Picard-Lefschetz Oscillators for the Drinfeld-Lafforgue-Vinberg Compactification

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Picard-Lefschetz oscillators for the Drinfeld-Lafforgue-Vinberg compactification

A dissertation presented

by

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Picard-Lefschetz oscillators for the Drinfeld-Lafforgue-Vinberg compactification

Abstract

We study the singularities of the Drinfeld-Lafforgue-Vinberg compactification $\overline{\mathrm{Bun}}_G$ of the moduli stack of G-bundles on a smooth projective curve for a reductive group G. The study of these compactifications was initiated by V. Drinfeld (for $G = \mathrm{GL}_2$) and continued by L. Lafforgue (for $G = \mathrm{GL}_n$) in their work on the Langlands correspondence for function fields; unlike the work of Drinfeld and Lafforgue, however, we focus on questions about the singularities of these compactifications which arise naturally in the geometric Langlands program. A definition of $\overline{\mathrm{Bun}}_G$ for a general reductive group G is also due to Drinfeld (unpublished) and relies on the Vinberg semigroup of G; this case will be dealt with in the forthcoming work [Sch]. In the present work we focus on the case $G = \mathrm{SL}_2$. In this case the compactification can alternatively be viewed as a canonical one-parameter degeneration of the moduli space of SL_2-bundles. We study the singularities of this one-parameter degeneration via the weight-monodromy theory of the associated nearby cycles construction: We give an explicit description of the nearby cycles sheaf together with its monodromy action in terms of certain novel perverse sheaves which we call "Picard-Lefschetz oscillators", and then use this description to determine the intersection cohomology sheaf and other invariants of the singularities. Our proofs rely on the construction of certain local models for the one-parameter degeneration which themselves form one-parameter families of spaces which are factorizable in the sense of Beilinson and Drinfeld. We also include a first application on the level of functions.

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Für meine Eltern

1. Introduction

1.1. Overview.

1.1.1. *The relative compactification* $\overline{\text{Bun}}_G$. Let X be a smooth projective curve over an algebraically closed field k, let G be a reductive group over k, and let Bun_G denote the moduli stack of G-bundles on X. In this work we begin the study of the singularities of a relative compactification $\overline{\text{Bun}}_G$ of Bun_G defined by Drinfeld. Recall first that the diagonal morphism

$$\Delta : \ \text{Bun}_G \ \xrightarrow{\Delta} \ \text{Bun}_G \times \text{Bun}_G$$

of Bun_G is not proper. Drinfeld has hence defined a larger stack $\overline{\text{Bun}}_G$ together with a factorization of the diagonal Δ as

$$\text{Bun}_G \longrightarrow \overline{\text{Bun}}_G \xrightarrow{\ \bar{\Delta}\ } \text{Bun}_G \times \text{Bun}_G .$$

where the map $\bar{\Delta}$ is proper. For $G = \text{GL}_2$ and for $G = \text{GL}_n$ certain open substacks of $\overline{\text{Bun}}_G$ were used by Drinfeld and by L. Lafforgue in their seminal work on the Langlands correspondence for function fields (see [Dr1], [Dr2], [Laf]). For a general reductive group Drinfeld's definition of $\overline{\text{Bun}}_G$ uses the Vinberg semigroup of G (see [V] for Vinberg's original work) and will appear in [Sch]. While the open substacks used by Drinfeld and Lafforgue are smooth, the stack $\overline{\text{Bun}}_G$ is already singular for $G = \text{SL}_2$. The need to understand the singularities of $\overline{\text{Bun}}_G$ arises naturally in the geometric Langlands program, such as in the study of the "miraculous duality" on Bun_G introduced by Drinfeld and Gaitsgory (see [G2]).

1.1.2. *The degeneration* VinBun_G. In the present work we study the singularities of $\overline{\mathrm{Bun}}_G$ in the special case $G = \mathrm{SL}_2$. The case of a general reductive group will be treated in [Sch]. For most of the article it is more convenient for us to work with a minor modification of $\overline{\mathrm{Bun}}_G$ which we denote by VinBun_G and refer to as the *Drinfeld-Lafforgue-Vinberg degeneration* of Bun_G. The degeneration VinBun_G is the total space of a $\mathbb{G}_m$-bundle over $\overline{\mathrm{Bun}}_G$, so that the singularities are not affected by this modification. For $G = \mathrm{SL}_2$ it is easy to state the definition of VinBun_G: It parametrizes triples (E_1, E_2, φ) consisting of two SL_2-bundles E_1, E_2 on the curve X together with a non-zero morphism of the associated vector bundles $\varphi : E_1 \to E_2$. Taking the determinant of the map φ defines a map

$$v : \mathrm{VinBun}_G \longrightarrow \mathbb{A}^1$$

whose fibers away from $0 \in \mathbb{A}^1$ are isomorphic to Bun_G, and whose fiber over 0 contains the singular locus of VinBun_G. It is in this sense that VinBun_G is a degeneration of Bun_G. Although the Vinberg semigroup will not explicitly appear in the present article, we remark that this degeneration of Bun_G is canonical in the sense that it is induced by the canonical Vinberg semigroup degeneration of the group G.

1.1.3. *The goal.* In the present article we are interested on the one hand in the singularities of the map v, and on the other hand in the singularities of its total space VinBun_G. More precisely, we on the one hand want to understand the nearby cycles sheaf of the family v together with its monodromy action, and on the other hand want to determine the IC-sheaf of the total space VinBun_G. These two tasks are however related: Our approach will be to first obtain an explicit description of the nearby cycles sheaf, and to then deduce information about the IC-sheaf from an explicit understanding of the monodromy action on the nearby cycles. In fact, accessing the

IC-sheaf via the nearby cycles was our original motivation for introducing the modification VinBun_G of $\overline{\mathrm{Bun}}_G$. A direct calculation of the IC-sheaf appears to be difficult, as we explain in Section 1.2.6 below.

1.2. **Main results.**

The main theorem of this article, Theorem 3.3.3 below, provides an explicit formula for the nearby cycles perverse sheaf Ψ of the degeneration VinBun_G, together with its monodromy action. More precisely, we will give an explicit formula for the associated graded $\mathrm{Gr}\Psi$ of the weight-monodromy filtration on Ψ as a representation of the Lefschetz-$\mathfrak{sl}_2$, so that the action of the lowering operator in $\mathfrak{sl}_2$ coincides with the action of the logarithm of the unipotent part of the monodromy. To give a rough idea of the ingredients of this formula, we first discuss a natural stratification of the special fiber $v^{-1}(0)$ of the family VinBun_G.

1.2.1. *The defect stratification.* To each point $\varphi : E_1 \to E_2$ of the special fiber $v^{-1}(0)$ of the family VinBun_G one can naturally associate an effective divisor D on the curve X which measures the "defect" of the map φ, i.e., it yields a measure of "how singular" the point $\varphi : E_1 \to E_2$ is. The special fiber of VinBun_G can then be stratified into strata $_k\mathrm{VinBun}_G$ on which the *defect* k, i.e., the degree of the associated *defect divisor* D, is equal to k. Associating to each point its defect divisor we obtain natural maps

$$p_k : {}_k\mathrm{VinBun}_G \longrightarrow X^{(k)}$$

where $X^{(k)}$ denotes the k-th symmetric power of the curve X. Finally, let

$$j_k : {}_k\mathrm{VinBun}_G \lhook\joinrel\longrightarrow \overline{{}_k\mathrm{VinBun}_G}$$

denote the inclusion of a stratum into its closure, and let $(j_k)_{!*}$ denote the corresponding intermediate extension functor for perverse sheaves.

1.2.2. *Main theorem about nearby cycles.* Broadly speaking, our main theorem about the nearby cycles of VinBun_G (Theorem 3.3.3) expresses the associated graded $\text{Gr}\Psi$ as the direct sum

$$\text{Gr }\Psi \;\; = \;\; \bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} (j_k)_{!*} \, p_k^* \, \mathcal{P}_k$$

where the $\mathcal{P}_k$ denote certain novel perverse sheaves on $X^{(k)}$ which we call *Picard-Lefschetz oscillators*; their definition is rather simple and originates in the classical Picard-Lefschetz theory (see Section 3.2 below). By definition the Picard-Lefschetz oscillators come equipped with actions of the Lefschetz-$\mathfrak{sl}_2$, and our theorem in fact asserts that the above isomorphism identifies the induced action of the Lefschetz-$\mathfrak{sl}_2$ on the right hand side with its monodromy action on the left hand side. Finally, the intermediate extension $(j_k)_{!*}$ can be explicitly computed via certain finite resolutions of singularities of the strata closures which are constructed using the smooth relative compactifications of the map $\text{Bun}_B \to \text{Bun}_G$ defined by Drinfeld ([BG1]) and Laumon ([Lau]); see the actual formulation of Theorem 3.3.3 for a precise statement.

1.2.3. *Intersection Cohomology.* From the above explicit formula for the nearby cycles one can deduce a description of the weight filtration of the restriction of the IC-sheaf of VinBun_G to the special fiber (see Theorem 3.4.1 below). Extracting this description from the above formula essentially amounts to computing the perverse kernel of the monodromy operator on the Picard-Lefschetz oscillators, which can be done systematically using a variant of the classical Schur-Weyl duality. Furthermore, exploiting the geometry of the defect stratification it is also possible to compute the IC-stalks from this description of the weight filtration (see Remark 7.1.7).

1.2.4. *Stalks of the ∗-extension of the constant sheaf.* Let

$$j: \ \mathrm{VinBun}_G\big|_{\mathbb{A}^1 \smallsetminus \{0\}} \ \hookrightarrow \ \mathrm{VinBun}_G$$

denote the open inclusion of the inverse image of $\mathbb{A}^1 \smallsetminus \{0\}$ under v. One ingredient in our proof of the nearby cycles theorem which might be of independent interest is the determination of the ∗-stalks of the ∗-extension j_* of the constant sheaf on $\mathrm{VinBun}_G\big|_{\mathbb{A}^1 \smallsetminus \{0\}}$ (see Section 6.4 below). In fact, we proceed by relating these stalks to the cohomology $\widetilde{\Omega}$ of the "open" Zastava spaces from [FFKM], [BFGM].

1.2.5. *Function-theoretic applications.* The above stalk computation can also be used, via the sheaf-function correspondence, to answer a natural question on the level of functions; this application is given at the end of the article in Section 8 below. The relation of this computation to Drinfeld's "strange" invariant bilinear form on the space of automorphic forms is explained in the forthcoming article [DW] by Drinfeld and Wang. Its relation to the "asymptotic map" appearing in the work [BK] of Bezrukavnikov and Kazhdan and the works [SakV], [Sak] of Sakellaridis and Venkatesh will be discussed in the forthcoming work [Sch].

1.2.6. *Non-factorization of the IC-sheaf.* To motivate the next section, we now briefly discuss one peculiarity exhibited by the IC-sheaf of VinBun_G, which sets it apart from similarly defined singular moduli spaces in the geometric Langlands program such as Drinfeld's relative compactifications $\overline{\mathrm{Bun}}_B$ (see [BG1], [BFGM]). As was mentioned in Section 1.2.1 above, one can naturally associate to any singular point of VinBun_G its defect divisor. The IC-stalk at the singular point will in fact essentially only depend on this divisor; let us temporarily denote the IC-stalk at a singular point with associated defect divisor D by IC_D. A natural expectation is then that the IC-sheaf "factorizes", i.e., if $D = D_1 + D_2$ for two effective divisors D_1 and D_2 on X

with disjoint supports, then (up to shifts and twists) we have

$$\text{IC}_D = \text{IC}_{D_1} \otimes \text{IC}_{D_2} .$$

Less formally, one might expect that distinct points "cannot see each other" in the sense that they contribute to the IC-stalk independently; this is indeed the case for Drinfeld's $\overline{\text{Bun}}_B$ (see [FFKM], [BFGM], and Section 1.3 below). However, it turns out that the IC-sheaf of VinBun_G does not factorize in this sense, which in turn appears to make it difficult to carry out the approach of [BFGM] to compute the IC-stalks directly, necessitating our approach via the nearby cycles. For an example where a direct computation of the IC-stalks is possible in our setting, see Section 6.6, where the case of defect $\leqslant 2$ is treated and used in our proof of the nearby cycles theorem. A geometric explanation for the lack of factorization of the IC-sheaf can be found in the nature of the local models for VinBun_G which we discuss next.

