processes to Whittaker processes is known in the literature, see [8, Thm. 4.2.4]. Since both dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$  constructed in §6 sample the  $q$ -Whittaker processes, we can employ them to give another proof of this limit transition. Moreover, we also establish the convergence of the corresponding *stochastic dynamics*.

Let us first define the appropriately scaled pre-limit dynamics. In what follows, for  $\epsilon > 0$  and  $\theta_j, \hat{\theta}_t$  as above, we set  $q := e^{-\epsilon}$ ,  $a_j = e^{-\theta_j \epsilon}$  and  $\alpha_t := e^{-\hat{\theta}_t \epsilon}$ .

**Definition 8.4** (Scaled  $\mathcal{Q}_{\text{row}}^q[\alpha]$  dynamics). Start the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  from the zero initial condition (that is,  $\lambda_i^{(j)}(0) \equiv 0$ ). Denote by  $r_{j,k}(t, \epsilon)$  the position of the  $k$ -th particle *from the right* on the  $j$ -th level of the array after  $t$  steps of the dynamics (at each time step  $t \rightarrow t + 1$ , apply the dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  with parameter  $\alpha = \alpha_{t+1}$ ). For  $t \geq k$ , define the

random variables  $\hat{R}_k^j(t, \epsilon)$  via

$$r_{j,k}(t, \epsilon) = (t + j - 2k + 1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log(\hat{R}_k^j(t, \epsilon)).$$

The reason for the restriction  $t \geq k$  comes from the fact that  $r_{j,k}(t, \epsilon) = 0$  for  $t < k$ .

We will view the collection of random variables  $\{\hat{R}_k^j(t, \epsilon)\}$  as a stochastic process  $\hat{R}(t, \epsilon)$  which at a fixed time  $t$  becomes an array  $\hat{R}_k^j(t, \epsilon)$ ,  $1 \leq k \leq j \leq n$  for  $t \geq n$ , or a truncated array  $\hat{R}_k^j(t, \epsilon)$ ,  $1 \leq k \leq \min\{t, j\} \leq n$  for  $0 < t < n$ .

**Definition 8.5** (Scaled  $\mathcal{Q}_{\text{col}}^q[\alpha]$  dynamics). Start the dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  from the zero initial condition, and denote by  $\ell_{j,k}(t, \epsilon)$  the position of the  $k$ -th particle *from the left* on the  $j$ -th level of the array after  $t$  steps of the dynamics (again, at each time step  $t \rightarrow t + 1$ , apply the dynamics  $\mathcal{Q}_{\text{col}}^q[\alpha]$  with parameter  $\alpha = \alpha_{t+1}$ ). For  $t \geq j - k + 1$ , define the random variable  $\hat{L}_k^j(t, \epsilon)$  via

$$\ell_{j,k}(t, \epsilon) = (t - j + 2k - 1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log(\hat{L}_k^j(t, \epsilon)).$$

The reason for the restriction  $t \geq j - k + 1$  comes from the fact that  $\ell_{j,k}(t, \epsilon) = 0$  for  $t < j - k + 1$ .

We will view the collection of random variables  $\{\hat{L}_k^j(t, \epsilon)\}$  as a stochastic process  $\hat{L}(t, \epsilon)$ , which at a fixed time  $t$  becomes an array  $\hat{L}_k^j(t, \epsilon)$ ,  $1 \leq k \leq j \leq n$  for  $t \geq n$ , or a truncated array  $\hat{L}_k^j(t, \epsilon)$ ,  $1 \leq k \leq j \leq \min\{n, k + t - 1\}$  for  $0 < t < n$ .

**Remark 8.6.** Observe that for a fixed time  $t$ , the array  $r_{j,k}(t, \epsilon)$  has the same distribution as the array  $\ell_{j,j-k+1}(t, \epsilon)$  (by Theorems 6.4 and 6.11, they are distributed as  $q$ -Whittaker processes). Hence the (possibly truncated) arrays  $\hat{R}_k^j(t, \epsilon)$  and  $1/\hat{L}_{j-k+1}^j(t, \epsilon)$  for  $1 \leq k \leq \min\{t, j\} \leq n$  have the same distribution. However, these arrays will not be identically distributed as *stochastic processes* in  $t$  since they come from different multivariate dynamics.

In the setting of polymer partition functions, define random processes  $\hat{R}(t)$  and  $\hat{L}(t)$  on (possibly truncated) arrays via

$$\hat{R}_k^j(t) := R_k^j(t)/R_{k-1}^j(t), \quad \text{for } 1 \leq k \leq \min\{t, j\} \leq n$$

and

$$\hat{L}_k^j(t) := L_k^j(t)/L_{k-1}^j(t), \quad \text{for } 1 \leq k \leq j \leq \min\{n, k+t-1\}.$$

They are well defined, because  $R_k^j(t), R_{k-1}^j(t) > 0$  for  $t \geq k$  and  $L_k^j(t), L_{k-1}^j(t) > 0$  for  $t \geq j - k + 1$ .

We are now in a position to formulate results on the limiting behavior of dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$ . In this section we prove the following:

**Theorem 8.7.** *As  $\epsilon \rightarrow 0$ , the process  $\hat{R}(t, \epsilon)$  of Definition 8.4 converges in distribution to the process  $\hat{R}(t)$ .*

**Theorem 8.8.** *As  $\epsilon \rightarrow 0$ , the process  $\hat{L}(t, \epsilon)$  of Definition 8.5 converges in distribution to the process  $\hat{L}(t)$ .*

**Corollary 8.9.** *The (possibly truncated) arrays  $\hat{R}_k^j(t)$  and  $1/\hat{L}_{j-k+1}^j(t)$  for  $1 \leq k \leq \min\{t, j\} \leq n$  have the same distribution.*

In particular,  $1/\hat{R}_j^j(t)$  and  $\hat{L}_1^j(t)$  have the same distribution. The latter fact was proven in [57], and was used to analyze the strict-weak polymer partition function via the geometric RSK row insertion (see §8.2.1 below), and to establish the Tracy-Widom asymptotics for the strict-weak polymer. See also [57] for the close relation between the log-gamma and strict weak polymers, where it is explained that one is the complement of the other. To the best of our knowledge, the full statement of Corollary 8.9 has not previously appeared in the literature.

Let us provide a brief outline of our proofs of Theorems 8.7 and 8.8 which are presented in the rest of this section. First, in §8.2 we describe the constructions of the *geometric RSK dynamics*, which will serve as  $\epsilon \rightarrow 0$  limits of elementary steps used in dynamics  $\mathcal{Q}_{\text{row}}^q[\alpha]$  and  $\mathcal{Q}_{\text{col}}^q[\alpha]$ . Then in §8.3 we prove a number of lemmas concerning  $\epsilon \rightarrow 0$  behavior of the  $q$ -distributions from §6.1. Finally, in §8.4 we use these ingredients to establish the desired statements.

**8.2. Geometric RSKs.** As we already know, the dynamics on  $q$ -Whittaker processes constructed in §6 degenerate for  $q = 0$  into the dynamics  $\mathcal{Q}_{\text{row}}^{q=0}[\alpha]$  and  $\mathcal{Q}_{\text{col}}^{q=0}[\alpha]$  based on the classical RSK row or column insertion, respectively. In this subsection we describe the corresponding *geometric Robinson–Schensted–Knuth insertions*, which will serve as building blocks for understanding  $q \nearrow 1$  limits of the dynamics on  $q$ -Whittaker processes.

The  $q = 0$  and  $q \nearrow 1$  pictures (i.e., the classical and the geometric RSK insertion tableau maps) are related via a certain procedure called *detropicalization*. Namely, the geometric RSK *row insertion* introduced in [42] is obtained by detropicalizing the classical RSK row insertion by replacing the  $(\max, +)$  operations in its definition by  $(+, \times)$ . About the geometric RSK row insertion see also, e.g., [52], [21], [59], and [18].

By analogy with the geometric RSK row insertion, one can define the *geometric RSK column insertion*, by detropicalizing the classical RSK column insertion, this time replacing the  $(\min, +)$  operations by  $(+, \times)$ .

**Remark 8.10** (Names and notation). The geometric RSK correspondences are also sometimes called *tropical RSK correspondences* [42], [52], [21], despite the fact that they come from the process of detropicalization. We adopt a convention of calling them the geometric RSK correspondences (following, e.g., [59], [18], [56]). The latter name arises in connection with geometric crystals (see [18] for more background).



Note that the word “geometric” in the name of the geometric RSK correspondences should be distinguished from the same word in the names of the geometric  $q$ -PushTASEP and the geometric  $q$ -TASEP (described in §6.3 and §6.6, respectively). The former refers to detropicalization of the classical RSK correspondences, while the latter is attached to the  $q$ -geometric jump distribution.

Below in this section, by  $\lambda, \nu, \dots$  we will denote vectors (words) with continuous components, and not signatures as before. To indicate the difference, we will use superscripts to denote their components.

8.2.1. *Geometric RSK row insertion.* Consider a triangular array  $z_k^j$  ( $1 \leq k \leq j \leq n$ ) of nonnegative real numbers, such that a word  $z_k = (z_k^k, \dots, z_k^n)$  either has all positive entries or is equal to  $(1, 0, \dots, 0)$  (in which case we call it an *empty word*).

First, define the *geometric row insertion* of a nonempty word  $a = (a^k, \dots, a^n)$  into a nonempty word  $\lambda = (\lambda^k, \dots, \lambda^n)$  as an operation that takes the pair  $\{\lambda, a\}$  as input, and produces a pair of words  $\{\nu = (\nu^k, \dots, \nu^n), b = (b^{k+1}, \dots, b^n)\}$  as output via the following rule:

$$\begin{array}{ccc}
 & a & \\
 & | & \\
 \lambda & \text{---} & \nu \\
 & | & \\
 & b & 
 \end{array}
 \qquad
 \begin{aligned}
 \nu^j &= \sum_{i=k}^j \lambda^i a^i \dots a^j \\
 b^j &= a^j \frac{\lambda^j \nu^{j-1}}{\lambda^{j-1} \nu^j}
 \end{aligned}$$

If  $\lambda$  is an empty word, then by definition  $b$  is not produced, while

$$\nu := (a^k, a^k a^{k+1}, \dots, a^k a^{k+1} \dots a^n)$$

is produced according to the same rule. The word  $b$  is also not produced for  $k = n$ . Observe that always  $\nu^j = (\lambda^j + \nu^{j-1})a^j$  for  $k < j \leq n$  and  $\nu^k = \lambda^k a^k$ .

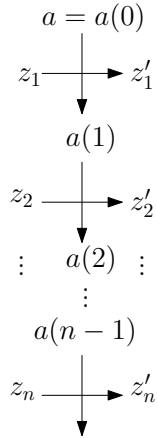


FIGURE 32. Geometric RSK row insertion.

**Definition 8.11.** The *geometric RSK row insertion* of a word  $a = (a^1, \dots, a^n)$  into an array  $z_k^j$  is defined by consecutively modifying the words  $z_1, \dots, z_n$  via the insertion according to the diagram on Fig. 32. The bottom output word  $a(1), a(2), \dots$  of each insertion is then used as a top input word for the next insertion. If after some insertion no bottom output word is produced, then no further insertions are performed.

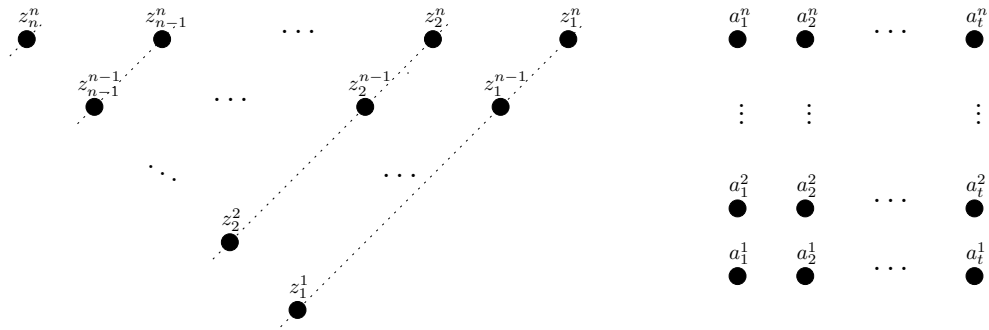


FIGURE 33. Array and strip for the geometric RSK row insertion.

The geometric RSK row insertion is related to the polymer partition functions of §8.1 in the following way:

**Proposition 8.12** ([52]). *If we start with an array  $z$  of empty words, and consecutively insert into it nonempty fixed words  $a_1, \dots, a_t$ ,  $a_i = (a_i^1, \dots, a_i^n)$ , via the geometric RSK row*

insertion, then in the obtained array we have

$$z_k^j(t) = \frac{R_k^j(t)(a_1, \dots, a_t)}{R_{k-1}^j(t)(a_1, \dots, a_t)} \quad \text{for all } t \geq k.$$

Here with a slight abuse of notation we denote by  $R_k^j(t)(a_1, \dots, a_t)$  the same weighted sum over  $k$ -tuples of nonintersecting paths as in Definition 8.2, but in a strip in which each node  $(s, i)$  has a deterministic weight  $a_s^i$  (see Fig. 33).

8.2.2. *Geometric RSK column insertion.* Consider a triangular array  $y_k^j$  ( $1 \leq k \leq j \leq n$ ) of nonnegative real numbers, such that in each word  $y_k = (y_k^k, \dots, y_k^n)$  either all entries are positive, or there is  $k \leq j \leq n$ , such that  $y_k^j = 1$ ,  $y_k^i = 0$  for  $j < i \leq n$  and  $y_k^i > 0$  for  $k \leq i \leq j$ . We again call  $(1, 0, \dots, 0)$  an empty word.

To define the *geometric RSK column insertion* first define the insertion of a word  $a = (a^k, \dots, a^n)$  with positive entries into a word  $\lambda = (\lambda^k, \dots, \lambda^n)$  as an operation that takes the pair  $\{\lambda, a\}$  as input, and produces a pair of words  $\{\nu = (\nu^k, \dots, \nu^n), b = (b^{k+1}, \dots, b^n)\}$  as output via the following rule:

$$\begin{array}{ccc} & a & \\ & \downarrow & \\ \lambda & \longrightarrow & \nu \\ & \downarrow & \\ & b & \end{array} \quad \begin{array}{l} \nu^k = a^k \lambda^k \\ \nu^j = \begin{cases} \lambda^j a^j \lambda^{j-1} \nu^{j-1} & \text{for } k < j \leq n \\ a^j \frac{\lambda^j \nu^{j-1}}{\lambda^{j-1} \nu^j}, & \text{if } \lambda^j > 0, \end{cases} \\ b^j = \begin{cases} a^j \nu^{j-1}, & \text{if } \lambda^j = 0 \text{ and } \lambda^{j-1} > 0, \\ a^j, & \text{if } \lambda^{j-1} = 0. \end{cases} \end{array}$$

**Definition 8.13.** The *geometric RSK column insertion* of a word into an array is defined similarly to the row insertion (Definition 8.11), by consecutively performing the column insertion operations defined above, in order as on Fig. 32.

Note that  $y_k^j$  in this definition corresponds to  $\lambda_{j-k+1}^{(j)}$  in classical RSK column insertion. We will need the following fact which is analogous to Proposition 8.12

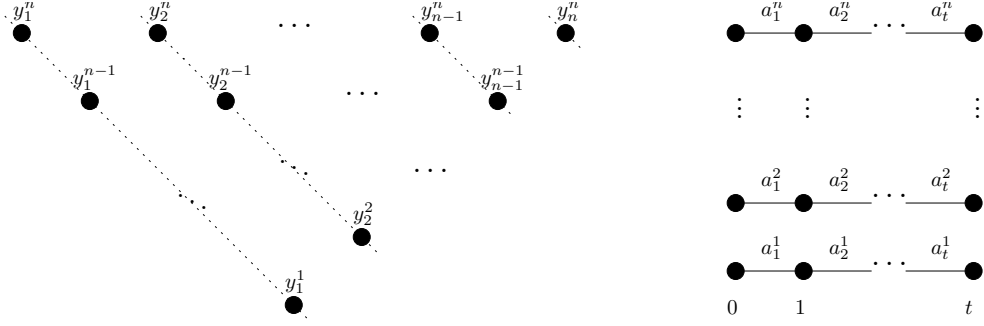


FIGURE 34. Array and strip for the geometric RSK column insertion.

**Proposition 8.14.** *If we start with an array  $y$  of empty words, and consecutively insert into it words  $a_1, \dots, a_t$  with positive entries via the geometric RSK column insertion, then in the obtained array we have*

$$y_k^j(t) = \frac{L_k^j(t)(a_1, \dots, a_t)}{L_{k-1}^j(t)(a_1, \dots, a_t)} \quad \text{for all } t \geq j - k + 1.$$

Here again we denote by  $L_k^j(t)(a_1, \dots, a_t)$  the same weighted sum over  $k$ -tuples of nonintersecting paths as in Definition 8.3, but in a strip in which each edge  $(s-1, i) \rightarrow (s, i)$  has a deterministic weight  $a_s^i$  (see Fig. 34).

*Proof.* Our proof is similar to that of Proposition 8.12 (the latter is given in [52]).

For  $a = (a^1, \dots, a^n)$ , denote by  $H(a)$  the  $n \times n$  matrix such that  $H(a)_{i,i} := a^i$ ,  $H(a)_{i,i+1} = 1$ , and other entries are 0. For  $a = (a^k, \dots, a^n)$ , denote by  $H_k(a)$  the  $n \times n$  matrix of the form  $\begin{pmatrix} Id_{k-1} & 0 \\ 0 & H(a) \end{pmatrix}$ . For  $\lambda = (\lambda^k, \dots, \lambda^n)$  such that  $\lambda^i > 0$  for  $k \leq i \leq j$  and  $\lambda^i = 0$  for  $j < i \leq n$ , denote by  $G(\lambda)$  the  $n \times n$  matrix of the form

$$\begin{pmatrix} Id_{k-1} & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & Id_{n-j} \end{pmatrix},$$

where  $G$  is the upper-triangular  $(j - k + 1) \times (j - k + 1)$  matrix with

$$G_{p,r} = \frac{\lambda^{r+k-1}}{\lambda^{p+k-2}} \quad \text{for } 1 \leq p \leq r \leq j - k + 1.$$

Assume  $\lambda^{k-1} = 1$ .

The key to the proof is the commutation relation

$$G(\lambda)H_k(a) = H_{k+1}(b)G(\nu), \quad (8.3)$$

whenever a pair of words  $\nu = (\nu^k, \dots, \nu^n), b = (b^{k+1}, \dots, b^n)$  is obtained by inserting  $a = (a^k, \dots, a^n)$  into  $\lambda = (\lambda^k, \dots, \lambda^n)$ .

To check (8.3), denote its left-hand side by  $\mathbf{L}$  and right-hand side by  $\mathbf{R}$ . Clearly,  $\mathbf{L}_{i,i} = \mathbf{R}_{i,i} = 1$  for  $1 \leq i \leq k - 1$  and  $\mathbf{L}_{i,i} = \mathbf{R}_{i,i} = a^i$  for  $j + 2 \leq i \leq n$ , and  $\mathbf{L}_{j+1,j+2} = \mathbf{R}_{j+1,j+2} = 1$ . Otherwise  $\mathbf{L}_{i,m} = \mathbf{R}_{i,m} = 0$  unless  $k \leq i \leq j + 1$  and  $k \leq m \leq j + 1$ . Let us thus assume that the two latter inequalities hold. On the diagonal, for  $k < i < j + 1$ , we have

$$\mathbf{L}_{i,i} = a^i \frac{\lambda^i}{\lambda^{i-1}} = b^i \frac{\nu^i}{\nu^{i-1}} = \mathbf{R}_{i,i}, \quad \mathbf{L}_{k,k} = a^k \lambda^k = \nu^k = \mathbf{R}_{k,k},$$

and

$$\mathbf{L}_{j+1,j+1} = a^{j+1} = \frac{b^{j+1}}{\nu^j} = \mathbf{R}_{j+1,j+1}.$$

Above the diagonal, for  $k < i < m \leq j + 1$ , we have

$$\mathbf{L}_{i,m} = \frac{\lambda^{m-1}}{\lambda^{i-1}} + \frac{\lambda^m}{\lambda^{i-1}} a^m = \frac{\nu^m}{\lambda^{i-1}} = b^i \frac{\nu^m}{\nu^{i-1}} + \frac{\nu^m}{\nu^i} = \mathbf{R}_{i,m},$$

since  $\frac{b^i}{\nu^{i-1}} + \frac{1}{\nu^i} = \frac{1}{\nu^i} (a^i \frac{\lambda^i}{\lambda^{i-1}} + 1) = \frac{1}{\lambda^{i-1}}$ , and finally

$$\mathbf{L}_{k,m} = \lambda^{m-1} + \lambda^m a^m = \nu^m = \mathbf{R}_{k,m}.$$

This completes the proof of the commutation relation (8.3).

By applying the commutation relation multiple times according to the geometric column RSK insertion (Definition 8.13), we get

$$G(y_n(t)) \cdots G(y_1(t)) = H(a_1) \cdots H(a_t). \quad (8.4)$$

Observe that the  $(i, j)$ -entry of the right-hand side above is equal to the sum of weights of all directed strict-weak (as on Fig. 31, right) paths from  $(0, i)$  to  $(t, j)$ , where the weight of a path is given by the product of weights of horizontal edges, as before. Indeed, this entry is equal to

$$\sum_{1 \leq i_1, \dots, i_{t+1} \leq n: i_1 = i, i_{t+1} = j} \prod_{\ell=1}^t H(a_\ell)_{i_\ell, i_{\ell+1}} = \sum_{1 \leq i_1, \dots, i_{t+1} \leq n: i_1 = i, i_{t+1} = j} \prod_{\ell=1}^t (\mathbf{1}_{i_\ell = i_{\ell+1} - 1} + a_\ell \mathbf{1}_{i_\ell = i_{\ell+1}}).$$

By the Lindström-Gessel-Viennot principle [49], [38], the determinant of the minor of the right-hand side at the intersection of the first  $k$  rows, and columns from  $(j - k + 1)$ -st to  $j$ -th, is  $L_k^j(t)(a_1, \dots, a_t)$ .