1.3. **Proofs via local models.**

A powerful technique in the study of singular moduli spaces in the geometric Langlands program is to construct local models which feature the same singularities but have the advantage of being *factorizable* in the sense of Beilinson and Drinfeld (see e.g. [BD1], [BD2]). The utility of the factorization property is that it allows for inductive calculations of sheaves on the moduli spaces which themselves factorize. A prime example of this technique, and a major influence on the present article, is the computation of the IC-sheaf of Drinfeld's relative compactifications $\overline{\text{Bun}}_B$ by Braverman, Finkelberg, Gaitsgory, and Mirkovic in [BFGM]; in this case the local models are the Zastava spaces introduced by Drinfeld, Feigin, Finkelberg, Kuznetsov and Mirkovic (see [FFKM], [BFGM]). That the Zastava spaces are indeed factorizable in fact implies that the IC-sheaf of Drinfeld's $\overline{\text{Bun}}_B$ factorizes in the above sense.

1.3.1. *Local models for* VinBun_G. To prove our main theorem about nearby cycles we will construct certain local models $(Y^n)_{n \in \mathbb{Z}_{\geqslant 1}}$ for the degeneration VinBun_G, which themselves form one-parameter families

$$ v : \ Y^n \ \longrightarrow \ \mathbb{A}^1 \, . $$

Their relationship with VinBun_G is completely analogous to the relationship between the Zastava spaces and Drinfeld's $\overline{\mathrm{Bun}}_B$ (see [BFGM], [BG2]): Broadly speaking, the local model Y^n features the same singularities as the open substack $_{\leqslant n}\mathrm{VinBun}_G$ of defect $\leqslant n$; hence the validity of the nearby cycles theorem and the IC-sheaf computation for VinBun_G is equivalent to the validity of the corresponding assertions for the local models Y^n for all $n \geqslant 1$. We will prove the nearby cycles theorem for Y^n by induction on n, making use of the following factorization property:

1.3.2. *Factorization in families.* The main difference between our local models Y^n and the Zastava spaces is that they actually do not factorize, but rather "factorize in families", i.e., the fibers of the map $v : Y^n \to \mathbb{A}^1$ are factorizable in compatible ways. In fact, our local models can be also viewed as canonical one-parameter "Vinberg degenerations" of the Zastava spaces. The fact that factorization holds only in families is a natural explanation for the fact that the IC-sheaf of VinBun_G and $\overline{\mathrm{Bun}}_G$ does not factorize either. Unlike the IC-stalks, the nearby cycles sheaf however does factorize also in the present setting, and is thus amenable to an inductive computation.

1.3.3. *The local models Y^n for small n.* As another ingredient of our proof of the nearby cycles theorem, we mention the possibility to describe our local models Y^n in very concrete terms; this is a special feature of the case $G = \mathrm{SL}_2$ considered in this article. For example, we construct natural embeddings of our local models Y^n into certain products of Beilinson-Drinfeld affine Grassmannians to derive explicit

equations in coordinates. In the simplest case of defect degree $\leqslant 1$, which is simultaneously the base case of the inductive proof, these formulas show that the resulting one-parameter degeneration

$$Y^1 \longrightarrow \mathbb{A}^1$$

essentially recovers the Picard-Lefschetz family $x \cdot y = t$ of hyperbolas degenerating to a node. Similarly, a somewhat more involved analysis of the equations in the case of defect degree $\leqslant 2$ can be used to prove the appearance of the Picard-Lefschetz oscillators in the formula for the nearby cycles. Although it is possible to give more abstract and possibly quicker proofs of these statements using Koszul duality for nearby cycles, we have tried in the current article to give a concrete proof when possible, and have postponed using more abstract methods to the case of an arbitrary reductive group G.

1.3.4. *The case of an arbitrary reductive group.* A generalization of the nearby cycles theorem to an arbitrary reductive group G will appear in [Sch]. In this case, the explicit formula for the nearby cycles is more complicated as it involves not only the Picard-Lefschetz oscillators but also the combinatorics of the Langlands dual group $\check{G}$ of G. Although several features of the computation in the general case are already visible for $G = \mathrm{SL}_2$, the SL_2-case is significantly simpler not only due to the reduced complexity and easier combinatorics: For example, the important Lemma 6.3.3 below, on which the approach of the proof in this article rests, fails for a general reductive group.

1.4. **Structure of the article.**

We now briefly discuss the content of the individual sections.

In Section 2 we define the compactification $\overline{\mathrm{Bun}}_G$ and the degeneration VinBun_G and explain their relationship. We then focus on VinBun_G and introduce the aforementioned defect stratification. To construct the stratification, but also to prepare for the statement of the main theorem about nearby cycles, we use Drinfeld's and Laumon's relative compactifications $\overline{\mathrm{Bun}}_B$ to compactify the inclusion maps of the individual strata.

In Section 3 we first recall some facts about nearby cycles, the weight-monodromy filtration, the action of the Lefschetz-$\mathfrak{sl}_2$, and the relationship between the nearby cycles and the IC-sheaf. We then define the Picard-Lefschetz oscillators and state our main theorems about the nearby cycles and the IC-sheaf of VinBun_G.

In Sections 4 and 5 we first construct the local models for VinBun_G and restate the analogous theorem about nearby cycles in this context. We then study their geometry: We discuss the aforementioned factorization in families, and construct embeddings into a product of Beilinson-Drinfeld affine Grassmannians. We use these embeddings on the one hand to construct $\mathbb{G}_m$-actions which contract the local models onto the strata of maximal defect, and on the other hand to derive the explicit equations for the local models mentioned above.

In Section 6 we give the proof of the main theorem about nearby cycles. In Section 7 we deduce from it the aforementioned description of the IC-sheaf. In Section 8 we give the application on the level of functions related to the stalk computation mentioned in Section 1.2.4 above.

1.5. **Notation and conventions.**

Since we will use a formalism of mixed sheaves, we for concreteness choose the following setup: We assume the curve X is defined over a finite field, and work with

Weil sheaves over the algebraic closure of the finite field. For a scheme or stack Y, we will denote by $D(Y)$ the derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on Y. We will frequently abuse terminology and refer to its objects as sheaves. We fix once and for all a square root $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ of the Tate twist $\overline{\mathbb{Q}}_\ell(1)$. We normalize all IC-sheaves to be pure of weight 0; for example, on a smooth variety Y the IC-sheaf is equal to $\overline{\mathbb{Q}}_\ell[\dim Y](\frac{1}{2}\dim Y)$. Given a local system E on a smooth dense open subscheme U of a scheme Y, we refer to the intermediate extension of the shifted and twisted local system $E[\dim Y](\frac{1}{2}\dim Y)$ to Y as the IC-extension of E. Our conventions for nearby cycles are stated in Section 3.1.1 below.

Although we restrict to the case $G = \mathrm{SL}_2$ throughout the article, we will continue to use the symbol G; we denote by B and B^- the standard Borel and opposite Borel subgroups of $G = \mathrm{SL}_2$, and by T the standard maximal torus. The arrow $F \hookleftarrow E$ denotes the inclusion of a subbundle F of a vector bundle E; a usual injective arrow $F \hookrightarrow E$ stands for an injection of coherent sheaves.

We will indicate the restriction of a space or a sheaf to a "disjoint locus" by the symbol $\circ$, whenever there is no confusion about what the disjointness is referring to. For example, we denote by
$$X^{(n_1)} \overset{\circ}{\times} X^{(n_2)}$$
the open subset of the product $X^{(n_1)} \times X^{(n_2)}$ of symmetric powers of the curve X consisting of those pairs of effective divisors with disjoint support, and refer to it as the disjoint locus of $X^{(n_1)} \times X^{(n_2)}$. Similarly, for objects $F_1 \in D(X^{(n_1)})$ and $F_2 \in D(X^{(n_1)})$ we denote by
$$F_1 \overset{\circ}{\boxtimes} F_2$$
the restriction of the exterior product $F_1 \boxtimes F_2$ to the disjoint locus of the above product. Finally, we denote by $\overset{\circ}{X}{}^{(n)}$ the open subscheme of $X^{(n)}$ obtained by removing

all diagonals, i.e., the open subscheme consisting of all effective divisors of the form $\sum_{i=1}^{n} x_i$ with all x_i distinct.

2. The Drinfeld-Lafforgue-Vinberg compactification

2.1. The definition of the degeneration and the compactification.

2.1.1. *Definition of* VinBun_G. We now define the Drinfeld-Lafforgue-Vinberg degeneration VinBun_G for $G = \text{SL}_2$. An S-point of VinBun_G consists of the data of two vector bundles E_1, E_2 of rank 2 on $X \times S$, together with trivializations of their determinant line bundles $\det E_1$ and $\det E_2$, and a map of coherent sheaves

$$\varphi : \ E_1 \ \longrightarrow \ E_2$$

satisfying the following condition: For each geometric point $\bar{s} \to S$ we require that the map

$$\varphi|_{X \times \bar{s}} : \ E_1|_{X \times \bar{s}} \ \longrightarrow \ E_2|_{X \times \bar{s}}$$

is not the zero map; in other words, the map $\varphi|_{X \times \bar{s}}$ is required to not vanish generically on the curve $X \times \bar{s}$.

The stack VinBun_G admits a natural map

$$v : \ \text{VinBun}_G \ \longrightarrow \ \mathbb{A}^1$$

which sends an S-point as above to the determinant

$$\det \varphi \ \in \ \Gamma(\mathcal{O}_{X \times S}) \ = \ \Gamma(\mathcal{O}_S) \ = \ \mathbb{A}^1(S).$$

It will follow from Lemma 2.1.6 below that VinBun_G is indeed an algebraic stack.

2.1.2. *Definition of* $\overline{\mathrm{Bun}}_G$. We now recall the definition of the Drinfeld-Lafforgue-Vinberg compactification $\overline{\mathrm{Bun}}_G$ for $G = \mathrm{SL}_2$, following Drinfeld. This definition, as well as Section 2.1.3 below, will not be used in the rest of the article, and is given for reasons of motivation only.

An S-point of $\overline{\mathrm{Bun}}_G$ consists of the following data: Two vector bundles E_1 and E_2 of rank 2 on $X \times S$ together with trivializations of their determinant line bundles $\det E_1$ and $\det E_2$; a line bundle L on S; and a map of coherent sheaves

$$\varphi : \ E_1 \ \longrightarrow \ E_2 \otimes \mathrm{pr}^* L,$$

where $\mathrm{pr}^* L$ denotes the pullback of L along the projection map $\mathrm{pr} : X \times S \to S$. Similarly to above we require the above data to satisfy the following condition: For each geometric point $\bar{s} \to S$ we require that the map

$$\varphi|_{X \times \bar{s}} : \ E_1|_{X \times \bar{s}} \ \longrightarrow \ (E_2 \otimes p^* L)|_{X \times \bar{s}}$$

is not the zero map; in other words, the map $\varphi|_{X \times \bar{s}}$ is required to not vanish generically on the curve $X \times \bar{s}$.

Similarly to VinBun_G, the stack $\overline{\mathrm{Bun}}_G$ admits a natural map

$$\bar{v} : \ \overline{\mathrm{Bun}}_G \ \longrightarrow \ \mathbb{A}^1/\mathbb{G}^m$$

to the quotient of $\mathbb{A}^1$ by $\mathbb{G}_m$ with respect to the quadratic action, defined by remembering only the line bundle L together with the global section of its square

$$\det \varphi : \ \mathcal{O}_S \ \longrightarrow \ L^{\otimes 2}.$$

2.1.3. *Compactifying the diagonal of* Bun_G. We now explain why we call the stack $\overline{\mathrm{Bun}}_G$ the Drinfeld-Lafforgue-Vinberg compactification. To do so, let

$$b : \mathrm{Bun}_G \ \longrightarrow \ \overline{\mathrm{Bun}}_G \quad \text{and} \quad \bar{\Delta} : \ \overline{\mathrm{Bun}}_G \ \longrightarrow \ \mathrm{Bun}_G \times \mathrm{Bun}_G$$

denote the natural maps. If the characteristic is not equal to 2, the map b is an etale map onto its image in $\overline{\mathrm{Bun}}_G$ of degree 2. If the characteristic is equal to 2, the map b is radicial onto its image in $\overline{\mathrm{Bun}}_G$. To explain the above terminology, note on the one hand that the diagonal morphism of Bun_G

$$\Delta : \ \mathrm{Bun}_G \ \longrightarrow \ \mathrm{Bun}_G \times \mathrm{Bun}_G$$

naturally factors as

$$\mathrm{Bun}_G \xrightarrow{\quad b \quad} \overline{\mathrm{Bun}}_G \xrightarrow{\quad \bar{\Delta} \quad} \mathrm{Bun}_G \times \mathrm{Bun}_G \ .$$

On the other hand we have the following lemma, which can be easily checked from the definitions:

Lemma 2.1.4. *The map*

$$\bar{\Delta} : \ \overline{\mathrm{Bun}}_G \ \longrightarrow \ \mathrm{Bun}_G \times \mathrm{Bun}_G$$

is schematic and proper. The fiber of the map $\bar{\Delta}$ over a point (E_1, E_2) in $\mathrm{Bun}_G \times \mathrm{Bun}_G$ is equal to the projectivization $\mathbb{P}(\mathrm{Hom}(E_1, E_2))$ of the vector space of all homomorphisms of coherent sheaves $E_1 \to E_2$.

2.1.5. *The relation between* VinBun_G *and* $\overline{\mathrm{Bun}}_G$. Next consider the natural map

$$\mathrm{VinBun}_G \longrightarrow \overline{\mathrm{Bun}}_G$$

defined by taking L to be the trivial line bundle $\mathcal{O}_S$ on S and by not changing the remaining data. Then the square

$$
\begin{array}{ccc}
\mathrm{VinBun}_G & \longrightarrow & \overline{\mathrm{Bun}}_G \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \bar{v}} \\
\mathbb{A}^1 & \longrightarrow & \mathbb{A}^1/\mathbb{G}^m
\end{array}
$$

commutes, where the bottom arrow is the natural projection map. In fact one sees directly from the definitions:

Lemma 2.1.6. *The above square is cartesian. Thus the map*

$$\mathrm{VinBun}_G \longrightarrow \overline{\mathrm{Bun}}_G$$

is a $\mathbb{G}_m$-bundle, and in particular the stacks $\overline{\mathrm{Bun}}_G$ and VinBun_G, as well as the maps v and $\bar{v}$, are smooth-locally isomorphic.

Note that Lemmas 2.1.6 and 2.1.4 imply that $\overline{\mathrm{Bun}}_G$ and VinBun_G are indeed algebraic stacks.

Remark 2.1.7. *Because of Lemma 2.1.6 above, we will restrict our attention to the Drinfeld-Lafforgue-Vinberg degeneration* VinBun_G *for the entire article. The study of the singularities of* $\overline{\mathrm{Bun}}_G$, *or the study of the map* $\bar{v}$, *immediately reduces to the study of* VinBun_G *and the study of the map* v *due to the cartesian square of Lemma 2.1.6.*

2.2. The G-locus, the B-locus, and the defect-free locus.

Consider again the natural map $v : \mathrm{VinBun}_G \to \mathbb{A}^1$, which on the level of k-points is defined by sending a triple (E_1, E_2, φ) to the determinant $\det \varphi$ of the map φ. We will call the fiber of the map v over $0 \in \mathbb{A}^1$ the B-*locus* of VinBun_G, and denote it by $\mathrm{VinBun}_{G,B}$. We will call the inverse image of $\mathbb{A}^1 \smallsetminus \{0\}$ under v the G-*locus* of VinBun_G and denote it by $\mathrm{VinBun}_{G,G}$. Thus, on the level of k-points, the G-locus $\mathrm{VinBun}_{G,G}$ consist precisely of those triples (E_1, E_2, φ) for which the map φ is an isomorphism. Similarly, the B-locus $\mathrm{VinBun}_{G,B}$ consist precisely of those triples for which the determinant

$$\det \varphi : \ \mathcal{O}_X \ \longrightarrow \ \mathcal{O}_X$$

equals the zero map, i.e., for which the induced maps on fibers

$$\varphi|_x : \ E_1|_x \ \longrightarrow \ E_2|_x$$

have rank $\leqslant 1$ at every point $x \in X$. In other words, the B-locus consists of those triples for which the map φ has generic rank 1 on the curve X.

The G-locus of VinBun_G in fact naturally decomposes as a product:

Lemma 2.2.1. *The natural map*

$$\mathrm{VinBun}_{G,G} \ \longrightarrow \ \mathrm{Bun}_G \times (\mathbb{A}^1 \smallsetminus \{0\})$$

$$(E_1, E_2, \varphi) \quad \longmapsto \quad (E_1, \det \varphi)$$

is an isomorphism.

Proof. By definition the G-locus of VinBun_G is equal to the mapping stack

$$\mathrm{Maps}(X\,,\ \mathrm{SL}_2 \backslash \mathrm{GL}_2 / \mathrm{SL}_2)$$

parametrizing maps from the curve X into the quotient stack $\mathrm{SL}_2 \backslash \mathrm{GL}_2 / \mathrm{SL}_2$ for the action by left and right translations. But after identifying the quotient $\mathrm{GL}_2 / \mathrm{SL}_2$ for the action from the right with $\mathbb{G}_m$ via the determinant map, we see that the remaining action of SL_2 from the left on this quotient is trivial, and the result follows. $\qquad\square$

2.2.2. *The defect-free locus.* We now define an open substack

$$_0\mathrm{VinBun}_G \ \subset \ \mathrm{VinBun}_G$$

which will be referred to as the *defect-free locus* of VinBun_G. This terminology is in line with the notion of defect defined in Section 2.3 below. To define the open substack we require the triple (E_1, E_2, φ) to additionally satisfy the following condition: For each $\bar{s} \to S$ we require the map

$$\varphi|_{X \times \bar{s}}: \ E_1|_{X \times \bar{s}} \longrightarrow (E_2)|_{X \times \bar{s}}$$

to not vanish at any point of the curve $X \times \bar{s}$. In particular the defect-free locus contains the G-locus $\mathrm{VinBun}_{G,G}$.

Proposition 2.2.3. *The restriction of the map $v : \mathrm{VinBun}_G \to \mathbb{A}^1$ to $_0\mathrm{VinBun}_G$ is smooth; in particular the open substack $_0\mathrm{VinBun}_G$ is smooth.*

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Proof. Let $\mathrm{Mat}_{2\times 2}^{\geqslant 1}$ denote the variety of 2×2 matrices over k of rank $\geqslant 1$, and abbreviate

$$Q \; := \; \mathrm{SL}_2 \backslash \mathrm{Mat}_{2\times 2}^{\geqslant 1} / \mathrm{SL}_2 \; .$$

By definition, the defect-free locus $_0\mathrm{VinBun}_G$ is equal to the mapping stack $\mathrm{Maps}(X, Q)$, and the above map

$$v : \quad _0\mathrm{VinBun}_G \; = \; \mathrm{Maps}(X, Q) \quad \longrightarrow \quad \mathrm{Maps}(X, \mathbb{A}^1) \; = \; \mathbb{A}^1$$

is equal to the map induced on mapping stacks by the determinant map

$$d : \; Q \; \longrightarrow \; \mathbb{A}^1 \, .$$

The above mapping stacks are objects of classical algebraic geometry; we will now consider the corresponding derived mapping stacks, which are objects of derived algebraic geometry. In the present context, this should be considered as nothing more than a convenient formalism when dealing with tangent complexes. We denote the derived mapping stacks and the map between them by

$$v_{der} : \quad \mathrm{Maps}_{der}(X, Q) \quad \longrightarrow \quad \mathrm{Maps}_{der}(X, \mathbb{A}^1) \, ,$$

and will show that the map v_{der} is smooth. Since the base change of v_{der} along the natural map

$$\mathbb{A}^1 \; = \; \mathrm{Maps}(X, \mathbb{A}^1) \quad \longrightarrow \quad \mathrm{Maps}_{der}(X, \mathbb{A}^1)$$

agrees with the map v between classical mapping stacks, establishing that v_{der} is smooth suffices to prove the lemma.

To prove that the map v_{der} is indeed smooth, we will show that the fiber of its relative tangent complex

$$T_{rel} \quad := \quad T_{\mathrm{Maps}_{der}(X,Q) \, / \, \mathrm{Maps}_{der}(X,\mathbb{A}^1)}$$

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at any geometric point of $\mathrm{Maps}_{der}(X,Q)$ is concentrated in degrees -1 and 0. To do so, let

$$T_{Q/\mathbb{A}^1} \longrightarrow T_Q \longrightarrow d^*T_{\mathbb{A}^1} \xrightarrow{+1}$$

denote the usual tangent complex triangle on the double quotient Q associated to the determinant map $d : Q \to \mathbb{A}^1$. We claim that the complex $T_{Q/\mathbb{A}^1}$ is concentrated in degree -1. Indeed, since the map d is smooth, it suffices to show that the tangent complex of each fiber of d is concentrated in degree -1. But since the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on any fiber of the determinant map

$$\mathrm{Mat}_{2\times 2}^{\geq 1} \longrightarrow \mathbb{A}^1$$

is transitive with smooth stabilizers, the fibers of the map d are classifying stacks of smooth groups, proving the claim that $T_{Q/\mathbb{A}^1}$ is concentrated in degree -1. We can now show that the fiber of the relative tangent complex T_{rel} at any given geometric point $f : X \to Q$ of $\mathrm{Maps}_{der}(X,Q)$ is concentrated in degrees -1 and 0. Namely, since we are using derived mapping stacks, taking the fiber of the usual tangent complex triangle

$$T_{rel} \longrightarrow T_{Maps_{der}(X,Q)} \longrightarrow v_{der}^* T_{\mathrm{Maps}_{der}(X,\mathbb{A}^1)} \xrightarrow{+1}$$

on $\mathrm{Maps}_{der}(X,Q)$ at the point f yields a triangle

$$T_{rel}|_f \longrightarrow R\Gamma(X, f^*T_Q) \longrightarrow R\Gamma(X, f^*d^*T_{\mathbb{A}^1}) \xrightarrow{+1} .$$

But pulling back the tangent complex triangle on Q above along the map f and then applying $R\Gamma(X,-)$ yields the triangle

$$R\Gamma(X, f^*T_{Q/\mathbb{A}^1}) \longrightarrow R\Gamma(X, f^*T_Q) \longrightarrow R\Gamma(X, f^*d^*T_{\mathbb{A}^1}) \xrightarrow{+1},$$

whose second map agrees with the second map of the previous triangle. Thus

$$T_{rel}|_f \ = \ R\Gamma(X, f^* T_{Q/\mathbb{A}^1}),$$

and hence $T_{rel}|_f$ is indeed concentrated in degrees -1 and 0 as desired. $\qquad\square$

2.2.4. *Remarks about the Vinberg semigroup.* The terminology "G-locus" and "B-locus" stems from the more general context of the Vinberg semigroup: The Vinberg semigroup of a reductive group admits a natural stratification indexed by the parabolic subgroups of the reductive group; this stratification induces a stratification of the degeneration VinBun_G, which specializes to the stratification into the G-locus and the B-locus in the case of $G = \mathrm{SL}_2$. For further motivation for this notation see Section 2.3 below.

2.3. The defect stratification.

2.3.1. *Definition of the defect.* The B-locus $\mathrm{VinBun}_{G,B}$ possesses a natural stratification by the following notion of *defect*. Let (E_1, E_2, φ) be a k-point of $\mathrm{VinBun}_{G,B}$. Then the map φ admits a unique factorization

$$E_1 \ \twoheadrightarrow \ M_1 \ \hookrightarrow \ M_2 \ \hookleftarrow \ E_2$$

where M_1 and M_2 are line bundles on the curve X, the first map is surjective, the middle map is an injection of coherent sheaves, and the last map is a subbundle map. We call the effective divisor on the curve X corresponding to the injection $M_1 \hookrightarrow M_2$ the *defect divisor*; its degree will be called the *defect*.

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2.3.2. *Stratification by defect.* We now stratify the B-locus $\mathrm{VinBun}_{G,B}$ into loci of constant defect, according to the factorization of the map φ above. We first set up the notation. Recall that the moduli stack Bun_B classifying B-bundles on X admits a natural map

$$\mathfrak{q} : \mathrm{Bun}_B \longrightarrow \mathrm{Bun}_T$$

which induces a bijection between the sets of connected components

$$\pi_0(\mathrm{Bun}_B) \;=\; \pi_0(\mathrm{Bun}_T) \;=\; \mathbb{Z}.$$

Let $\mathrm{Bun}_{T,n}$ denote the connected component of Bun_T consisting of degree n line bundles, and define $\mathrm{Bun}_{B,n}$ and $\mathrm{Bun}_{B^-,n}$ in the same way. Furthermore, let $k \in \mathbb{Z}_{\geqslant 0}$ be a non-negative integer and let $X^{(k)}$ denote the k-th symmetric power of the curve X.