Next, observe that for  $1 \leq s \leq k$ ,  $j - k + 1 \leq p \leq j$ , the  $(s, p)$ -entry of the left-hand side of (8.4) is equal to the sum of weights of directed up/right (as on Fig. 31, left) lattice paths from  $(k + 1 - s, s)$  to  $(\min\{k + t + 1 - p, k\}, p)$  in the array as on Fig. 35 (the left picture if  $t \geq j$ , and the right one if  $j - k + 1 \leq t < j$ ). The weight of each path is defined to be the product of weights of all nodes along the path. By Lindström-Gessel-Viennot principle, the determinant of the minor of the left-hand side of (8.4) at the intersection of the first  $k$  rows, and columns from  $(j - k + 1)$ -st to  $j$ -th, is equal to the sum of weights of all  $k$ -tuples of non-intersecting paths from  $(k, 1), \dots, (2, k - 1), (1, k)$  to  $(\min\{2k + t - j, k\}, j - k + 1), \dots, (\min\{k + t + 2 - j, k\}, j - 1), (\min\{k + t + 1 - j, k\}, j)$ . There is only one such tuple, which covers all points on Figure (35) and has weight  $\prod_{i=1}^k y_i^{\min\{i+t, j\}}$ .

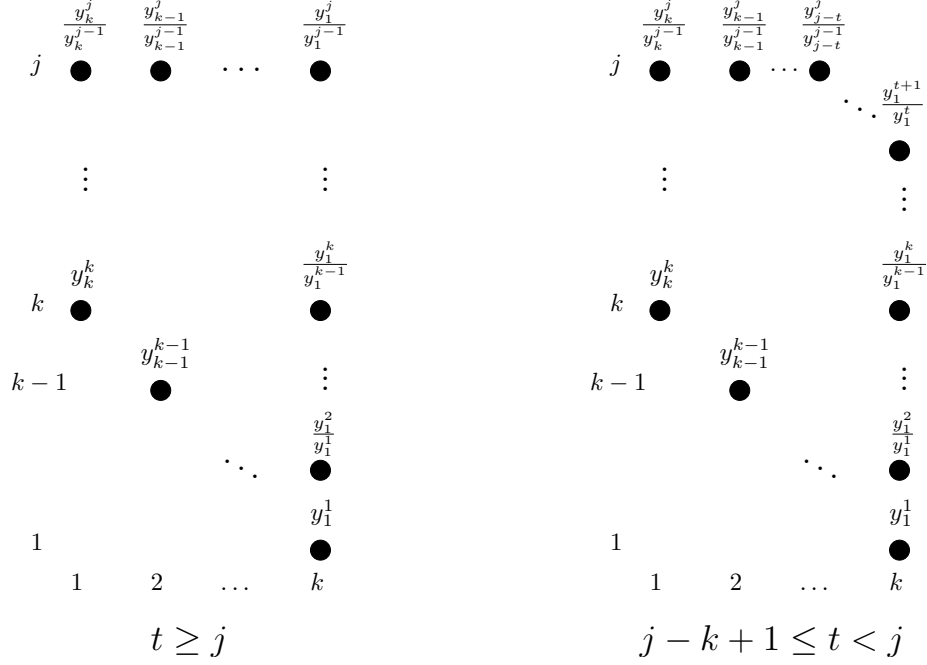


FIGURE 35. Arrays used in the proof of Proposition 8.14.

Therefore,

$$L_k^j(t)(a_1, \dots, a_t) = \prod_{i=1}^k y_i^{\min\{i+t, j\}}, \quad t \geq j - k + 1,$$

which establishes the desired statement.  $\square$

**8.3. Asymptotics of  $q$ -deformed Beta-binomial distributions.** We will need several lemmas about the limiting properties of the distributions  $\varphi_{q, \xi, \eta}(s | y)$  (6.1).

**Lemma 8.15.** *Let  $X^\epsilon$  be a  $\mathbb{Z}_{\geq 0}$ -valued random variable with*

$$\text{Prob}(X^\epsilon = j) = (\alpha; q)_\infty \frac{\alpha^j}{(q; q)_j} \quad \text{for } \alpha = e^{-\theta\epsilon} \text{ and } q = e^{-\epsilon}.$$

*Then as  $\epsilon \rightarrow 0$ ,  $\epsilon \exp\{\epsilon X^\epsilon\}$  converges in distribution to  $\text{Gamma}^{-1}(\theta)$ .*

**Lemma 8.16.** *Let  $n(\epsilon)$  be a function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ , such that*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \exp\{-\epsilon n(\epsilon)\} = \varphi.$$

*Let  $X^\epsilon$  be a  $\mathbb{Z}_{\geq 0}$ -valued random variable with*

$$\text{Prob}(X^\epsilon = j) = \varphi_{q,\alpha,0}(j \mid n(\epsilon)) \quad \text{for } \alpha = e^{-\theta\epsilon} \text{ and } q = e^{-\epsilon}.$$

*Then as  $\epsilon \rightarrow 0$ ,  $\epsilon^{-1} \exp\{-\epsilon X^\epsilon\}$  converges in distribution to  $\varphi + \text{Gamma}(\theta)$ .*

These two lemmas were both proven in [24] (Lemma 2.1 and a part of proof of Theorem 1.4, respectively). In the next three lemmas, parameters of distributions which are not explicitly fixed are assumed to depend on  $\epsilon$ , and sometimes might also be random themselves.

**Lemma 8.17.** *Fix  $C$  and  $0 < \sigma < 1$ . Let  $Y^\epsilon$  be a  $\mathbb{Z}_{\geq 0}$ -valued random variable distributed according to  $\varphi_{q^{-1},\xi,\eta}(\cdot \mid n)$  with  $q = e^{-\epsilon}$ ,  $\xi q^{-n} \rightarrow \sigma$ ,  $n \geq \epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1}C$ , and  $\log \eta + 2n\epsilon \leq \log \sigma$ . Then as  $\epsilon \rightarrow 0$ ,  $\epsilon Y^\epsilon \rightarrow \log(1 + \sigma)$ .*

*Proof.* The fact that  $\varphi_{q^{-1},\xi,\eta}$  with such parameters is indeed a probability distribution for  $\epsilon$  small enough follows from inequalities in the statement of the lemma. Let  $A(\epsilon) := (e^{-\epsilon}; e^{-\epsilon})_\infty$ . By [8, Corollary 4.1.10],

$$(e^{-\epsilon}; e^{-\epsilon})_{\lfloor \epsilon^{-1} \log \epsilon^{-1} - C\epsilon^{-1} \rfloor} \leq A(\epsilon)C'$$

for all  $\epsilon$  small enough and some constant  $C'$  that depends only on  $C$ . As  $\epsilon \rightarrow 0$ ,

$$\epsilon \log(e^{-\epsilon}; e^{-\epsilon})_{\lceil r/\epsilon \rceil} \rightarrow \int_0^r \log(1 - e^{-x}) dx,$$

since the left-hand side is a Riemann sum for the right-hand side integral, hence we have

$$\epsilon \log \frac{(e^{-\epsilon}; e^{-\epsilon})_\infty}{(e^{-r}; e^{-\epsilon})_\infty} \rightarrow \int_0^r \log(1 - e^{-x}) dx.$$



(Note that although this integral blows up at 0, it is still finite and convergence holds.)

Fix  $\delta > 0$ . For  $\epsilon$  small enough,

$$\begin{aligned} \text{Prob}(Y^\epsilon = k) &= (\xi q^{-n+k})_k \frac{(\eta/\xi; q^{-1})_k (\xi; q^{-1})_{n-k}}{(\eta; q^{-1})_n} \binom{n}{k}_q \leq \frac{(\xi q^{-n})^k e^{-\epsilon k^2} C'}{(e^{-\epsilon}; e^{-\epsilon})_k} \\ &\leq \frac{(2\sigma)^k e^{-\epsilon k^2} C'}{(e^{-\epsilon}; e^{-\epsilon})_k} \leq C' \exp\left(k \log 2\sigma - \epsilon k^2 - \frac{1}{\epsilon} \int_0^{k\epsilon} \log(1 - e^{-x}) dx\right) \\ &\leq e^{-T^2/2\epsilon} \quad \text{for } T \text{ large enough and } k \geq T/\epsilon. \end{aligned}$$

Hence

$$\text{Prob}(Y^\epsilon \geq T/\epsilon) \leq \sum_{k=\lceil T/\epsilon \rceil}^{\infty} e^{-\epsilon k^2/2} \leq e^{-T^2/2\epsilon} \sum_{i=0}^{\infty} e^{-Ti},$$

which can be made less than  $\delta/2$  for all  $\epsilon$  small enough by choosing sufficiently large  $T$ .

Observe that for  $k \leq T/\epsilon$  and  $\epsilon$  small enough there is some constant  $C_0$  that depends only on  $C$  and  $T$ , such that

$$C_0^{-1} \leq \frac{(\eta/\xi; q^{-1})_k}{(\xi q; q)_\infty (\eta; q^{-1})_n} \frac{(q; q)_n}{(q; q)_{n-k}} \leq C_0.$$

Let

$$f(\psi) := -\psi^2 + (\log \sigma)\psi - \int_0^\psi \log(1 - e^{-x}) dx - \int_0^{\psi - \log \sigma} \log(1 - e^{-x}) dx$$

for  $\psi \geq 0$ . Then

$$f'(\psi) = -2\psi + \log \sigma - \log(1 - e^{-\psi}) - \log(1 - e^{-\psi + \log \sigma}),$$

which is strictly decreasing, and  $f'(\log(1 + \sigma)) = 0$ . Hence  $f$  attains a unique maximum at  $\log(1 + \sigma)$ , so one can choose  $M_1 > M_2 > M_3 > M_4$  such that

$$f(\psi) > M_1 \quad \text{for } \psi \in (\log(1 + \sigma) - \delta/2, \log(1 + \sigma) + \delta/2)$$

and

$$f(\psi) < M_4 \quad \text{for } \psi \notin (\log(1 + \sigma) - \delta, \log(1 + \sigma) + \delta),$$

and for  $\epsilon$  small enough

$$\text{Prob}(\epsilon Y^\epsilon \in (\log(1 + \sigma) - \delta/2, \log(1 + \sigma) + \delta/2)) \geq C_0^{-1} \left( \frac{\delta}{\epsilon} - 1 \right) A(\epsilon) e^{M_2/\epsilon}$$

and

$$\text{Prob}(\epsilon Y^\epsilon \notin (\log(1 + \sigma) - \delta, \log(1 + \sigma) + \delta) \cup [T, \infty)) \leq C_0 \left( \frac{T - 2\delta}{\epsilon} + 2 \right) A(\epsilon) e^{M_3/\epsilon}.$$

Therefore, for  $\epsilon$  small enough

$$\text{Prob}(\epsilon Y^\epsilon \in (\log(1 + \sigma) - \delta, \log(1 + \sigma) + \delta)) \geq 1 - \delta,$$

and this completes the proof. □

**Lemma 8.18.** Fix  $C$  and  $0 \leq \sigma < 1$ . Let  $Y^\epsilon$  be a  $\mathbb{Z}_{\geq 0}$ -valued random variable with

$$\text{Prob}(Y^\epsilon = j) = \varphi_{q, \alpha, 0}(j \mid n)$$

for  $\alpha \rightarrow \sigma$  as  $\epsilon \rightarrow 0$ ,  $q = e^{-\epsilon}$  and  $n \geq \epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} C$ . Then as  $\epsilon \rightarrow 0$ ,  $\epsilon Y^\epsilon \rightarrow -\log(1 - \sigma)$ .