Next define a map

$$X^{(k)} \times \mathrm{Bun}_B \;\longrightarrow\; \mathrm{Bun}_T$$

as the composition

$$X^{(k)} \times \mathrm{Bun}_B \;\overset{\mathrm{id}\times\mathfrak{q}}{\longrightarrow}\; X^{(k)} \times \mathrm{Bun}_T \;\overset{\mathrm{twist}}{\longrightarrow}\; \mathrm{Bun}_T \,,$$

where the second map sends a pair (D, L) consisting of an effective divisor D and a line bundle L to the twisted line bundle $L(-D)$.

Using the previous map we now form the fiber product

$$\mathrm{Bun}_{B^-} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \mathrm{Bun}_B \right),$$

from which we will now construct a map to the B-locus $\mathrm{VinBun}_{G,B}$. By definition, a point of this fiber product consists of a B^--bundle $E_1 \twoheadrightarrow M_1$, an effective divisor D, a B-bundle $M_2 \hookleftarrow E_2$, and an identification $M_1 \cong M_2(-D)$. Thus, given two

integers n_1, n_2 with $n_1 = n_2 - k$ we can define a map

$$f_{n_1,k,n_2} : \ \operatorname{Bun}_{B^-,n_1} \underset{\operatorname{Bun}_T}{\times} \left(X^{(k)} \times \operatorname{Bun}_{B,n_2} \right) \ \longrightarrow \ \operatorname{VinBun}_{G,B}$$

by sending the above point to the triple (E_1, E_2, φ) where the map φ is defined as the composition

$$\varphi : \quad E_1 \ \twoheadrightarrow \ M_1 = M_2(-D) \ \hookrightarrow \ M_2 \ \hookleftarrow \ E_2 .$$

We then have the following stratification of the B-locus $\operatorname{VinBun}_{G,B}$:

Proposition 2.3.3.

(a) *The map f_{n_1,k,n_2} is a locally closed immersion and thus defines an isomorphism onto a smooth locally closed substack*

$$_{(n_1,k,n_2)}\operatorname{VinBun}_{G,B} \ \hookrightarrow \ \operatorname{VinBun}_{G,B} .$$

(b) *On the level of k-points, the B-locus $\operatorname{VinBun}_{G,B}$ is equal to the disjoint union*

$$\operatorname{VinBun}_{G,B} \ = \ \bigcup_{(n_1,k,n_2)} {}_{(n_1,k,n_2)}\operatorname{VinBun}_{G,B} ,$$

where the union runs over all triples (n_1, k, n_2) with $n_1, n_2 \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geqslant 0}$, and $n_1 = n_2 - k$.

(c) *On the level of k-points, the closure of a stratum $_{(n_1,k,n_2)}\operatorname{VinBun}_{G,B}$ is equal to the union of strata*

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$$_{(n_1,k,n_2)}\overline{\mathrm{VinBun}}_{G,B} \quad = \quad \bigcup_{\substack{d_1 \geqslant 0 \\ d_2 \geqslant 0}} {}_{(n_1-d_1,k+d_1+d_2,n_2+d_2)}\mathrm{VinBun}_{G,B} \ .$$

(d) Let $k \in \mathbb{Z}_{\geqslant 0}$. Then the locus in $\mathrm{VinBun}_{G,B}$ obtained by requiring that the defect is at most k naturally forms an open substack

$$_{\leqslant k}\mathrm{VinBun}_{G,B} \quad \subset \quad \mathrm{VinBun}_{G,B} \ .$$

(e) The union of all strata $_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B}$ of fixed defect $k \in \mathbb{Z}_{\geqslant 0}$ naturally forms a locally closed substack

$$_{k}\mathrm{VinBun}_{G,B} \quad \hookrightarrow \quad \mathrm{VinBun}_{G,B}$$

which is isomorphic as stacks to the disjoint union

$$_{k}\mathrm{VinBun}_{G,B} \quad = \quad \coprod_{n_1,n_2} {}_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B} \ .$$

We will prove Proposition 2.3.3 in Section 2.4.7 below, using certain compactifications $\bar{f}_{n_1,k,n_2}$ of the maps f_{n_1,k,n_2} that we introduce next. A posteriori, these compactified maps are in fact resolutions of singularities of the strata closures (see Corollary 2.4.8 below).

2.4. Compactifying the maps f_{n_1,k,n_2}.

2.4.1. *Overview.* The goal of this section is to compactify the maps f_{n_1,k,n_2} introduced above. These compactifications will be used to to prove Proposition 2.3.3 above, and

are constructed using the relative compactifications $\overline{\mathrm{Bun}}_B$ of Drinfeld and Laumon. Since the compactifications of the maps f_{n_1,k,n_2} are also used in the description of the nearby cycles sheaf in Section 3 below, we begin with a brief review of the relative compactifications of Drinfeld and Laumon.

Recall first that the map $\mathrm{Bun}_B \to \mathrm{Bun}_G$ is schematic but not proper; relative compactifications have been defined by G. Laumon for $G = \mathrm{GL}_n$ (see [Lau]) and by V. Drinfeld for an arbitrary reductive group G (see [BG1]), and have been of great importance in the geometric Langlands program. For $G = \mathrm{GL}_n$ and $n \geqslant 2$, Laumon's compactification and Drinfeld's compactification differ. However, in the case of interest $G = \mathrm{SL}_2$ of the present paper, the two compactifications agree; we will denote them by $\overline{\mathrm{Bun}}_B$. We now recall the definition of $\overline{\mathrm{Bun}}_B$ for $G = \mathrm{SL}_2$ and then use it to compactify the maps f_{n_1,k,n_2} from Section 2.3.2 above. For more details on $\overline{\mathrm{Bun}}_B$ we refer the reader to [BG1].

2.4.2. *Definition of* $\overline{\mathrm{Bun}}_B$. Let $G = \mathrm{SL}_2$. An S-point of $\overline{\mathrm{Bun}}_B$ consists of the data of a vector bundle E of rank 2 on $X \times S$ with trivialized determinant, a line bundle L on $X \times S$, and an injection of coherent sheaves $L \hookrightarrow E$ which remains injective after being restricted to $X \times \bar{s}$ for any geometric point $\bar{s} \to S$. The definition of $\overline{\mathrm{Bun}}_{B^-}$ is analogous.

2.4.3. *Basic properties.* The open substack of $\overline{\mathrm{Bun}}_B$ obtained by requiring that the above injection of sheaves is a subbundle map is naturally identified with Bun_B and is dense in $\overline{\mathrm{Bun}}_B$. Furthermore, the maps $\mathrm{Bun}_B \to \mathrm{Bun}_G$ and $\mathrm{Bun}_B \to \mathrm{Bun}_T$ naturally extend to $\overline{\mathrm{Bun}}_B$, and the extended map $\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ is schematic and proper when restricted to connected components of $\overline{\mathrm{Bun}}_B$. Finally, for $G = \mathrm{SL}_2$ the map $\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ is in fact smooth.

2.4.4. *Stratification of* $\overline{\mathrm{Bun}}_B$. The stack $\overline{\mathrm{Bun}}_B$ possesses the following stratification.
For a connected component $\overline{\mathrm{Bun}}_{B,n}$ with

$$n \ \in \ \mathbb{Z} \ = \ \pi_0(\mathrm{Bun}_B) \ = \ \pi_0(\overline{\mathrm{Bun}}_B)$$

and an integer $k \in \mathbb{Z}_{\geqslant 0}$, consider the map

$$X^{(k)} \times \mathrm{Bun}_{B,n+k} \ \longrightarrow \ \overline{\mathrm{Bun}}_{B,n}$$

defined as

$$(D, L \longleftarrow E) \quad \longmapsto \quad (L(-D) \hookrightarrow L \longleftarrow E) \, .$$

This map is in fact a locally closed immersion, and as k ranges over $\mathbb{Z}_{\geqslant 0}$ the corre-
sponding locally closed substacks stratify $\overline{\mathrm{Bun}}_{B,n}$:

$$\overline{\mathrm{Bun}}_{B,n} \ = \ \bigcup_{k \in \mathbb{Z}_{\geqslant 0}} (X^{(k)} \times \mathrm{Bun}_{B,n+k})$$

Finally, note that the map

$$X^{(k)} \times \mathrm{Bun}_B \ \longrightarrow \ \mathrm{Bun}_T$$

from Section 2.3.2 above is in fact equal to the composition

$$X^{(k)} \times \mathrm{Bun}_B \ \longrightarrow \ \overline{\mathrm{Bun}}_B \ \xrightarrow{\ \bar{q}\ } \ \mathrm{Bun}_T \, .$$

2.4.5. *Compactifying the maps* f_{n_1,k,n_2}. We now define the above-mentioned compact-
ifications $\bar{f}_{n_1,k,n_2}$ of the maps f_{n_1,k,n_2} from Section 2.3.2 above. To do so, observe first
that the maps

$$X^{(k)} \times \mathrm{Bun}_{B,n+k} \ \longrightarrow \ \overline{\mathrm{Bun}}_{B,n}$$

from Section 2.4.4 above naturally extend to maps

$$X^{(k)} \times \overline{\mathrm{Bun}}_{B,n+k} \longrightarrow \overline{\mathrm{Bun}}_{B,n} \, .$$

We can therefore enlarge the fiber product from Section 2.3.2 by replacing Bun_B and Bun_{B^-} by $\overline{\mathrm{Bun}}_B$ and $\overline{\mathrm{Bun}}_{B^-}$, and define the compactified map

$$\bar{f}_{n_1,k,n_2} : \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right) \longrightarrow \mathrm{VinBun}_{G,B}$$

in the exact same fashion as the map f_{n_1,k,n_2}. We then have:

Lemma 2.4.6. *The map $\bar{f}_{n_1,k,n_2}$ is finite.*

Proof. We first show that the map is quasifinite. This can easily be deduced from the definitions and from the stratification of $\overline{\mathrm{Bun}}_B$ in Section 2.4.4, as follows. Consider the induced stratification of the fiber product

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right)$$

with the strata

$$\left(\mathrm{Bun}_{B^-,n_1-d_1} \times X^{(d_1)} \right) \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times X^{(d_2)} \times \mathrm{Bun}_{B,n_2+d_2} \right),$$

where the integers $d_1, d_2 \in \mathbb{Z}_{\geqslant 0}$ are varying. We claim that the fiber over any k-point of $\mathrm{VinBun}_{G,B}$ can meet at most finitely many of the above strata. Indeed, any k-point of $\mathrm{VinBun}_{G,B}$ admits a unique factorization

$$\varphi : \ E_1 \ \twoheadrightarrow \ M_1 \ \hookrightarrow \ M_2 \ \twoheadleftarrow \ E_2$$

as in Section 2.3.1 above; if $m \in \mathbb{Z}_{\geqslant 0}$ denotes its defect, then one sees from the definition of $\bar{f}_{n_1, k, n_2}$ that only the strata with

$$d_1 + k + d_2 \;=\; m$$

can meet its fiber.

Hence it suffices to prove that $\bar{f}_{n_1, k, n_2}$ is quasifinite when restricted to any such stratum; this follows from the unique factorization of the map φ above together with the fact that the map

$$X^{(d_1)} \times X^{(k)} \times X^{(d_2)} \;\longrightarrow\; X^{(d_1 + k + d_2)}$$

defined by adding effective divisors is quasifinite.

To show that the map $\bar{f}_{n_1, k, n_2}$ is finite it now suffices to show that it is proper. To do so, consider first the fiber product

$$\overline{\mathrm{Bun}}_{B^-, n_1} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_G \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B, n_2}$$

where the two maps $\mathrm{VinBun}_G \to \mathrm{Bun}_G$ are the two maps remembering only the bundles E_1 and E_2, respectively. Thus the fiber product parametrizes the data of a point (E_1, E_2, φ) of VinBun_G together with a point $E_1 \to M_1$ of $\overline{\mathrm{Bun}}_{B^-, n_1}$ and a point $M_2 \to E_2$ of $\overline{\mathrm{Bun}}_{B, n_2}$.

Consider now the closed substack $\mathcal{Y}$ of the above fiber product obtained by requiring that the map φ factors through the map $E_1 \to M_1$ and also through the map $M_2 \to E_2$:

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$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ \downarrow & & \downarrow \\ M_1 & & M_2 \end{array}$$

We claim that the closed substack $\mathcal{Y}$ is in fact isomorphic to the stack

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right).$$

Indeed, given an S-point of $\mathcal{Y}$ as above, the map φ is forced to factor as

$$E_1 \longrightarrow M_1 \xrightarrow{i} M_2 \longrightarrow E_2,$$

and the datum of the map $i : M_1 \to M_2$ is equivalent to the datum of the map φ. Moreover, the definition of VinBun_G forces the map $i : M_1 \to M_2$ to be injective when restricted to $X \times \bar{s}$ for any geometric point $\bar{s} \to S$. Since M_1 and M_2 have degrees n_1 and n_2 when restricted to each $X \times \bar{s}$ and since $n_1 = n_2 - k$, the datum of the map i above is in turn equivalent to the datum of an S-point of $X^{(k)}$.