*Proof.* Suppose  $\sigma > 0$  and fix  $\delta > 0$ . We can write

$$\begin{aligned} \text{Prob}(Y^\epsilon = k) &= \frac{\alpha^k}{(e^{-\epsilon}; e^{-\epsilon})_k} \frac{(\alpha; e^{-\epsilon})_{n-k} (e^{-\epsilon}, e^{-\epsilon})_n}{(e^{-\epsilon}; e^{-\epsilon})_{n-k}} \leq \frac{\alpha^k}{(e^{-\epsilon}; e^{-\epsilon})_k} \\ &\leq \exp \left( \frac{1}{\epsilon} \left( \left( \frac{1}{2} \log \sigma \right) T - \int_0^T \log(1 - e^{-x}) dx \right) \right) \leq e^{(\log \sigma) T / 4\epsilon} \end{aligned}$$

for  $T$  large enough and  $\epsilon$  small enough, such that  $k \geq T/\epsilon$ . Hence

$$Prob(Y^\epsilon \geq T/\epsilon) \leq e^{(\log \sigma)T/4\epsilon} \sum_{i=0}^{\infty} e^{\frac{i \log \sigma}{4}}$$

which is less than  $\delta/2$  for  $T$  large enough.

For  $\epsilon$  small enough,  $k \leq T/\epsilon$ , and some constant  $C_0$  that depends only on  $C$ ,  $T$ , and  $\sigma$ , one can write

$$C_0^{-1}(\alpha; e^{-\epsilon})_\infty \leq \frac{(\alpha; e^{-\epsilon})_{n-k}(e^{-\epsilon}; e^{-\epsilon})_n}{(e^{-\epsilon}; e^{-\epsilon})_{n-k}} \leq C_0(\alpha; e^{-\epsilon})_\infty.$$

Let

$$f(\psi) := (\log \sigma)\psi - \int_0^\psi \log(1 - e^{-x})dx$$

for  $\psi \geq 0$ . Then

$$f'(\psi) = \log \sigma - \log(1 - e^{-\psi}),$$

which is strictly decreasing, and  $f'(-\log(1 - \sigma)) = 0$ . Hence  $f$  attains a unique maximum at  $-\log(1 - \sigma)$ , so one can choose  $M_1 > M_2 > M_3 > M_4$  such that

$$f(\psi) > M_1 \quad \text{for } \psi \in (-\log(1 - \sigma) - \delta/2, -\log(1 - \sigma) + \delta/2)$$

and

$$f(\psi) < M_4 \quad \text{for } \psi \notin (-\log(1 - \sigma) - \delta, -\log(1 - \sigma) + \delta),$$

and for  $\epsilon$  small enough

$$Prob(\epsilon Y^\epsilon \in (-\log(1 - \sigma) - \delta/2, -\log(1 - \sigma) + \delta/2)) \geq C_0^{-1} \left( \frac{\delta}{\epsilon} - 1 \right) (\alpha; e^{-\epsilon})_\infty e^{M_2/\epsilon}$$

and

$$Prob(\epsilon Y^\epsilon \notin (-\log(1 - \sigma) - \delta, -\log(1 - \sigma) + \delta) \cup [T, \infty)) \leq C_0 \left( \frac{T - 2\delta}{\epsilon} + 2 \right) (\alpha; e^{-\epsilon})_\infty e^{M_3/\epsilon}.$$

Therefore, for  $\epsilon$  small enough we can write

$$\text{Prob}(\epsilon Y^\epsilon \in (-\log(1 - \sigma) - \delta, -\log(1 - \sigma) + \delta)) \geq 1 - \delta,$$

and this completes the proof for the case  $\sigma > 0$ .

If  $\sigma = 0$ , fix arbitrary  $u > 0$ . For all large enough  $U$ , such that  $Uu > \frac{1}{2}Uu - \int_0^\infty \log(1 - e^{-x})dx$ ,  $\epsilon$  small enough, and  $k \geq \frac{u}{\epsilon}$ ,

$$\text{Prob}(Y^\epsilon = k) \leq \frac{\alpha^k}{(e^{-\epsilon}; e^{-\epsilon})_k} \leq \exp\left(\frac{1}{\epsilon}(-Uk\epsilon - \int_0^\infty \log(1 - e^{-x})dx)\right) \leq e^{-\frac{Uu}{2\epsilon}}.$$

So,

$$\text{Prob}(\epsilon Y^\epsilon \geq u) \leq e^{-\frac{Uu}{2\epsilon}} \sum_{i=0}^{\infty} e^{-Ui/2},$$

which can be made less than any given  $\delta > 0$  for  $U$  large enough. Thus, we have convergence  $\epsilon Y^\epsilon \rightarrow 0$  in distribution.  $\square$

**Lemma 8.19.** Fix  $\alpha, \sigma \in (0, 1)$  such that  $\alpha(1 + \sigma) < 1$ . Let  $Y^\epsilon$  be a  $\mathbb{Z}_{\geq 0}$ -valued random variable with  $\text{Prob}(Y^\epsilon = j) = \varphi_{q^{-1}, \alpha, 0}(j | n)$  for  $n\epsilon \rightarrow \log(1 + \sigma)$  as  $\epsilon \rightarrow 0$ ,  $q = e^{-\epsilon}$ . Then as  $\epsilon \rightarrow 0$ ,  $\epsilon Y^\epsilon \rightarrow \log(1 + \alpha\sigma)$ .

*Proof.*  $\alpha(1 + \sigma) < 1$  ensures that this is indeed a probability distribution for  $\epsilon$  small enough.

This distribution looks as

$$\text{Prob}(Y^\epsilon = k) = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} (\alpha q^{k-n})^k (\alpha; q^{-1})_{n-k}.$$

Let

$$\begin{aligned} f(\psi) &:= - \int_0^\psi \log(1 - e^{-x})dx - \int_0^{\log(1+\sigma)-\psi} \log(1 - e^{-x})dx + \psi \log \alpha - \psi^2 + \psi \log(1 + \sigma) \\ &+ \int_{-\log \alpha - \log(1+\sigma) + \psi}^{-\log \alpha} \log(1 - e^{-x})dx \quad \text{for } \log(1 + \sigma) \geq \psi \geq 0. \end{aligned}$$

Then

$$f'(\psi) = -\log(1 - e^{-\psi}) + \log\left(1 - \frac{e^\psi}{1 + \sigma}\right) + \log \alpha - 2\psi + \log(1 + \sigma) - \log(1 - \alpha(1 + \sigma)e^{-\psi}),$$

which is strictly decreasing, and  $f'(\log(1 + \alpha\sigma)) = 0$ . Hence  $f$  attains a unique maximum at  $\log(1 + \alpha\sigma)$ , so one can choose  $M_1 > M_2$  such that

$$f(\psi) > M_1 \quad \text{for } \psi \in (\log(1 + \alpha\sigma) - \delta/2, \log(1 + \alpha\sigma) + \delta/2)$$

and

$$f(\psi) < M_2 \quad \text{for } \psi \notin (\log(1 + \alpha\sigma) - \delta, \log(1 + \alpha\sigma) + \delta).$$

Hence for  $\epsilon$  small enough,

$$\text{Prob}(\epsilon Y^\epsilon \in (\log(1 + \alpha\sigma) - \delta/2, \log(1 + \alpha\sigma) + \delta/2)) \geq \left(\frac{\delta}{\epsilon} - 1\right) (e^{-\epsilon}; e^{-\epsilon})_n e^{M_1/\epsilon},$$

and

$$\text{Prob}(\epsilon Y^\epsilon \notin (\log(1 + \alpha\sigma) - \delta, \log(1 + \alpha\sigma) + \delta)) \leq \left(\frac{\log(1 + \sigma) - 2\delta}{\epsilon} + 2\right) (e^{-\epsilon}; e^{-\epsilon})_n e^{M_2/\epsilon}.$$

Thus, for  $\epsilon$  small enough, we have

$$\text{Prob}(\epsilon Y^\epsilon \in (\log(1 + \alpha\sigma) - \delta, \log(1 + \alpha\sigma) + \delta)) \geq 1 - \delta,$$

and this completes the proof. □

**8.4. Proofs of Theorems 8.7 and 8.8.** In our proofs, we denote by

$$A(\epsilon), B(\epsilon), C(\epsilon), D(\epsilon), E(\epsilon), F(\epsilon)$$

possibly random positive valued functions tending to deterministic constants  $A, B, C, D, E, F$ , respectively, as  $\epsilon \rightarrow 0$ . This notation will be repeatedly used in the arguments below for defining conditional probabilities.

8.4.1. *Proof of Theorem 8.7.* We must show that for fixed  $(t, k, j)$ , such that  $k \leq \min\{t, j\}$  and  $j \leq n$ , the random variable  $\hat{R}_k^j(t, \epsilon)$  conditioned on  $\hat{R}_K^J(T, \epsilon) \rightarrow X_K^J(T)$  for all  $(T, K, J) < (t, k, j)$  in the lexicographic order,<sup>28</sup> converges as  $\epsilon \rightarrow 0$  to  $\hat{R}_k^j(t)$  conditioned on  $\hat{R}_K^J(T) = X_K^J(T)$  for all  $(T, K, J) < (t, k, j)$  in the lexicographic order. Here  $X_K^J(T)$  are some fixed constants. In the rest of the proof we will always assume this conditioning.

**Right edge ( $k = 1$ ).** The Markovian projection of the  $\mathcal{Q}_{\text{row}}^q[\alpha]$  dynamics on the right edge is the geometric  $q$ -PushTASEP (§6.3), hence the proof of the theorem for the right edge is the same as showing that suitably rescaled positions of particles in the geometric  $q$ -PushTASEP converge to the partition functions of the log-Gamma polymer (Definition 8.2).

**a)** If  $t = 1$ , then  $r_{j,1}(1, \epsilon) = r_{j-1,1}(1, \epsilon) +$  an independent random movement  $d$  distributed according to  $\varphi_{q, a_j \alpha_1, 0}(d \mid \infty)$  (assume  $r_{0,1}(1, \epsilon) = 0$  and  $X_1^0(1) = 1$ ). By Lemma 8.15,

$$\begin{aligned} \log(\hat{R}_1^j(1, \epsilon)) &= r_{j,1}(1, \epsilon)\epsilon - j \log \epsilon^{-1} \\ &= r_{j-1,1}(1, \epsilon)\epsilon - (j-1) \log \epsilon^{-1} + d\epsilon - \log \epsilon^{-1} \\ &\rightarrow \log(X_1^{j-1}(1)) + \log(\Gamma) \end{aligned}$$

for an independent random variable  $\Gamma = a_1^j$  distributed according to  $\text{Gamma}^{-1}(\theta_j + \hat{\theta}_1)$ , which is consistent with  $\hat{R}_1^j(1) = X_1^{j-1}(1)a_1^j$ .

---

<sup>28</sup>That is,  $T < t$ , or  $T = t$  and  $K < k$ , or  $T = t, K = k$  and  $J < j$ .