Finally, note that the map $\bar{f}_{n_1,k,n_2}$ is equal to the composition of the inclusion map of

$$\mathcal{Y} = \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right)$$

into the fiber product

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_G \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B,n_2}$$

with the projection of the latter to VinBun_G. Since the inclusion map is a closed immersion and the projection map is proper by Section 2.4.3, we conclude that $\bar{f}_{n_1,k,n_2}$ is proper, finishing the proof. $\qquad\square$

2.4.7. *Proof of stratification results.* Using Lemma 2.4.6 we can now prove the strat-
ification results of Proposition 2.3.3 above. Before doing so, we state the following
corollary, which follows from the fact that the map

$$\overline{\text{Bun}}_B \longrightarrow \text{Bun}_T$$

is smooth for $G = \text{SL}_2$ (see Section 2.4.3 above).

Corollary 2.4.8 (of Lemma 2.4.6). *The compactified map*

$$\bar{f}_{n_1,k,n_2} : \quad \overline{\text{Bun}}_{B^-,n_1} \underset{\text{Bun}_T}{\times} \left(X^{(k)} \times \overline{\text{Bun}}_{B,n_2} \right) \longrightarrow {}_{(n_1,k,n_2)}\overline{\text{VinBun}}_{G,B}$$

is a resolution of singularities of the closure of the stratum ${}_{(n_1,k,n_2)}\text{VinBun}_{G,B}$.

Finally we prove Proposition 2.3.3:

Proof of Proposition 2.3.3 (a). We use the same notation as in Section 2.3.2. We first
show that the map f_{n_1,k,n_2} is a monomorphism of algebraic stacks. Thus we need to
check that the data

$$E_1 \longrightarrow M_1 \longrightarrow M_2 \longleftarrow E_2$$

on $X \times S$ can be reconstructed from the composite map $\varphi : E_1 \longrightarrow E_2$. Indeed, the
line bundle M_1 can be recovered as the image $\text{im}(\varphi)$, and the factorization through
M_1 corresponds to the factorization

$$E_1 \longrightarrow \text{im}(\varphi) \longrightarrow E_2 .$$

One can argue dually for M_2, and hence f_{n_1,k,n_2} is a monomorphism.

We now show that f_{n_1,k,n_2} is in fact a locally closed immersion. Let $\mathcal{B}$ denote the
boundary of

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$$\mathcal{Y} = \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right),$$

i.e., the closed complement in $\mathcal{Y}$ of the open substack

$$\mathrm{Bun}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \mathrm{Bun}_{B,n_2} \right).$$

Since the map $\bar{f}_{n_1,k,n_2}$ is proper, the image of the boundary $\mathcal{Z}$ under $\bar{f}_{n_1,k,n_2}$ is a closed substack of VinBun_G; let $\mathcal{U}$ denote its open complement. We claim that taking the inverse image of $\mathcal{U}$ under $\bar{f}_{n_1,k,n_2}$ yields the cartesian square

$$
\begin{array}{ccc}
\mathrm{Bun}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \mathrm{Bun}_{B,n_2} \right) & \overset{\mathrm{open}}{\lhook\joinrel\longrightarrow} & \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right) \\
\Big\downarrow & {\scriptstyle f_{n_1,k,n_2}} \!\!\!\!\!\searrow & \Big\downarrow {\scriptstyle \bar{f}_{n_1,k,n_2}} \\
\mathcal{U} & \overset{\mathrm{open}}{\lhook\joinrel\longrightarrow} & \mathrm{VinBun}_G
\end{array}
$$

This follows from the fact that any point of VinBun_G lying in the image of the boundary $\mathcal{Z}$ must have defect strictly greater than k.

The diagonal map of the above square is precisely the map f_{n_1,k,n_2}, which has already been shown to be a monomorphism. Thus the left vertical arrow is also a monomorphism; but being the base change of the proper map $\bar{f}_{n_1,k,n_2}$, the left vertical arrow is also proper, and hence it must be a closed immersion. This establishes the desired factorization of the map f_{n_1,k,n_2}, showing that it is indeed a locally closed immersion.

Finally, the assertion about smoothness follows from the fact that the map $\mathrm{Bun}_{B^-} \longrightarrow \mathrm{Bun}_T$ is smooth. $\qquad\square$

Proof of Proposition 2.3.3 (b) through (e). Part (b) follows immediately from the fact that every map $\varphi : E_1 \to E_2$ factors uniquely as

$$\varphi : \ E_1 \twoheadrightarrow M_1 \hookrightarrow M_2 \twoheadleftarrow E_2$$

as in Section 2.3.1 above. Part (c) follows from the definition of the map $\bar{f}_{n_1,k,n_2}$ together with the stratifications of $\overline{\mathrm{Bun}}_B$ and $\overline{\mathrm{Bun}}_{B^-}$ in Section 2.4.4. Part (d) follows from the formula for the strata closure in part (c). For part (e), note that by part (c) each stratum $_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B}$ is closed in the open substack $_{\leqslant k}\mathrm{VinBun}_{G,B}$. Thus the natural map

$$\coprod_{n_1,n_2} {}_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B} \ \longrightarrow \ {}_{\leqslant k}\mathrm{VinBun}_{G,B}$$

is a closed immersion, and the claim follows. $\qquad\square$

3. Statement of main theorems

3.1. Preliminaries about nearby and vanishing cycles.

3.1.1. *Conventions.* Given a map $Y \to \mathbb{A}^1$ we will denote by

$$\Psi : \ \mathrm{D}(Y|_{\mathbb{A}^1 \smallsetminus \{0\}}) \ \longrightarrow \ \mathrm{D}(Y|_{\{0\}})$$

the unipotent nearby cycles functor in the perverse and Verdier-self dual renormalization, i.e., we shift and twist the usual unipotent nearby cycles functor by $[-1](-\frac{1}{2})$ so that it is t-exact for the perverse t-structure, preserves the weights, and commutes with Verdier duality literally and not just up to twist. We will refer to Ψ simply as *the nearby cycles*, and we denote the analogously shifted and twisted unipotent vanishing cycles functor simply by Φ. We denote the logarithm of the unipotent part of the monodromy operator by

$$N : \ \Psi \ \longrightarrow \ \Psi(-1) \,,$$

and will refer to it simply as *the monodromy operator*. The monodromy operator N admits the factorization $N = \mathrm{var} \circ \mathrm{can}$ into the natural maps

$$\mathrm{can} : \ \Psi \longrightarrow \Phi \qquad \text{and} \qquad \mathrm{var} : \ \Phi \longrightarrow \Psi(-1) \,.$$

In the above normalization, the usual triangle relating Ψ and Φ reads

$$F|_{Y|_{\{0\}}}^*[-1](-\tfrac{1}{2}) \ \longrightarrow \ \Psi(F) \ \xrightarrow{\ \mathrm{can}\ } \ \Phi(F) \ \xrightarrow{\ +1\ }$$

for any object $F \in \mathrm{D}(Y)$. We refer the reader to [B] and [BB, Sec. 5] for more background on unipotent nearby and vanishing cycles.

3.1.2. *Monodromy and weight filtrations and Gabber's theorem.* We now recall some facts about the monodromy and weight filtrations on nearby cycles; we refer the reader to [De, Sec. 1.6] and [BB, Sec. 5] for proofs.

Given a perverse sheaf F on $Y|_{\mathbb{A}^1 \smallsetminus \{0\}}$, the endomorphism N acts nilpotently on the perverse sheaf $\Psi(F)$, and thus induces the *monodromy filtration* on $\Psi(F)$. The latter filtration is the unique finite filtration

$$\Psi(F) = M_n \;\supseteq\; M_{n-1} \;\supseteq\; \cdots \;\supseteq\; M_{-n} \;\supseteq\; 0$$

by perverse sheaves M_i satisfying that

$$N(M_i) \;\subset\; M_{i-2}(-1)$$

for all i, and that the induced maps

$$N^i : \; M_i/M_{i-1} \;\longrightarrow\; \big(M_{-i}/M_{-i-1}\big)(-i)$$

are isomorphisms for all $i \geqslant 0$. In particular the operator N acts on the associated graded perverse sheaf $\mathrm{Gr}(\Psi(F))$, and we have the following well-known lemma:

Lemma 3.1.3. *The action of N on the associated graded $\mathrm{Gr}(\Psi(F))$ extends canonically to an action of the "Lefschetz-$\mathfrak{sl}_2$", i.e.: There exists a unique action of the Lie algebra $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$ on $\mathrm{Gr}(\Psi(F))$ such that the action of the lowering operator of $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$ coincides with the action of N, and such that the Cartan subalgebra of $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$ acts on the summand $\mathrm{Gr}(\Psi(F))_i = M_i/M_{i-1}$ with Cartan weight i. Thus the decomposition*

$$\mathrm{Gr}(\Psi(F)) \;=\; \bigoplus_i M_i/M_{i-1}$$

agrees with the decomposition of the $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$-representation $\mathrm{Gr}(\Psi(F))$ according to Cartan weights. We will refer to the Lie algebra $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$ in this context as the Lefschetz-$\mathfrak{sl}_2$.

In the case that F is a pure perverse sheaf, the monodromy filtration satisfies Gabber's theorem, which we state for the case of weight 0:

Proposition 3.1.4 (Gabber)**.** *Assume that F is a pure perverse sheaf of weight 0.*
Then the subquotients of the monodromy filtration on $\Psi(F)$ are pure, and the weight
of the subquotient $\mathrm{Gr}(\Psi(F))_i = M_i/M_{i-1}$ is equal to i. In other words, the monodromy
filtration agrees with the weight filtration of $\Psi(F)$. In particular, the weight of each
subquotient as a Weil sheaf agrees with its Cartan weight with respect to the action
of the Lefschetz-$\mathfrak{sl}_2$.

Finally, recall on the one hand that the *i-th primitive part P_i* of $\Psi(F)$ is defined
as the kernel of the map

$$N: \; \mathrm{Gr}_i(\Psi(F)) \; \longrightarrow \; \mathrm{Gr}_{i-2}(\Psi(F)) \,.$$

On the other hand, consider the filtration induced on the kernel

$$\ker\big(N : \Psi(F) \to \Psi(F)(-1)\big) \;\; \subset \;\; \Psi(F)$$

by the monodromy filtration on $\Psi(F)$ by means of intersecting the kernel with the
monodromy filtration. We then have the following well-known lemma:

Lemma 3.1.5. *The i-th subquotient of the latter filtration is canonically isomorphic*
to the i-th primitive part P_i. Less precisely, the associated graded of the kernel of
$N : \Psi(F) \to \Psi(F)(-1)$ agrees with the kernel of N acting on the associated graded
$\mathrm{Gr}(\Psi(F))$.

3.1.6. *Intersection cohomology from nearby cycles.* As above let Y be a scheme or
stack, let $Y \to \mathbb{A}^1$ be a map, and let IC_Y denote the IC-sheaf of Y, normalized to be
pure of weight 0 as mentioned in Section 1.5. Consider the monodromy operator

$$N: \; \Psi(\mathrm{IC}_Y) \; \longrightarrow \; \Psi(\mathrm{IC}_Y)(-1)$$

acting on the nearby cycles of the IC-sheaf of Y, and let $\ker(N) \subset \Psi(\mathrm{IC}_Y)$ and $\mathrm{im}(N) \subset \Psi(\mathrm{IC}_Y)(-1)$ denote its perverse kernel and image. Note furthermore that for the IC-sheaf of Y, the usual triangle for the map $\mathrm{can} : \Psi \to \Phi$ is in fact a short exact sequence of perverse sheaves

$$0 \longrightarrow \mathrm{IC}_Y\,|^*_{Y|_{\{0\}}}[-1](-\tfrac{1}{2}) \longrightarrow \Psi(\mathrm{IC}_Y) \xrightarrow{\;can\;} \Phi(\mathrm{IC}_Y) \longrightarrow 0\,.$$

For example from Beilinson's gluing description ([B]) applied to the IC-sheaf of Y one verifies:

Lemma 3.1.7. *The above short exact sequence in fact coincides with the short exact sequence*

$$0 \longrightarrow \ker(N) \longrightarrow \Psi(\mathrm{IC}_Y) \xrightarrow{\;N\;} \mathrm{im}(N) \longrightarrow 0\,,$$

i.e., we have

$$\ker(N) \;=\; \mathrm{IC}_Y\,|^*_{Y|_{\{0\}}}[-1](-\tfrac{1}{2})\,,$$

$$\mathrm{im}(N) \;=\; \Phi(\mathrm{IC}_Y)\,.$$

In particular, one can obtain the restriction $\mathrm{IC}_Y\,|^*_{Y|_{\{0\}}}[-1](-\tfrac{1}{2})$ *and the vanishing cycles* $\Phi(\mathrm{IC}_Y)$ *from understanding the monodromy action on* $\Psi(\mathrm{IC}_Y)$.