**b)** If  $j = 1$  and  $t \geq 2$ , then  $r_{1,1}(t, \epsilon) = r_{1,1}(t-1, \epsilon) +$  an independent random movement  $d$  distributed according to  $\varphi_{q, a_1 \alpha_t, 0}(d | \infty)$ . By Lemma 8.15,

$$\begin{aligned} \log(\hat{R}_1^1(t, \epsilon)) &= r_{1,1}(t, \epsilon)\epsilon - t \log \epsilon^{-1} \\ &= r_{1,1}(t-1, \epsilon)\epsilon - (t-1) \log \epsilon^{-1} + d\epsilon - \log \epsilon^{-1} \\ &\rightarrow \log(X_1^1(t-1)) + \log(\Gamma) \end{aligned}$$

for an independent random variable  $\Gamma = a_t^1$  distributed according to  $\text{Gamma}^{-1}(\theta_1 + \hat{\theta}_t)$ , which is consistent with  $\hat{R}_1^1(t) = X_1^1(t-1)a_t^1$ .

**c)** Assume  $t \geq 2$  and  $j \geq 2$ . Condition on

$$\begin{aligned} r_{j,1}(t-1, \epsilon) &= (t+j-2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log A(\epsilon), \quad \text{so } X_1^j(t-1) = A; \\ r_{j-1,1}(t-1, \epsilon) &= (t+j-3)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log B(\epsilon), \quad \text{so } X_1^{j-1}(t-1) = B; \\ r_{j-1,1}(t, \epsilon) &= (t+j-2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log C(\epsilon), \quad \text{so } X_1^{j-1}(t) = C. \end{aligned}$$

The movement of the rightmost particle on the  $j$ -th level during the time step  $t-1 \rightarrow t$  which happens due to the pushing by the rightmost particle at level  $j-1$  behaves as  $\epsilon^{-1} \log(1 + \frac{C}{A})$  (by Lemma 8.17). The independent movement of the rightmost particle on the  $j$ -th level behaves as  $\epsilon^{-1} \log \Gamma + \epsilon^{-1} \log \epsilon^{-1}$  (by Lemma 8.15), for an independent random variable  $\Gamma = a_t^j$  distributed according to  $\text{Gamma}^{-1}(\theta_j + \hat{\theta}_t)$ . Therefore,

$$\log(\hat{R}_1^j(t, \epsilon)) \rightarrow \log A + \log\left(1 + \frac{C}{A}\right) + \log \Gamma = \log((A+C)\Gamma),$$

which is consistent with  $\hat{R}_1^j(t) = (X_1^j(t-1) + X_1^{j-1}(t))a_t^j$ .

**$k$ -th edge from the right for  $k \geq 2$ .**

a) Assume  $t = k$ . We have  $r_{j,k}(k, \epsilon) = r_{j-1,k}(k, \epsilon) +$  a movement due to pulling from the  $(k-1)$ -st particle from the right on the  $(j-1)$ -st level (assume  $r_{k-1,k}(k, \epsilon) = 0$  and  $X_k^{k-1}(k) = 1$ ). Condition on

$$\begin{aligned} r_{j-1,k-1}(k-1, \epsilon) &= (j-k+1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log D(\epsilon), \quad \text{so } X_{k-1}^{j-1}(k-1) = D; \\ r_{j-1,k-1}(k, \epsilon) &= (j-k+2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log E(\epsilon), \quad \text{so } X_{k-1}^{j-1}(k) = E; \\ r_{j,k-1}(k-1, \epsilon) &= (j-k+2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log F(\epsilon), \quad \text{so } X_{k-1}^j(k-1) = F. \end{aligned}$$

By Lemma 8.17, this movement times  $\epsilon$  and minus  $\log \epsilon^{-1}$  converges to  $\log(E/D) - \log(1 + E/F)$ . Therefore,

$$\begin{aligned} \log(\hat{R}_k^j(k, \epsilon)) &= r_{j,k}(k, \epsilon)\epsilon - (j-k+1) \log \epsilon^{-1} \\ &\rightarrow \log(X_k^{j-1}(k)) + \log(E/D) - \log(1 + E/F) \\ &= \log\left(\frac{X_k^{j-1}(k)EF}{(F+E)D}\right), \end{aligned}$$

which is consistent with

$$\hat{R}_k^j(k) = \frac{X_k^{j-1}(k)X_{k-1}^{j-1}(k)X_{k-1}^j(k-1)}{(X_{k-1}^j(k-1) + X_{k-1}^{j-1}(k))X_{k-1}^{j-1}(k-1)}.$$

Indeed, if we insert (via the geometric row insertion) a nonempty word  $b = (b^k, \dots, b^n)$  into the empty word  $\lambda_k = (1, 0, \dots, 0)$ , where  $b$  is itself the bottom output of the insertion of  $a = (a^{k-1}, \dots, a^n)$  into  $\lambda_{k-1} = (\lambda_{k-1}^{k-1}, \dots, \lambda_{k-1}^n) = (X_{k-1}^{k-1}(k-1), \dots, X_{k-1}^n(k-1))$ , then we get

$$\nu_k^j = \nu_k^{j-1}b^j = \nu_k^{j-1} \frac{a^j \lambda_{k-1}^j \nu_{k-1}^{j-1}}{\lambda_{k-1}^{j-1} \nu_{k-1}^j} = \nu_k^{j-1} \frac{\lambda_{k-1}^j}{\lambda_{k-1}^{j-1}} \cdot \frac{\nu_{k-1}^{j-1}}{\nu_{k-1}^{j-1} + \lambda_{k-1}^j},$$

which is the same as the expression for  $\hat{R}_k^j(k)$  above.



**b)** Assume  $t \geq k + 1$  and  $k < j \leq n$ . Condition on

$$r_{j,k}(t-1, \epsilon) = (t+j-2k)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log A(\epsilon), \quad \text{so } X_k^j(t-1) = A;$$

$$r_{j-1,k}(t-1, \epsilon) = (t+j-2k-1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log B(\epsilon), \quad \text{so } X_k^{j-1}(t-1) = B;$$

$$r_{j-1,k}(t, \epsilon) = (t+j-2k)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log C(\epsilon), \quad \text{so } X_k^{j-1}(t) = C;$$

$$r_{j-1,k-1}(t-1, \epsilon) = (t+j-2k+1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log D(\epsilon), \quad \text{so } X_{k-1}^{j-1}(t-1) = D;$$

$$r_{j-1,k-1}(t, \epsilon) = (t+j-2k+2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log E(\epsilon), \quad \text{so } X_{k-1}^{j-1}(t) = E;$$

$$r_{j,k-1}(t-1, \epsilon) = (t+j-2k+2)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log F(\epsilon), \quad \text{so } X_{k-1}^j(t-1) = F.$$

Movement of  $r_{j,k}$  during time step  $t-1 \rightarrow t$  due to pushing by the  $k$ -th particle from the right on the  $(j-1)$ -st level behaves as  $\epsilon^{-1} \log(1 + \frac{C}{A})$  (by Lemma 8.17). The movement due to the pulling (by the  $(k-1)$ -st particle from the right on the  $(j-1)$ -st level) times  $\epsilon^{-1}$  minus  $\log \epsilon^{-1}$  by the same lemma tends to  $\log(\frac{E}{D}) - \log(1 + \frac{E}{F})$ . Hence we may write

$$\log(\hat{R}_k^j(t, \epsilon)) \rightarrow \log A + \log\left(1 + \frac{C}{A}\right) + \log\left(\frac{E}{D}\right) - \log\left(1 + \frac{E}{F}\right) = \log\left(\frac{(A+C)EF}{(F+E)D}\right),$$

which is consistent with

$$\hat{R}_k^j(t) = \frac{(X_k^j(t-1) + X_k^{j-1}(t)) X_{k-1}^{j-1}(t) X_{k-1}^j(t-1)}{(X_{k-1}^j(t-1) + X_{k-1}^{j-1}(t)) X_{k-1}^{j-1}(t-1)}.$$

Indeed, if we insert (via the geometric row insertion) a nonempty word  $b = (b^k, \dots, b^n)$  into a nonempty word  $\lambda_k = (\lambda_k^k, \dots, \lambda_k^n) = (X_k^k(t-1), \dots, X_k^n(t-1))$ , where  $b$  is itself the bottom output of the insertion of  $a = (a^{k-1}, \dots, a^n)$  into  $\lambda_{k-1} = (\lambda_{k-1}^{k-1}, \dots, \lambda_{k-1}^n) = (X_{k-1}^{k-1}(t-1), \dots, X_{k-1}^n(t-1))$ , then we get

$$\nu_k^j = (\nu_k^{j-1} + \lambda_k^j) b^j = (\nu_k^{j-1} + \lambda_k^j) \frac{a^j \lambda_{k-1}^j \nu_{k-1}^{j-1}}{\lambda_{k-1}^{j-1} \nu_{k-1}^j} = (\nu_k^{j-1} + \lambda_k^j) \frac{\lambda_{k-1}^j}{\lambda_{k-1}^{j-1}} \cdot \frac{\nu_{k-1}^{j-1}}{\nu_{k-1}^{j-1} + \lambda_{k-1}^j},$$

which is the same as the expression for  $\hat{R}_k^j(t)$  above.

c) Finally, if  $j = k$  and  $t \geq k + 1$ , then the previous argument carries out with the exception that in this case the leftmost particle on the  $j$ -th level experiences only pulling of the leftmost particle on the  $(j - 1)$ -st level, hence we should take  $C = 0$  in the formulas from part b).

This completes the proof of Theorem 8.7.

8.4.2. *Proof of Theorem 8.8.* We must show that for fixed  $(t, k, j)$ , such that  $k \leq j \leq t + k - 1$  and  $j \leq n$ , the random variable  $\hat{L}_k^j(t, \epsilon)$ , conditioned on  $\hat{L}_K^J(T, \epsilon) \rightarrow Y_K^J(T)$  for all  $(T, K, J) < (t, k, j)$  in the lexicographic order, converges as  $\epsilon \rightarrow 0$  to  $\hat{L}_k^j(t)$ , conditioned on  $\hat{L}_K^J(T) = Y_K^J(T)$  for  $(T, K, J) < (t, k, j)$  in the lexicographic order (here  $Y_K^J(T)$  are some fixed constants). In the rest of the proof we will always assume this conditioning.

**Left edge** ( $k = 1$ ). The Markovian projection of the  $\mathcal{Q}_{\text{col}}^q[\alpha]$  dynamics on the left edge is the geometric  $q$ -TASEP (§6.6), hence the proof of the theorem for the left edge is the same as showing that suitably rescaled positions of particles in the geometric  $q$ -TASEP converge to the partition functions of the strict-weak polymer (Definition 8.3). This was already done in [24], but we still include this part in the proof for the reader's convenience.

**a)** If  $t = j = 1$ , then by Lemma 8.15,  $\log(\hat{L}_1^1(1, \epsilon)) = \log \epsilon^{-1} - \ell_{1,1}(1, \epsilon)\epsilon$  converges in distribution to  $\log \Gamma$  for a random variable  $\Gamma = a_1^1$  distributed according to  $\text{Gamma}(\theta_1 + \hat{\theta}_1)$ .

**b)** Assume  $t = j > 1$ . Then  $m = \ell_{j,1}(j, \epsilon)$  is distributed according to  $\varphi_{q, a_j \alpha_j, 0}(m \mid \ell_{j-1,1}(j-1, \epsilon))$ . If we condition on

$$\ell_{j-1,1}(j-1, \epsilon) = \epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log F(\epsilon), \quad \text{so } Y_1^{j-1}(j-1) = F,$$

then by Lemma 8.16,

$$\log(\hat{L}_1^j(j, \epsilon)) = \log \epsilon^{-1} - \ell_{j,1}(j, \epsilon)\epsilon \rightarrow \log(F + \Gamma)$$

for an independent random variable  $\Gamma = a_j^j$  distributed according to  $\text{Gamma}(\theta_j + \hat{\theta}_j)$ , which is consistent with  $\hat{L}_1^j(j) = a_j^j + Y_1^{j-1}(j-1)$ .

c) Let  $t > j = 1$ . By Lemma 8.15, the quantities

$$\log(\hat{L}_1^1(t, \epsilon)) - \log(\hat{L}_1^1(t-1, \epsilon)) = \log \epsilon^{-1} - (\ell_{1,1}(t, \epsilon) - \ell_{1,1}(t-1, \epsilon)) \epsilon$$

converge in distribution to  $\log \Gamma$  for a random variable  $\Gamma = a_t^1$  distributed according to  $\text{Gamma}(\theta_1 + \hat{\theta}_t)$ , which is consistent with  $\hat{L}_1^1(t) = a_t^1 Y_1^1(t-1)$ .

d) Assume  $t > j > 1$ . Condition on

$$\ell_{j,1}(t-1, \epsilon) = (t-j)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log A(\epsilon), \quad \text{so } Y_1^j(t-1) = A;$$

$$\ell_{j-1,1}(t-1, \epsilon) = (t-j+1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log F(\epsilon), \quad \text{so } Y_1^{j-1}(t-1) = F.$$

By Lemma 8.16, the movement of the leftmost particle on the  $j$ -th level during the time step  $t-1 \rightarrow t$  times  $\epsilon$  and minus  $\log \epsilon^{-1}$  converges to  $-\log(\frac{F}{A} + \Gamma)$  for an independent random variable  $\Gamma = a_t^j$  distributed according to  $\text{Gamma}(\theta_j + \hat{\theta}_t)$ . Therefore,

$$\log(\hat{L}_1^j(t, \epsilon)) \rightarrow \log A + \log(F/A + \Gamma) = \log(F + A\Gamma),$$

which is consistent with  $\hat{L}_1^j(t) = Y_1^{j-1}(t-1) + a_t^j Y_1^j(t-1)$ .

**Second edge from the left ( $k = 2$ ).**

a) If  $j = 2, t = 1$ , then we have to look at  $\ell_{2,2}(1, \epsilon) = \ell_{1,1}(1, \epsilon) +$  a jump  $m$  distributed according to  $\varphi_{q, a_2 \alpha_1, 0}(m | \infty)$ . By Lemma 8.15,

$$\log(\hat{L}_2^2(1)) - \log(\hat{L}_1^1(1)) = \log \epsilon^{-1} - m\epsilon \rightarrow \log \Gamma,$$

where  $\Gamma = a_1^2$  is an independent random variable distributed according to  $\text{Gamma}(\theta_2 + \hat{\theta}_1)$ , which is consistent with  $\hat{L}_2^2(1) = a_1^2 Y_1^1(1)$ .

**b)** Assume  $j > 2$ ,  $t = j - 1$ . We have  $\ell_{j,2}(j - 1, \epsilon) = m_1 + m_2$ , where  $m_1$  is an independent move distributed according to  $\varphi_{q, a_j \alpha_{j-1}, 0}(m_1 \mid \ell_{j-1,2}(j - 2, \epsilon))$ , and  $m_2$  is the push from the leftmost particle on the  $(j - 1)$ -st level distributed according to

$$\varphi_{q^{-1}, q^{\ell_{j-1,1}(j-1, \epsilon)}, q^{\ell_{j-1,2}(j-2, \epsilon)}}(\ell_{j-1,2}(j - 2, \epsilon) - m_1 - m_2 \mid \ell_{j-1,2}(j - 2, \epsilon) - m_1).$$

Condition on

$$\ell_{j-1,1}(j - 1, \epsilon) = \epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E(\epsilon), \quad \text{so } Y_1^{j-1}(j - 1) = E;$$

$$\ell_{j-1,2}(j - 2, \epsilon) = 2\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log F(\epsilon), \quad \text{so } Y_2^{j-1}(j - 2) = F.$$

By Lemma 8.16,  $\log \epsilon^{-1} - m_1 \epsilon \rightarrow \log \Gamma$ , where  $\Gamma = a_{j-1}^j$  is an independent random variable distributed according to  $\text{Gamma}(\theta_j + \hat{\theta}_{j-1})$ . By Lemma 8.17,

$$\log(\hat{L}_2^j(j - 1, \epsilon)) = 2 \log \epsilon^{-1} - \ell_{j,2}(j - 1, \epsilon) \epsilon \rightarrow \log F + \log\left(1 + \frac{E\Gamma}{F}\right) = \log(F + E\Gamma),$$

which is consistent with  $Y_2^j(j - 1) = Y_2^{j-1}(j - 2) + a_{j-1}^j Y_1^{j-1}(j - 1)$ .

**c)** Let  $j = 2$ ,  $t > 1$ . Then

$$\ell_{2,2}(t, \epsilon) = \ell_{2,2}(t - 1, \epsilon) + \ell_{1,1}(t, \epsilon) - \ell_{1,1}(t - 1, \epsilon) + m,$$

where the jump  $m$  is distributed according to

$$\varphi_{q, a_2 \alpha_t q^{\ell_{1,1}(t-1, \epsilon) - \ell_{2,1}(t, \epsilon)}, 0}(m \mid \infty).$$

Condition on

$$\ell_{2,2}(t - 1, \epsilon) = t\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log A(\epsilon), \quad \text{so } Y_2^2(t - 1) = A;$$

$$\ell_{1,1}(t - 1, \epsilon) = (t - 1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log B(\epsilon), \quad \text{so } Y_1^1(t - 1) = B;$$

$$\ell_{1,1}(t, \epsilon) = t\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log C(\epsilon), \quad \text{so } Y_1^1(t) = C;$$

$$\ell_{2,1}(t, \epsilon) = (t-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E(\epsilon), \quad \text{so } Y_1^2(t) = E.$$

By Lemma 8.18,  $m\epsilon \rightarrow -\log\left(1 - \frac{B}{E}\right)$ , hence

$$\log(\hat{L}_2^2(t, \epsilon)) = \ell_{2,2}(t, \epsilon)\epsilon - (t+1) \log \epsilon^{-1} \rightarrow \log\left(A \left(1 - \frac{B}{E}\right) \frac{C}{B}\right),$$

which is consistent with  $\hat{L}_2^2(t) = Y_2^2(t-1) \left(1 - \frac{Y_1^1(t-1)}{Y_1^2(t)}\right) \frac{Y_1^1(t)}{Y_1^1(t-1)}$ . Indeed, we have

$$\begin{aligned} \hat{L}_2^2(t) &= \hat{L}_2^2(t-1) a_t^2 \frac{\hat{L}_1^2(t-1) \hat{L}_1^1(t)}{\hat{L}_1^1(t-1) \hat{L}_1^2(t)} = \hat{L}_2^2(t-1) \frac{\hat{L}_1^2(t) - \hat{L}_1^1(t-1)}{\hat{L}_1^2(t-1)} \frac{\hat{L}_1^2(t-1) \hat{L}_1^1(t)}{\hat{L}_1^1(t-1) \hat{L}_1^2(t)} \\ &= \hat{L}_2^2(t-1) \left(1 - \frac{\hat{L}_1^1(t-1)}{\hat{L}_1^2(t)}\right) \frac{\hat{L}_1^1(t)}{\hat{L}_1^1(t-1)}. \end{aligned}$$

**d)** Assume  $j > 2$ ,  $t > j-1$ . Condition on

$$\ell_{j,2}(t-1, \epsilon) = (t-j+2)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log A(\epsilon), \quad \text{so } Y_2^j(t-1) = A;$$

$$\ell_{j-1,1}(t-1, \epsilon) = (t-j+1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log B(\epsilon), \quad \text{so } Y_1^{j-1}(t-1) = B;$$

$$\ell_{j-1,1}(t, \epsilon) = (t-j+2)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log C(\epsilon), \quad \text{so } Y_1^{j-1}(t) = C;$$

$$\ell_{j,1}(t, \epsilon) = (t-j+1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E(\epsilon), \quad \text{so } Y_1^j(t) = E;$$

$$\ell_{j-1,2}(t-1, \epsilon) = (t-j+3)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log F(\epsilon), \quad \text{so } Y_2^{j-1}(t-1) = F.$$

Denote by  $m$  the independent move of the particle which is the second from the left on the  $j$ -th level. This move is distributed according to

$$\varphi_{q, a_j \alpha_t q^{\ell_{j-1,1}(t-1, \epsilon) - \ell_{j,1}(t, \epsilon)}, 0}(m \mid \ell_{j-1,2}(t-1, \epsilon) - \ell_{j,2}(t-1, \epsilon)).$$

As in the previous case, by Lemma 8.18,  $m\epsilon \rightarrow -\log\left(1 - \frac{B}{E}\right)$ .

Thus, we see that  $\ell_{j,2}(t, \epsilon) = \ell_{j-1,2}(t-1, \epsilon) - M$ , where  $M$  is distributed according to

$$\varphi_{q^{-1}, q^{\ell_{j-1,1}(t, \epsilon) - \ell_{j-1,1}(t-1, \epsilon)}, q^{\ell_{j-1,2}(t-1, \epsilon) - \ell_{j-1,2}(t-1, \epsilon)}}(M \mid \ell_{j-1,2}(t-1, \epsilon) - \ell_{j,2}(t-1, \epsilon) - m).$$

Hence by Lemma 8.17,  $M\epsilon \rightarrow (1 - \frac{B}{E}) \frac{1}{F} \frac{C}{B} A + 1$ . Therefore,

$$\log(\hat{L}_2^j(t, \epsilon)) = \ell_{j,2}(t, \epsilon) - (t - j + 3)\epsilon^{-1} \log \epsilon^{-1} \rightarrow \log \left( \left(1 - \frac{B}{E}\right) \frac{C}{B} A + F \right),$$

which is consistent with  $\hat{L}_2^j(t) = \left(1 - \frac{Y_1^{j-1}(t-1)}{Y_1^j(t)}\right) \frac{Y_1^{j-1}(t)}{Y_1^{j-1}(t-1)} Y_2^j(t-1) + Y_2^{j-1}(t-1)$ . Indeed, one checks that

$$\begin{aligned} \hat{L}_2^j(t) &= \hat{L}_2^j(t-1) a_t^j \frac{\hat{L}_1^j(t-1) \hat{L}_1^{j-1}(t)}{\hat{L}_1^{j-1}(t-1) \hat{L}_1^j(t)} + \hat{L}_2^{j-1}(t-1) \\ &= \hat{L}_2^j(t-1) \frac{\hat{L}_1^j(t) - \hat{L}_1^{j-1}(t-1)}{\hat{L}_1^j(t-1)} \frac{\hat{L}_1^j(t-1) \hat{L}_1^{j-1}(t)}{\hat{L}_1^{j-1}(t-1) \hat{L}_1^j(t)} + \hat{L}_2^{j-1}(t-1) \\ &= \hat{L}_2^j(t-1) \left(1 - \frac{\hat{L}_1^{j-1}(t-1)}{\hat{L}_1^j(t)}\right) \frac{\hat{L}_1^{j-1}(t)}{\hat{L}_1^{j-1}(t-1)} + \hat{L}_2^{j-1}(t-1). \end{aligned}$$