3.1.8. *Nearby cycles of the Picard-Lefschetz family of hyperbolas.* We now recall the well-known computation of the nearby cycles for the family of hyperbolas $xy = t$, i.e., the nearby cycles for the map

$$d :\; \mathbb{A}^2 \longrightarrow \mathbb{A}^1, \quad (x,y) \longmapsto x \cdot y\,.$$

This example in fact plays a key role in the nearby cycles computation for VinBun$_G$, as will become clear in the next two sections and in the proof of the main theorem about nearby cycles in Section 6 below.

To state the result, let $C = d^{-1}(\{0\})$ be the fiber of d over $0 \in \mathbb{A}^1$, i.e., the reducible node in $\mathbb{A}^2$ formed by the union $C = C_x \cup C_y$ of the two coordinate axes C_x and C_y. Let p denote the origin of $\mathbb{A}^2$, i.e., the intersection of C_x and C_y, and let δ_p denote the pushforward of the constant sheaf $\overline{\mathbb{Q}}_\ell$ along the inclusion $p \hookrightarrow C$. Let $(\overline{\mathbb{Q}}_\ell)_C$ denote the constant sheaf on C and let $i_{x,*}(\overline{\mathbb{Q}}_\ell)_{C_x}$ and $i_{y,*}(\overline{\mathbb{Q}}_\ell)_{C_y}$ denote the pushforwards of the constant sheaves from C_x and C_y to C. Thus the IC-sheaf of C is equal to

$$\mathrm{IC}_C \;=\; i_{x,*}(\overline{\mathbb{Q}}_\ell)_{C_x} \;\oplus\; i_{y,*}(\overline{\mathbb{Q}}_\ell)_{C_y} \,.$$

Applying the nearby cycles functor to the IC-sheaf $\mathrm{IC}_{\mathbb{A}^2} = \overline{\mathbb{Q}}_\ell[2](1)$ of $\mathbb{A}^2$ one finds:

Lemma 3.1.9. *The weight-monodromy filtration on $\Psi(\mathrm{IC}_{\mathbb{A}^2})$ is equal to*

$$\Psi(\mathrm{IC}_{\mathbb{A}^2}) \;\supsetneqq\; (\overline{\mathbb{Q}}_\ell)_C[1](\tfrac{1}{2}) \;\supsetneqq\; \delta_p(\tfrac{1}{2}) \;\supsetneqq\; 0 \,,$$

and the corresponding associated graded object equals

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{\mathbb{A}^2}) \;=\; \delta_p(-\tfrac{1}{2}) \;\oplus\; \mathrm{IC}_C \;\oplus\; \delta_p(\tfrac{1}{2}) \,.$$

Furthermore, the action of the monodromy operator N on $\mathrm{Gr}\,\Psi(\mathrm{IC}_{\mathbb{A}^2})$ identifies $\mathrm{Gr}_1 = \delta_p(-\tfrac{1}{2})$ with $\mathrm{Gr}_{-1}(-1) = \delta_p(\tfrac{1}{2})(-1) = \delta_p(-\tfrac{1}{2})$, and the action on $\mathrm{Gr}_0 = \mathrm{IC}_C$ is trivial. In particular, as a representation of the Lefschetz-$\mathfrak{sl}_2$ the direct sum $\mathrm{Gr}_{-1} \oplus \mathrm{Gr}_1$ is isomorphic to the standard representation of the Lefschetz-$\mathfrak{sl}_2$.

3.2. **Picard-Lefschetz oscillators.**

3.2.1. *Factorization structures.* Assume we are given, for each $n \in \mathbb{Z}_{\geqslant 0}$, a perverse sheaf $F_n \in D(X^{(n)})$ on the symmetric power $X^{(n)}$ of the curve X. Here and below we denote by

$$\text{add}: \ X^{(n_1)} \times X^{(n_2)} \ \longrightarrow \ X^{(n)}$$

the map defined by adding effective divisors. Then we define a *factorization structure* on the collection of perverse sheaves F_n to be a collection of compatible isomorphisms

$$(\text{add}^* F_n)\big|^*_{X^{(n_1)} \overset{\circ}{\times} X^{(n_2)}} \ \cong \ F_{n_1} \overset{\circ}{\boxtimes} F_{n_2}$$

for any $n_1 + n_2 = n$. If there is no ambiguity about which factorization structure is being considered on a given collection of perverse sheaves F_n, then we also abuse terminology and refer to the collection of perverse sheaves F_n as *factorizable.*

3.2.2. *External exterior powers.* Recall that to any local system E on the curve X, placed in cohomological degree 0, one can associate its *n-th external exterior power* $\Lambda^{(n)}(E)$ on the symmetric power of the curve $X^{(n)}$. Namely, the n-fold exterior product $E \boxtimes \cdots \boxtimes E$ on the n-th power X^n carries a natural equivariant structure with respect to the action of the symmetric group S^n on X^n. Thus its pushforward $p_*(E \boxtimes \cdots \boxtimes E)$ along the natural map

$$p: \ X^n \ \longrightarrow \ X^{(n)}$$

carries a natural action of S^n, and we define $\Lambda^{(n)}(E)$ by taking S^n-invariants of the pushforward $p_*(E \boxtimes \cdots \boxtimes E)$ against the sign character of S^n.

This construction is functorial and satisfies the basic properties listed in the next lemma (see for example [G1, Sec. 5] for proofs).

Lemma 3.2.3.

(a) Over the disjoint locus $\mathring{X}^{(n)}$ the n-th external exterior power $\Lambda^{(n)}(E)$ is again a local system.

(b) The shifted object $\Lambda^{(n)}(E)[n]$ is a perverse sheaf. In fact, it is equal to the intermediate extension of its restriction to the disjoint locus.

(c) Let $D = \sum_k n_k x_k \in X^{(n)}$ be a divisor on X, with the points x_k distinct. Then the $*$-stalk of $\Lambda^{(n)}(E)$ at the point D is equal to

$$\bigotimes_k \Lambda^{n_k}(E).$$

(d) The collection of perverse sheaves $\Lambda^{(n)}(E)[n]$ is factorizable in the sense of Section 3.2.1 above.

3.2.4. *Definition of Picard-Lefschetz oscillators.* Let V denote the 2-dimen-sional standard representation of the Lefschetz-$\mathfrak{sl}_2$:

$$V \;=\; \overline{\mathbb{Q}}_\ell(\tfrac{1}{2}) \oplus \overline{\mathbb{Q}}_\ell(-\tfrac{1}{2})$$

We denote by

$$\underline{V} \;:=\; V \otimes \overline{\mathbb{Q}}_{\ell X}$$

the corresponding constant local system of rank 2 on the curve X together with the induced action of the Lefschetz-$\mathfrak{sl}_2$. For any integer $n \geqslant 1$ we then define the *Picard-Lefschetz oscillator* $\mathcal{P}_n$ on $X^{(n)}$ to be the n-th external exterior power of $\underline{V}$, shifted and twisted as follows:

$$\mathcal{P}_n \;:=\; \Lambda^{(n)}(\underline{V})\,[n](\tfrac{n}{2})$$

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Thus by Lemma 3.2.3 above $\mathcal{P}_n$ is a perverse sheaf on $X^{(n)}$, and carries an action of the Lefschetz-$\mathfrak{sl}_2$ by the functoriality of the external exterior power construction. Furthermore the definition and Lemma 3.2.3 together show:

Lemma 3.2.5. *Let the symmetric group S^n act on the n-fold tensor power $V \otimes \cdots \otimes V$ by permuting the factors and additionally multiplying by the sign of the permutation, and consider the local system on the disjoint locus $\overset{\circ}{X}{}^{(n)}$ associated to this represen-tation. Then the IC-extension of this local system is equal to the Picard-Lefschetz oscillator $\mathcal{P}_n$. In particular the perverse sheaf $\mathcal{P}_n$ is semisimple. Finally, the natural factorization structure on the collection of Picard-Lefschetz oscillators $\mathcal{P}_n$ respects the action of the Lefschetz-$\mathfrak{sl}_2$.*

Our choice of the term Picard-Lefschetz oscillators is due, on the one hand, to the appearance of the sign character in the action of the symmetric group in Lemma 3.2.5 above; and, on the other hand, due to the appearance of the representation V: For $n = 1$ the Picard-Lefschetz oscillator $\mathcal{P}_1$ equals, up to shifts and twists, the constant rank-2 local system on the curve X whose fiber is equal to the standard representation V of the Lefschetz-$\mathfrak{sl}_2$; the latter is precisely the summand of the associated graded of the nearby cycles sheaf of the Picard-Lefschetz family from Section 3.1.8 above consisting of those summands supported on the singular locus $\{p\}$ of the map $d : \mathbb{A}^2 \to \mathbb{A}^1$.

3.3. Nearby cycles for VinBun_G.

To state our main theorem about nearby cycles for VinBun_G we will need the following definition:

3.3.1. *Placing Picard-Lefschetz oscillators on* VinBun_G. We now define versions of the Picard-Lefschetz oscillators on the strata closures of the defect stratification of the B-locus from Section 2.3 above. More precisely, we define versions $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ of the Picard-Lefschetz oscillators $\mathcal{P}_k$ on the relative compactifications

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right)$$

from Section 2.4 above; the latter map onto the strata closures in VinBun_G via the compactified maps

$$\bar{f}_{n_1,k,n_2} : \ \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right) \ \longrightarrow \ \mathrm{VinBun}_{G,B}$$

introduced in Section 2.4.5 above.

To state the definition, let (n_1, k, n_2) be any triple with $n_1 = n_2 - k$, as before. Then we define $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ as

$$\widetilde{\mathcal{P}}_{n_1,k,n_2} \ := \ \mathrm{IC}_{\overline{\mathrm{Bun}}_{B^-,n_1}} \underset{\mathrm{Bun}_T}{\boxtimes} \mathcal{P}_k \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_{B,n_2}} \, ,$$

i.e., as the $*$-restriction of the external product

$$\mathrm{IC}_{\overline{\mathrm{Bun}}_{B^-,n_1}} \boxtimes \mathcal{P}_k \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_{B,n_2}}$$

from the product space

$$\overline{\mathrm{Bun}}_{B^-,n_1} \times X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2}$$

to the fiber product

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right),$$

shifted by $[-\dim \mathrm{Bun}_T]$ and twisted by $(-\frac{\dim \mathrm{Bun}_T}{2})$. Since the IC-sheaf of $\overline{\mathrm{Bun}}_B$ is constant for $G = \mathrm{SL}_2$ by Section 2.4.3, we can rephrase the definition of $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ as

follows. Let g denote the genus of the curve X and let s_k denote the integer

$$s_k \; := \; \dim \overline{\mathrm{Bun}}_{B^-,n_1} + \dim \overline{\mathrm{Bun}}_{B,n_2} - \dim \mathrm{Bun}_T \; = \; 3g - 3 + 2k \,.$$

Furthermore let

$$p_{n_1,k,n_2} : \; \overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right) \; \longrightarrow \; X^{(k)}$$

denote the natural forgetful map. Then we can equivalently define:

$$\widetilde{\mathcal{P}}_{n_1,k,n_2} \; := \; p^*_{n_1,k,n_2} \mathcal{P}_k \, [s_k](\tfrac{s_k}{2})$$

Finally, the action of the Lefschetz-$\mathfrak{sl}_2$ on the Picard-Lefschetz oscillator $\mathcal{P}_k$ induces an analogous action on $\widetilde{\mathcal{P}}_{n_1,k,n_2}$.