**k-th edge from the left for  $k > 2$ .**

a) Start with assuming that  $j = k$ ,  $t = 1$ . We have  $\ell_{k,k}(1, \epsilon) = \ell_{k-1,k-1}(1, \epsilon) +$  a jump  $m$  distributed according to  $\varphi_{q, a_k \alpha_1, 0}(m \mid \infty)$ . By Lemma 8.15,

$$\log(\hat{L}_k^k(1)) - \log(\hat{L}_{k-1}^{k-1}(1)) = \log \epsilon^{-1} - m\epsilon \rightarrow \log \Gamma,$$

where  $\Gamma = a_1^k$  is an independent random variable distributed according to  $\text{Gamma}(\theta_k + \hat{\theta}_1)$ , which is consistent with  $\hat{L}_k^k(1) = a_1^k Y_{k-1}^{k-1}(1)$ .

b) Let  $j > k$ ,  $t = j - k + 1$ . We have  $\ell_{j,k}(j - k + 1, \epsilon) = m_1 + m_2$ , where  $m_1$  is an independent move distributed according to  $\varphi_{q, a_j \alpha_{j-k+1}, 0}(m_1 \mid \ell_{j-1,k}(j - k, \epsilon))$ , and  $m_2$  is the push from the  $(k-1)$ -st particle from the left on the  $(j-1)$ -st level distributed according

to

$$\varphi_{q^{-1}, q}^{\ell_{j-1, k-1}(j-k+1, \epsilon), \ell_{j-1, k}(j-k, \epsilon)}(\ell_{j-1, k}(j-k, \epsilon) - m_1 - m_2 \mid \ell_{j-1, k}(j-k, \epsilon) - m_1).$$

Condition on

$$\ell_{j-1, k-1}(j-k+1, \epsilon) = (k-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E(\epsilon), \quad \text{so } Y_{k-1}^{j-1}(j-k+1) = E;$$

$$\ell_{j-1, k}(j-k, \epsilon) = k\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log F(\epsilon), \quad \text{so } Y_k^{j-1}(j-k) = F.$$

By Lemma 8.15,  $\log \epsilon^{-1} - m_1 \epsilon \rightarrow \log \Gamma$ , where  $\Gamma = a_{j-k+1}^j$  is an independent random variable distributed according to  $\text{Gamma}(\theta_j + \hat{\theta}_{j-k+1})$ . By Lemma 8.17,

$$\log(\hat{L}_k^j(j-k+1, \epsilon)) = k \log \epsilon^{-1} - \ell_{j, k}(j-k+1, \epsilon) \epsilon \rightarrow \log F + \log \left( 1 + \frac{E\Gamma}{F} \right) = \log(F + E\Gamma),$$

which is consistent with  $Y_k^j(j-k+1) = Y_k^{j-1}(j-k) + a_{j-k+1}^j Y_{k-1}^{j-1}(j-k+1)$ .

c) Assume  $j = k$ ,  $t \geq k$ . Condition on (for  $1 \leq i \leq k-1$ )

$$\ell_{k-1, i}(t-1, \epsilon) = (t-k+2i-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log B_i(\epsilon), \quad \text{so } Y_i^{k-1}(t-1) = B_i;$$

$$\ell_{k-1, i}(t, \epsilon) = (t-k+2i)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log C_i(\epsilon), \quad \text{so } Y_i^{k-1}(t) = C_i;$$

$$\ell_{k, i}(t-1, \epsilon) = (t-k+2i-2)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log D_i(\epsilon), \quad \text{so } Y_i^k(t-1) = D_i;$$

$$\ell_{k, i}(t, \epsilon) = (t-k+2i-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E_i(\epsilon), \quad \text{so } Y_i^k(t) = E_i;$$

By Lemma 8.18, the independent move of the rightmost particle on the  $k$ -th level times  $\epsilon$  converges to 0, while the push from the previous particles times  $\epsilon$  and minus  $\log \epsilon^{-1}$  converges to

$$-\log \left( \frac{\prod_{i=2}^{k-1} D_i \prod_{i=1}^{k-1} C_i}{\prod_{i=2}^{k-1} E_i \prod_{i=1}^{k-1} B_i} \left( 1 - \frac{B_1}{E_1} \right) \right),$$

which is consistent with

$$Y_k^k(t) = \hat{L}_k^k(t-1) \frac{\prod_{i=1}^{k-1} D_i \prod_{i=1}^{k-1} C_i}{\prod_{i=1}^{k-1} E_i \prod_{i=1}^{k-1} B_i} \cdot \frac{E_1 - B_1}{D_1}.$$

For  $t = k$  we take  $D_1 = 1$ .

d) Let  $j = k$ ,  $k > t > 1$ . Make the same conditioning as in the previous part, but with different ranges of indices:  $k - t + 1 \leq i \leq k - 1$  for  $D_i$ ,  $E_i$ , and  $k - t \leq i \leq k - 1$  for  $B_i$ ,  $C_i$ . Take  $B_{k-t} = D_{k-t+1} = 1$ . The independent move of the rightmost particle on the  $k$ -th level times  $\epsilon$  converges to 0, while the push from the previous particles times  $\epsilon$  and minus  $\log \epsilon^{-1}$  converges to

$$-\log \left( \frac{\prod_{i=k-t+1}^{k-1} D_i \prod_{i=k-t}^{k-1} C_i}{\prod_{i=k-t+1}^{k-1} E_i \prod_{i=k-t}^{k-1} B_i} a_t^k \right),$$

which is consistent with

$$Y_k^k(t) = \hat{L}_k^k(t-1) \frac{\prod_{i=k-t+1}^{k-1} D_i \prod_{i=k-t+1}^{k-1} C_i}{\prod_{i=k-t+1}^{k-1} E_i \prod_{i=k-t+1}^{k-1} B_i} (E_{k-t+1} - B_{k-t+1}).$$

e) Let  $j > k$ ,  $t \geq j$ . Condition on

$$\ell_{j-1,i}(t-1, \epsilon) = (t-j+2i-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log B_i(\epsilon), \quad \text{so } Y_i^{j-1}(t-1) = B_i \text{ for } 1 \leq i \leq k;$$

$$\ell_{j-1,i}(t, \epsilon) = (t-j+2i)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log C_i(\epsilon), \quad \text{so } Y_i^{j-1}(t) = C_i \text{ for } 1 \leq i \leq k-1;$$

$$\ell_{j,i}(t-1, \epsilon) = (t-j+2i-2)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log D_i(\epsilon), \quad \text{so } Y_i^j(t-1) = D_i \text{ for } 1 \leq i \leq k;$$

$$\ell_{j,i}(t, \epsilon) = (t-j+2i-1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log E_i(\epsilon), \quad \text{so } Y_i^j(t) = E_i \text{ for } 1 \leq i \leq k-1.$$

The independent move of the  $k$ -th particle from the left on the  $j$ -th level times  $\epsilon$  converges to 0. Denote by  $m_1$  the distance from this particle to  $\ell_{j-1,k}(t-1, \epsilon)$  after the push from the  $(k-1)$ -st particle from the left on the  $(j-1)$ -st level. Let  $m_2$  be this distance after



pushes from other particles. By Lemma 8.17,  $m_1\epsilon \rightarrow \log\left(1 + \frac{C_{k-1}D_k}{B_{k-1}B_k}\right)$ . By Lemma 8.19,

$$m_2\epsilon \rightarrow \log\left(1 + \frac{C_{k-1}D_k}{B_{k-1}B_k} \frac{\prod_{i=2}^{k-1} D_i \prod_{i=1}^{k-2} C_i}{\prod_{i=2}^{k-1} E_i \prod_{i=1}^{k-2} B_i} \left(1 - \frac{B_1}{E_1}\right)\right).$$

This is consistent with

$$\hat{L}_k^j(t) = B_k + Y_k^j(t-1) \frac{\prod_{i=1}^{k-1} D_i \prod_{i=1}^{k-1} C_i}{\prod_{i=1}^{k-1} E_i \prod_{i=1}^{k-1} B_i} \frac{E_1 - B_1}{D_1}.$$

f) Finally, assume that  $j > k$ ,  $j > t > j - k + 1$ . Make the same conditioning as in the previous part, but with different ranges of indices:  $j - t + 1 \leq i \leq k$  for  $D_i$ ,  $D_{j-t+1} = 1$ ,  $j - t + 1 \leq i \leq k - 1$  for  $E_i$ ,  $j - t \leq i \leq k$  for  $B_i$ ,  $B_{j-t} = 1$ , and  $j - t \leq i \leq k - 1$  for  $C_i$ . The independent move of the  $k$ -th particle from the left on the  $j$ -th level times  $\epsilon$  converges to 0. Denote by  $m_1$  the distance from this particle to  $\ell_{j-1,k}(t-1, \epsilon)$  after the push from the  $(k-1)$ -st particle from the left on the  $(j-1)$ -st level. Let  $m_2$  be this distance after pushes from other particles. By Lemma 8.17,  $m_1\epsilon \rightarrow \log\left(1 + \frac{C_{k-1}D_k}{B_{k-1}B_k}\right)$ . By lemma 8.19,

$$m_2\epsilon \rightarrow \log\left(1 + \frac{C_{k-1}D_k}{B_{k-1}B_k} \frac{\prod_{i=j-t+1}^{k-1} D_i \prod_{i=j-t}^{k-2} C_i}{\prod_{i=j-t+1}^{k-1} E_i \prod_{i=j-t}^{k-2} B_i} a_t^j\right).$$

This is consistent with

$$\hat{L}_k^j(t) = B_k + Y_k^j(t-1) \frac{\prod_{i=j-t+1}^{k-1} D_i \prod_{i=j-t+1}^{k-1} C_i}{\prod_{i=j-t+1}^{k-1} E_i \prod_{i=j-t+1}^{k-1} B_i} (E_{j-t+1} - B_{j-t+1}).$$

This completes the proof of Theorem 8.8.

## REFERENCES

- [1] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *Journal of the American Mathematical Society*, 12(4):1119–1178, 1999. arXiv:math/9810105 [math.CO].
- [2] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geometric And Functional Analysis*, 10(4):702–731, 2000. arXiv:math/9901118 [math.CO].
- [3] R. Baxter. *Exactly solved models in statistical mechanics*. Courier Dover Publications, 2007.
- [4] H. Bethe. Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette. (On the theory of metals. I. Eigenvalues and eigenfunctions of the linear atom chain). *Zeitschrift fur Physik*, 71:205–226, 1931.
- [5] P. Biane, P. Bougerol, and N. O’Connell. Littelmann paths and brownian paths. *Duke J. Math.*, 130(1):127–167, 2005. arXiv:math/0403171 [math.RT].
- [6] A. Borodin. Schur dynamics of the Schur processes. *Advances in Mathematics*, 228(4):2268–2291, 2011. arXiv:1001.3442 [math.CO].
- [7] A. Borodin and I. Corwin. Discrete time q-TASEPs. *Intern. Math. Research Notices*, 05 2013. arXiv:1305.2972 [math.PR], doi: 10.1093/imrn/rnt206.
- [8] A. Borodin and I. Corwin. Macdonald processes. *Prob. Theory Rel. Fields*, 158:225–400, 2014. arXiv:1111.4408 [math.PR].
- [9] A. Borodin, I. Corwin, L. Petrov, and T. Sasamoto. *Compositio Mathematica*.
- [10] A. Borodin, I. Corwin, and D. Remenik. Log-Gamma polymer free energy fluctuations via a Fredholm determinant identity. *Comm. Math. Phys.*, 324(1):215–232, 2013. arXiv:1206.4573 [math.PR].

- [11] A. Borodin, I. Corwin, and T. Sasamoto. From duality to determinants for  $q$ -TASEP and ASEP. *Ann. Probab.*, 42(6):2314–2382, 2014. arXiv:1207.5035 [math.PR].
- [12] A. Borodin and P. Ferrari. Anisotropic growth of random surfaces in 2+1 dimensions. *Comm. Math. Phys.*, 325:603–684, 2014. arXiv:0804.3035 [math-ph].
- [13] A. Borodin and V. Gorin. Lectures on integrable probability. 2012. arXiv:1212.3351 [math.PR].
- [14] A. Borodin and L. Petrov. Nearest neighbor Markov dynamics on Macdonald processes. 2013. arXiv:1305.5501 [math.PR], to appear in Adv. Math.
- [15] A. Borodin and L. Petrov. Integrable probability: From representation theory to Macdonald processes. *Probability Surveys*, 11:1–58, 2014. arXiv:1310.8007 [math.PR].
- [16] A. Bufetov. The central limit theorem for extremal characters of the infinite symmetric group. *Functional Analysis and Its Applications*, 46(2):83–93, 2012. arXiv:1105.1519 [math.RT].
- [17] A. Bufetov and L. Petrov. Law of Large Numbers for Infinite Random Matrices over a Finite Field. 2014. arXiv:1402.1772 [math.PR], to appear in Selecta Math.
- [18] R. Chhaibi. *Littelman path model for geometric crystals, Whittaker functions on Lie groups and Brownian motion*. PhD thesis, 2013. arXiv:1302.0902 [math.PR].
- [19] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1, 2012. arXiv:1106.1596 [math.PR].
- [20] I. Corwin. The  $q$ -Hahn Boson process and  $q$ -Hahn TASEP. *Intern. Math. Research Notices*, page rnu094, 2014. arXiv:1401.3321 [math.PR].
- [21] I. Corwin, N. O’Connell, T. Seppäläinen, and N. Zygouras. Tropical Combinatorics and Whittaker functions. *Duke J. Math.*, 163(3):513–563, 2014. arXiv:1110.3489 [math.PR].
- [22] I. Corwin and L. Petrov. The  $q$ -PushASEP: A New Integrable Model for Traffic in 1+1 Dimension. 2013. arXiv:1308.3124 [math.PR], to appear in Journal of Statistical Physics.

- [23] I. Corwin and L. Petrov. Stochastic higher spin vertex models on the line. 2015. arXiv:1502.07374 [math.PR].
- [24] I. Corwin, T. Seppäläinen, and H. Shen. The strict-weak lattice polymer. 2014. arXiv:1409.1794 [math.PR].
- [25] P. Diaconis and J. Fill. Strong stationary times via a new form of duality. *Ann. Probab.*, 18:1483–1522, 1990.
- [26] P. Diaconis and M. Shahshahani. Generating a random permutation with random transpositions. *Z. Wahr. verw. Gebiete*, 57(2):159–179, 1981.
- [27] F. Dyson. A Brownian motion model for the eigenvalues of a random matrix. *Journal of Mathematical Physics*, 3(6):1191–1198, 1962.
- [28] P. Etingof. Whittaker functions on quantum groups and  $q$ -deformed toda operators. *Amer. Math. Soc. Transl. Ser. 2*, 194:9–25, 1998.
- [29] P. Ferrari and B. Veto. Tracy-Widom asymptotics for  $q$ -TASEP. 2013. arXiv:1310.2515 [math.PR], *Ann. Inst. Henri Poincar Probab. Stat.*, to appear.
- [30] S. Fomin. Two-dimensional growth in Dedekind lattices. Master’s thesis, Leningrad State University, 1979.
- [31] S. Fomin. Generalized Robinson-Schnsted-Knuth correspondence. *Zapiski Nauchnykh Seminarov LOMI*, 155:156–175, 1986. in Russian.
- [32] S. Fomin. Duality of graded graphs. *Journal of Algebraic Combinatorics*, 3(4):357–404, 1994.
- [33] S. Fomin. Schensted algorithms for dual graded graphs. *Journal of Algebraic Combinatorics*, 4(1):5–45, 1995.
- [34] W. Fulton. *Young Tableaux with Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.

- [35] G. Gasper and M. Rahman. *Basic hypergeometric series*. Cambridge University Press, 2004.
- [36] I. Gelfand and M. Tsetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk SSSR*, 71:828–828, 1950.
- [37] A. Gerasimov, D. Lebedev, and S. Oblazin. On a classical limit of  $q$ -deformed Whittaker functions. *Letters in Mathematical Physics*, 100(3):279–290, 2012. arXiv:1101.4567 [math.AG].
- [38] I. Gessel and G. Viennot. Binomial determinants, paths, and hook length formulae. *Advances in mathematics*, 58(3):300–321, 1985.
- [39] A. Gnedin and G. Olshanski. A  $q$ -analogue of de Finetti’s theorem. *The electronic journal of combinatorics*, 16:R16, May 2009. arXiv:0905.0367 [math.PR].
- [40] K. Johansson. Shape fluctuations and random matrices. *Communications in mathematical physics*, 209(2):437–476, 2000. arXiv:math/9903134 [math.CO].
- [41] S. Kerov. *Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis*, volume 219. AMS, Translations of Mathematical Monographs, 2003.
- [42] A. Kirillov. Introduction to tropical combinatorics. In A. Kirillov and N. Liskova, editors, *Physics and Combinatorics, Proceedings of the Nagoya 2000 International Workshop*, pages 82–150, Singapore, 2001. World Scientific.
- [43] D. Knuth. Permutations, matrices, and generalized young tableaux. *Pacific J. Math.*, 34(3):709–727, 1970.
- [44] R. Koekoek and R. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its  $q$ -analogue. Technical report, Delft University of Technology and Free University of Amsterdam, 1996.
- [45] W. König. Orthogonal polynomial ensembles in probability theory. *Probab. Surv.*, 2:385–447, 2005. arXiv:math/0403090 [math.PR].

- [46] W. König, N. O’Connell, and S. Roch. Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electron. J. Probab.*, 7(5):1–24, 2002.
- [47] B. Kostant. Quantisation and representation theory. in: Representation theory of lie groups. *Proc. SRC/LMS Research Symposium, LMS Lecture Notes*, 34:287–316, 1977.
- [48] E. Lieb. The residual entropy of square ice. *Phys. Rev.*, 162:162–172, 1967.
- [49] B. Lindström. On the vector representations of induced matroids. *Bulletin of the London Mathematical Society*, 5(1):85–90, 1973.
- [50] M. Lothaire. *The Plactic Monoid, in Algebraic Combinatorics on Words*. Cambridge University Press, 2002.
- [51] I. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 2nd edition, 1995.
- [52] M. Noumi and Y. Yamada. Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 371–442. Math. Soc. Japan, Tokyo, 2004. arXiv:math-ph/0203030.
- [53] N. O’Connell. A path-transformation for random walks and the Robinson-Schensted correspondence. *Transactions of the American Mathematical Society*, 355(9):3669–3697, 2003.
- [54] N. O’Connell. Conditioned random walks and the RSK correspondence. *J. Phys. A*, 36(12):3049–3066, 2003.
- [55] N. O’Connell. Directed polymers and the quantum Toda lattice. *Ann. Probab.*, 40(2):437–458, 2012. arXiv:0910.0069 [math.PR].
- [56] N. O’Connell. Geometric RSK and the Toda lattice. *Illinois J. Math.*, 57(3):883–918, 2013. arXiv:1308.4631 [math.PR].
- [57] N. O’Connell and J. Ortmann. Tracy-Widom asymptotics for a random polymer model with gamma-distributed weights. 2014. arXiv:1408.5326 [math.PR].

- [58] N. O’Connell and Y. Pei. A  $q$ -weighted version of the Robinson-Schensted algorithm. *Electron. J. Probab.*, 18(95):1–25, 2013. arXiv:1212.6716 [math.CO].
- [59] N. O’Connell, T. Seppäläinen, and N. Zygouras. Geometric RSK correspondence, Whittaker functions and symmetrized random polymers. *Invent. Math.*, 197:361–416, 2014. arXiv:1110.3489 [math.PR].
- [60] N. O’Connell and M. Yor. Brownian analogues of Burke’s theorem. *Stochastic Processes and their Applications*, 96(2):285–304, 2001.
- [61] N. O’Connell and M. Yor. A representation for non-colliding random walks. *Electronic Communications in Probability*, (7):1–12, 2002.
- [62] A. Okounkov. Infinite wedge and random partitions. *Selecta Mathematica, New Series*, 7(1):57–81, 2001. arXiv:math/9907127 [math.RT].
- [63] A. Okounkov and N. Reshetikhin. Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. *Journal of the American Mathematical Society*, 16(3):581–603, 2003. arXiv:math/0107056 [math.CO].
- [64] Y. Pei. A symmetry property for  $q$ -weighted Robinson-Schensted algorithms and other branching insertion algorithms. *Journal of Algebraic Combinatorics*, 40:743–770, 2013. arXiv:1306.2208 [math.CO].
- [65] A. Povolotsky. On integrability of zero-range chipping models with factorized steady state. *J. Phys. A*, 46:465205, 2013.
- [66] B. Sagan. *The symmetric group: representations, combinatorial algorithms, and symmetric functions*. Springer Verlag, 2001.
- [67] T. Sasamoto and M. Wadati. Exact results for one-dimensional totally asymmetric diffusion models. *J. Phys. A*, 31:6057–6071, 1998.
- [68] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

- [69] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, 40(1):19–73, 2012. arXiv:0911.2446 [math.PR].
- [70] R. Stanley. *Enumerative Combinatorics. Vol. 2.* Cambridge University Press, Cambridge, 2001. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [71] E. Thoma. Die unzerlegbaren, positive-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Zeitschr.*, 85:40–61, 1964.
- [72] A. Vershik and S. Kerov. The characters of the infinite symmetric group and probability properties of the Robinson-Shensted-Knuth algorithm. *SIAM J. Alg. Disc. Math.*, 7(1):116–124, 1986.
- [73] J. Warren. Dyson’s Brownian motions, intertwining and interlacing. *Electron. J. Probab.*, 12(19):573–590, 2007. arXiv:math/0509720 [math.PR].
- [74] H. Weyl. *The Classical Groups. Their Invariants and Representations.* Princeton University Press, 1997.