3.3.2. *Main theorem about nearby cycles.* We can now state our main theorem about nearby cycles for VinBun_G. Recall from Section 2.2 that the G-locus $\mathrm{VinBun}_{G,G}$ is smooth, so that its IC-sheaf is constant up to shifts and twists. Applying the nearby cycles functor Ψ to this shifted constant sheaf and passing to the associated graded of its weight-monodromy filtration, we obtain a perverse sheaf on the B-locus $\mathrm{VinBun}_{G,B}$ carrying the monodromy action of the Lefschetz $\mathfrak{sl}_2$. The result then is:

Theorem 3.3.3. *There exists an isomorphism of perverse sheaves*

$$\mathrm{Gr}\, \Psi(\mathrm{IC}_{\mathrm{VinBun}_{G,G}}) \; \cong \; \bigoplus_{(n_1,k,n_2)} \bar{f}_{n_1,k,n_2,*} \, \widetilde{\mathcal{P}}_{n_1,k,n_2}$$

which identifies the action of the Lefschetz-$\mathfrak{sl}_2$ on the right hand side via the Picard-Lefschetz oscillators with the monodromy action on the left hand side. As before the direct sum runs over all triples (n_1,k,n_2) with $n_1, n_2 \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geqslant 0}$, and $n_1 = n_2 - k$.

3.3.4. *Remark.* By Section 2.4 above, the summands $\bar{f}_{n_1,k,n_2,*}\ \widetilde{\mathcal{P}}_{n_1,k,n_2}$ on the right hand side of Theorem 3.3.3 are equal to the intermediate extension of the perverse sheaf

$$\mathrm{IC}_{\mathrm{Bun}_{B^-,n_1}} \underset{\mathrm{Bun}_T}{\boxtimes} \mathcal{P}_k \boxtimes \mathrm{IC}_{\mathrm{Bun}_{B,n_2}}$$

from the stratum $_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B}$ to its closure in $\mathrm{VinBun}_{G,B}$.

3.4. **Intersection cohomology of** VinBun_G.

To state our main theorem about the IC-sheaf of VinBun_G, we introduce the following notation. Given a representation ρ of the symmetric group S^k, we denote by $\mathrm{IC}(\rho)$ the IC-extension of the corresponding local system on the disjoint locus of $X^{(k)}$. Furthermore, using the same notation as in the definition of $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ above, we define

$$\widetilde{\mathrm{IC}}(\rho)_{n_1,k,n_2} \quad := \quad \mathrm{IC}_{\overline{\mathrm{Bun}}_{B^-,n_1}} \underset{\mathrm{Bun}_T}{\boxtimes} \mathrm{IC}(\rho) \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_{B,n_2}}$$

on the fiber product

$$\overline{\mathrm{Bun}}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \overline{\mathrm{Bun}}_{B,n_2} \right).$$

Using the projection maps p_{n_1,k,n_2} from Section 3.3.1 above one can equivalently define

$$\widetilde{\mathrm{IC}}(\rho)_{n_1,k,n_2} \quad := \quad p_{n_1,k,n_2}^* \, \mathrm{IC}(\rho) \, [s_k](\tfrac{s_k}{2})$$

where the integers s_k are defined as in Section 3.3.1 above. We can then state:

Theorem 3.4.1. *The associated graded with respect to the weight filtration of the restriction* $\mathrm{IC}_{\mathrm{VinBun}_G}\big|^{*}_{\mathrm{VinBun}_{G,B}}[-1](-\tfrac{1}{2})$ *is equal to:*

$$\bigoplus_{(n_1,k,n_2,r)} \bar{f}_{n_1,k,n_2,*}\ \widetilde{\mathrm{IC}}(\rho_{k-r,r})_{n_1,k,n_2} \otimes \overline{\mathbb{Q}}_\ell(\tfrac{k}{2}-r)\,.$$

Here we denote by $\rho_{(k-r,r)}$ *the irreducible representation of* S_k *corresponding to the Young diagram with* $k-r$ *boxes in the first column and* r *boxes in the second column. The direct sum runs over all quadruples* (n_1,k,n_2,r) *where* $n_1, n_2 \in \mathbb{Z}$ *and* $k,r \in \mathbb{Z}_{\geqslant 0}$, *satisfying that* $n_1 = n_2 - k$ *and* $0 \leqslant r \leqslant \tfrac{k}{2}$.

Theorem 3.4.1 yields an explicit answer for the primitive parts P_i; in general it is however not clear how to compute the IC-stalks from the P_i. In the present situation it is however possible, due to a geometric fact visible on the level of the local models we will construct in Section 4 below. For this reason we comment on the computation of IC-stalks in Remark 7.1.7 below.

3.5. Stalks of the $*$-extension.

The next result is related to the study via $\overline{\mathrm{Bun}}_G$ of Drinfeld's and Gaitsgory's "miraculous duality" functor (see [G2] for the latter). To state it, let

$$j_G : \mathrm{VinBun}_{G,G} \longhookrightarrow \mathrm{VinBun}_G$$

denote the open immersion of the G-locus, and let i_{n_1,k,n_2} denote the inclusion of the stratum

$$_{(n_1,k,n_2)}\mathrm{VinBun}_{G,B} \;=\; \mathrm{Bun}_{B^-,n_1} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times \mathrm{Bun}_{B,n_2} \right)$$

into the B-locus $\mathrm{VinBun}_{G,B}$. Our result then expresses the $*$-restriction along i_{n_1,k,n_2} of the $*$-extension along j_G of the constant sheaf on the G-locus $\mathrm{VinBun}_{G,G}$ in terms of a certain complex $\widetilde{\Omega}_k$. This complex has already appeared implicitly in the work [BG2] of Braverman and Gaitsgory, and is defined as follows.

3.5.1. *Definition of* $\widetilde{\Omega}_k$. Let $_0Z^k$ denote the *open Zastava space* from [FFKM], [BFGM]; see Section 4.1.5 below for its definition. As is explained in Section 4.1.5, the space $_0Z^k$ is smooth and comes equipped with a projection map to the k-th symmetric power of the curve

$$\pi_Z : \; _0Z^k \; \longrightarrow \; X^{(k)} \, .$$

We then define $\widetilde{\Omega}_k$ as the pushforward

$$\widetilde{\Omega}_k \; := \; \pi_{Z,!}\big(\mathrm{IC}_{_0Z^k}\big) \; = \; \pi_{Z,!}\big((\overline{\mathbb{Q}}_\ell)_{_0Z^k}[\dim_{_0Z^k}](\tfrac{1}{2}\dim_{_0Z^k})\big) \, .$$

We refer the reader to Section 6.4.1 below for a more detailed discussion of the complex $\widetilde{\Omega}_k$. We will in fact express the above $*$-restriction in terms of the Verdier dual

$$\mathbb{D}\,\widetilde{\Omega}_k \; = \; \pi_{Z,*}\big(\mathrm{IC}_{_0Z^k}\big)$$

of $\widetilde{\Omega}_k$.

Using the same notation as before, our result reads:

Theorem 3.5.2. *The $*$-restriction of the $*$-extension of the IC-sheaf of the G-locus*

$$i^*_{n_1,k,n_2}\, j_{G,*}\,\mathrm{IC}_{\mathrm{VinBun}_{G,G}}$$

is equal to

$$\mathrm{IC}_{\mathrm{Bun}_{B^-,n_1}} \underset{\mathrm{Bun}_T}{\boxtimes} \left(\left(\mathbb{D}\widetilde{\Omega}_k[2k](k) \otimes H^*(\mathbb{A}^1 \smallsetminus \{0\})[1](\tfrac{1}{2}) \right) \boxtimes \mathrm{IC}_{\mathrm{Bun}_{B,n_2}} \right).$$

Following a suggestion of Drinfeld, we have also used this theorem to give an application on the level of functions via the sheaf-function correspondence for ℓ-adic sheaves; see Section 8 below. Furthermore, using Koszul duality for nearby cycles one can deduce from Theorem 3.5.2 above a simple description of the stalks of the nearby cycles Ψ in terms of the complex $\widetilde{\Omega}$, which in turn yields an application to the work of Sakellaridis and Venkatesh. This will be discussed, in the context of an arbitrary reductive group, in the forthcoming article [Sch].

4. Local models for VinBun$_G$

4.1. The absolute and relative local models.

4.1.1. *Definition of the relative model.* Let $n \in \mathbb{Z}_{\geqslant 1}$. An S-point of the *relative local model* Y_{rel}^n consists of the data of a triple (E_1, E_2, φ) on $X \times S$ as in the definition of VinBun_G, together with a line subbundle $L_1 \hookrightarrow E_1$ and a line quotient bundle $E_2 \twoheadrightarrow L_2$, satisfying the following conditions: For every geometric point $\bar{s} \to S$ we require the restriction to $X \times \bar{s}$ of the composite map

$$L_1 \hookrightarrow E_1 \xrightarrow{\varphi} E_2 \twoheadrightarrow L_2$$

to be an isomorphism generically on the curve $X \times \bar{s}$. Furthermore, for each $\bar{s} \to S$ we require the resulting injection of line bundles

$$L_1|_{X \times \bar{s}} \hookrightarrow L_2|_{X \times \bar{s}}$$

to be of relative degree n, i.e., we require it to correspond to an effective divisor of degree n on the curve $X \times \bar{s}$. Note that these conditions in particular imply that the composite map $L_1 \to L_2$ is an injection of coherent sheaves on $X \times S$.

4.1.2. *Definition of the absolute model.* Next consider the natural map

$$Y_{rel}^n \longrightarrow \mathrm{Bun}_T$$

defined by remembering only the line bundle L_2. We define the *absolute local model* Y^n as the fiber of this map over the trivial line bundle $\mathcal{O}_X$; i.e., the absolute model Y^n is obtained from the relative model Y_{rel}^n by requiring the "background" line bundle L_2 to be the trivial line bundle. It is not hard to see that the absolute local model Y^n is in fact a scheme.

4.1.3. *More definitions.* The following definitions and notation apply to both Y^n and Y_{rel}^n. We only state them for Y^n, the case of Y_{rel}^n being analogous. By construction the space Y^n admits a forgetful map to VinBun_G, and in particular a natural maps

to $\mathbb{A}^1 = T^+_{adj}$. Using the latter map we define the *G-locus* Y^n_G and the *B-locus* Y^n_B as for VinBun$_G$ in Section 2.2 above. The *defect-free locus* $_0Y^n$ of Y^n is defined exactly as for VinBun$_G$; i.e., it is the inverse image of $_0$VinBun$_G$ under the forgetful map $Y^n \to$ VinBun$_G$. As for VinBun$_G$ we have:

Lemma 4.1.4. *The restriction of the map $Y^n_G \to \mathbb{A}^1$ to the defect-free locus $_0Y^n_G$ is smooth; in particular the open subscheme $_0Y^n_G$ of Y^n is smooth.*

The space Y^n furthermore admits a natural projection map to the n-th symmetric power of the curve

$$\pi : \ Y^n \ \longrightarrow \ X^{(n)}.$$

Namely, recall that an S-point of $X^{(n)}$ consists of a line bundle L on $X \times S$ together with a map of coherent sheaves $L \to \mathcal{O}_{X \times S}$ which is injective of relative degree n whenever restricted to $X \times \bar{s}$ for every geometric point $\bar{s} \to S$. The map π is then defined by only remembering the composite map of line bundles $L_1 \to L_2 = \mathcal{O}_{X \times S}$.

As for VinBun$_{G,B}$ every k-point in the B-locus Y^n_B admits a unique factorization

$$L_1 \ \lhook\joinrel\longrightarrow \ E_1 \longrightarrow\!\!\!\!\rightarrow M_1 \longrightarrow M_2 \ \lhook\joinrel\longrightarrow E_2 \ \longrightarrow\!\!\!\!\rightarrow L_2$$

with notation as above. As before we call the effective divisor corresponding to $M_1 \hookrightarrow M_2$ the *defect divisor* and its degree the *defect*. The B-locus Y^n_B is stratified according to defect degrees just like VinBun$_{G,B}$. To state the analogous result, we first recall:

4.1.5. *Zastava spaces.* In [FFKM], [BFGM] certain local models for the relative compactifications $\overline{\text{Bun}}_B$ from Section 2.4, the *Zastava spaces*, were introduced. We recall now their definition for $G = \text{SL}_2$.

Let $n \in \mathbb{Z}_{\geqslant 0}$. Then an S-point of the relative Zastava space Z_{rel}^n consists of an S-point $L \hookrightarrow E$ of $\overline{\mathrm{Bun}}_B$ together with a "background" line bundle L' on $X \times S$ and a surjection $E \twoheadrightarrow L'$, subject to the following conditions. First, one requires that for every geometric point $\bar{s} \to S$ the restriction of the composite map

$$L \longhookrightarrow E \longtwoheadrightarrow L'$$

to $X \times \bar{s}$ is an isomorphism generically on the curve $X \times \bar{s}$. Second, the resulting injective map of line bundles $L \hookrightarrow L'$ on $X \times S$ is required to be of relative degree n on each $X \times \bar{s}$.

As before one defines an absolute version Z^n of Z_{rel}^n by forcing the "background" line bundle L' to be the trivial line bundle. The absolute Zastava space Z^n is in fact a scheme. Next, the notation

$$Z_{(\mathrm{Bun}_T, d)}^n$$

refers to the relative version, but with the degree of the "background" line bundle L' being required to be equal to the integer d. Similarly, for the opposite Borel B^-, the space

$$Z_{(\mathrm{Bun}_T, -n)}^{-,n}$$

parametrizes the data

$$L' \longhookrightarrow E \longrightarrow L$$

where now L' is a "background" line subbundle of E of degree $-n$, and the composite map $L' \to L$ is required to be an isomorphism generically on X of relative degree n.

We denote by $_0Z^n$ the open subscheme of the absolute Zastava space obtained by requiring that the injection $L \hookrightarrow E$ is in fact a subbundle map, and similary for the relative versions. We will refer to $_0Z^n$ and its relative versions as the *open Zastava*

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space. Finally, the absolute and relative Zastava spaces afford natural maps

$$\pi_Z : \ Z^n \ \longrightarrow \ X^{(n)}$$

defined as in Section 4.1.3 above; the absolute Zastava spaces are in fact *factorizable*, in the sense of Section 5.1, with respect to these maps.

We refer the reader to [FM], [FFKM], and [BFGM] for more background on the Zastava spaces.

4.1.6. *The G-locus of Y^n in terms of Zastava spaces.* Let $Y^n_{c=1}$ denote the fiber of the natural map $Y^n \to \mathbb{A}^1$ over the element $c = 1 \in \mathbb{A}^1$. Then directly from the definitions one sees that $Y^n_{c=1}$ agrees with the open Zastava space $_0 Z^n$. In fact we have the following analog for the G-locus Y^n_G of Lemma 2.2.1 for VinBun$_G$. Given an S-point

$$L_1 \ \overset{\propto}{\longleftrightarrow} \ E_1 \ \overset{\cong}{\longrightarrow} \ E_2 \ \longrightarrow\!\!\!\!\to \ \mathcal{O}_{X \times S}$$

of Y^n_G we can define an S-point

$$L_1 \ \overset{\propto}{\longleftrightarrow} \ E_1 \ \longrightarrow\!\!\!\!\to \ \mathcal{O}_{X \times S}$$

of $_0 Z^n$ by composing the last two maps; furthermore, we obtain the S-point $\det \varphi$ of $\mathbb{A}^1$. Then Lemma 2.2.1 above implies:

Lemma 4.1.7. *The natural map*

$$Y^n_G \ \longrightarrow \ _0 Z^n \times (\mathbb{A}^1 \smallsetminus \{0\})$$

defined by the above association is an isomorphism. Under this isomorphism, the natural map $Y^n_G \to \mathbb{A}^1 \smallsetminus \{0\}$ corresponds on the right hand side to the projection onto the second factor. The projection map $\pi : Y^n_G \to X^{(n)}$ corresponds on the

right hand side to the projection onto the first factor followed by the projection map
$\pi_Z : {}_0Z^n \to X^{(n)}$.

4.1.8. *Stratification of the B-locus of the local model.* The stratification of the *B*-locus of VinBun_G from Proposition 2.3.3 above takes the following form for the local model Y^n. Let $n_1, k, n_2 \in \mathbb{Z}_{\geqslant 0}$ be non-negative integers satisfying $n_1 + k + n_2 = n$. Then as in Section 2.3.2 above there exist natural maps

$$\bar{f}_{n_1,k,n_2} : \quad Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times Z^{n_2} \right) \quad \longrightarrow \quad Y^n_B,$$

and the analogous result is:

Corollary 4.1.9. *The maps $\bar{f}_{n_1,k,n_2}$ are proper, and their restrictions*

$$f_{n_1,k,n_2} : \quad {}_0Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times {}_0Z^{n_2} \right) \quad \longrightarrow \quad Y^n_B$$

are isomorphisms onto smooth locally closed substacks

$$_{(n_1,k,n_2)}Y^n_B \quad \longhookrightarrow \quad Y^n_B \, .$$

As the triples (n_1, k, n_2) range over all triples of non-negative integers satisfying $n_1 + k + n_2 = n$, the substacks $_{(n_1,k,n_2)}Y^n_B$ form a stratification of the B-locus Y^n_B. On the level of k-points the closure of a stratum is equal to the finite disjoint union of strata

$$_{(n_1,k,n_2)}\overline{Y^n_B} \quad = \quad \bigcup_{\substack{d_1 \geqslant 0 \\ d_2 \geqslant 0}} {}_{(n_1-d_1,k+d_1+d_2,n_2-d_2)}Y^n_B$$

Given any non-negative integer $k \in \mathbb{Z}_{\geqslant 0}$, the locus $_{\leqslant k}Y^n_B$ in Y^n_B obtained by requiring the defect to be at most k is open in Y^n_B. The locus $_kY^n_B$ obtained by requiring the

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defect to be exactly k is locally closed, and isomorphic as schemes to the disjoint union

$$_kY_B^n \quad = \quad \coprod_{n_1,n_2} {}_{(n_1,k,n_2)}Y_B^n \, .$$

Finally, the locus $_nY_B^n$ of maximal defect is closed in Y_B^n and isomorphic to the symmetric power $X^{(n)}$.

4.2. Restatements of the main theorems for the local models.

We claim that to prove the main theorem about nearby cycles, Theorem 3.3.3 above, it suffices to establish its analog for the absolute local models Y^n, for all integers $n \geqslant 0$. This can be shown analogously as in the work [BFGM], where a similar interplay between Drinfeld's compactification $\overline{\mathrm{Bun}}_B$ and the Zastava spaces is used. In our context, it can be achieved by first comparing the absolute and the relative local model to each other; the relative model then allows for a direct comparison to VinBun_G.

To the state this analog of Theorem 3.3.3 for Y^n, let $n_1, k, n_2 \in \mathbb{Z}_{\geqslant 0}$ be non-negative integers satisfying $n_1 + k + n_2 = n$, and recall the compactified maps

$$\bar{f}_{n_1,k,n_2} : \quad Z^{-,n_1}_{(\mathrm{Bun}_T,-n)} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times Z^{n_2} \right) \quad \longrightarrow \quad Y_B^n$$

from Section 4.1.8. Similarly as before let

$$\widetilde{\mathcal{P}}_{n_1,k,n_2} \quad := \quad \mathrm{IC}_{Z^{-,n_1}_{\mathrm{Bun}_T,-n}} \underset{\mathrm{Bun}_T}{\boxtimes} \mathcal{P}_k \boxtimes \mathrm{IC}_{Z^{n_2}}$$

denote the $*$-restriction of the external product

$$\mathrm{IC}_{Z^{-,n_1}_{\mathrm{Bun}_T,-n}} \boxtimes \mathcal{P}_k \boxtimes \mathrm{IC}_{Z^{n_2}}$$

from the product space

$$Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \times X^{(k)} \times Z^{n_2}$$

to the fiber product

$$Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times Z^{n_2} \right),$$

shifted by $[\dim \mathrm{Bun}_T]$ and twisted by $(\frac{\dim \mathrm{Bun}_T}{2})$. Since the Zastava spaces are smooth for $G = \mathrm{SL}_2$, we can equivalently define $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ as

$$\widetilde{\mathcal{P}}_{n_1,k,n_2} \quad = \quad p^*_{n_1,k,n_2} \, \mathcal{P}_k \, [2n - 2k](n - k)$$

where similarly to above we denote by

$$p_{n_1,k,n_2} : \quad Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times Z^{n_2} \right) \quad \longrightarrow \quad X^{(k)}$$

the forgetful map. The analog of Theorem 3.3.3 then reads:

Theorem 4.2.1. *There exists an isomorphism of perverse sheaves*

$$\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y^n_G}) \quad \cong \quad \bigoplus_{(n_1,k,n_2)} \bar{f}_{n_1,k,n_2,*} \, \widetilde{\mathcal{P}}_{n_1,k,n_2}$$

which identifies the action of the Lefschetz-$\mathfrak{sl}_2$ on the right hand side via the Picard-Lefschetz oscillators with the monodromy action on the left hand side. Here the direct sum runs over all triples (n_1, k, n_2) of non-negative integers satisfying $n_1 + k + n_2 = n$.

5. GEOMETRY OF THE LOCAL MODELS

5.1. **Factorization in families.**

Unlike the Beilinson-Drinfeld affine Grassmannian (see [BD1]) or the Zastava spaces (see [BFGM]), the local models Y^n are not literally factorizable. Instead, they are *factorizable in families*, i.e., the fibers of the map $Y^n \to \mathbb{A}^1$ are factorizable in a compatible way:

5.1.1. *Factorization in families.* The spaces Y^n are *factorizable in families* in the sense of the following lemma.

Proposition 5.1.2. *For any integers $n_1 + n_2 = n$ the natural map*

$$X^{(n_1)} \overset{\circ}{\times} X^{(n_2)} \longrightarrow X^{(n)}$$

defined by adding effective divisors induces a cartesian square

$$
\begin{array}{ccc}
Y^{n_1} \overset{\circ}{\underset{\mathbb{A}^1}{\times}} Y^{n_2} & \longrightarrow & Y^n \\
{\scriptstyle \pi_{n_1} \times \pi_{n_2}} \downarrow & & \downarrow {\scriptstyle \pi_n} \\
X^{(n_1)} \overset{\circ}{\times} X^{(n_2)} & \longrightarrow & X^{(n)}
\end{array}
$$

where the top horizontal arrow commutes with the natural maps to $\mathbb{A}^1$.

Broadly speaking, Proposition 5.1.2 follows from the fact that generically on the curve X, the datum of a point of Y^n is the trivial datum except for the determinant of the middle map φ. More precisely, Proposition 5.1.2 will be a direct consequence of the following easy lemma:

Lemma 5.1.3. *Let k be a non-negative integer, let*

$$L \xleftarrow{\;\;\;\;} E_1 \xrightarrow{\;\varphi\;} E_2 \twoheadrightarrow \mathcal{O}_{X \times S}$$

be an S-point of the local model Y^k, and let

$$d := \det(\varphi) \;\in\; \Gamma(X \times S, \mathcal{O}_{X \times S}) \;=\; \Gamma(S, \mathcal{O}_S) \;=\; \mathbb{A}^1(S)$$

denote its image under the usual map $Y^k \to \mathbb{A}^1$. Furthermore let $U \subset X \times S$ denote the dense open subscheme of $X \times S$ on which the composite map $L \to \mathcal{O}_{X \times S}$ is an isomorphism. Then over U the data of the above S-point takes the simple form

$$\mathcal{O}_U \xleftarrow{\;i_1\;} \mathcal{O}_U \oplus \mathcal{O}_U \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & d|_U \end{pmatrix}} \mathcal{O}_U \oplus \mathcal{O}_U \xrightarrow{\;pr_1\;} \mathcal{O}_U \,.$$

Proof. Composing the middle map φ either with the rightmost or the leftmost arrow we obtain the splittings

$$L|_U \hookrightarrow E_1|_U \twoheadrightarrow \mathcal{O}_U \qquad \text{and} \qquad L|_U \hookrightarrow E_2|_U \twoheadrightarrow \mathcal{O}_U$$

over the open subscheme U. These splittings in turn induce trivializations of the SL_2-bundles E_1 and E_2 which are compatible with the middle map φ, so that φ must be of the matrix form as above, but with an a priori unknown entry in the lower right corner. The fact that $\det(\varphi) = d$ on $X \times S$ however forces the entry in the lower right corner to be equal to $d|_U \in \Gamma(U, \mathcal{O}_U)$. $\qquad\square$

We can now prove Proposition 5.1.2:

Proof of Proposition 5.1.2. We need to construct a natural isomorphism

$$Y^{n_1} \overset{\circ}{\underset{\mathbb{A}^1}{\times}} Y^{n_2} \cong \left(X^{(n_1)} \overset{\circ}{\times} X^{(n_2)} \right) \underset{X^{(n)}}{\times} Y^n$$

which respects the forgetful maps to $X^{(n_1)} \overset{\circ}{\times} X^{(n_2)}$ and to $\mathbb{A}^1$. To do so, let us first define a map from the right hand side to the left hand side. Thus we are given an S-point

$$L \overset{\alpha}{\longleftrightarrow} E_1 \overset{\varphi}{\longrightarrow} E_2 \longrightarrow\!\!\!\!\rightarrow \mathcal{O}_{X \times S}$$

of Y^n, an S-point $L_1 \hookrightarrow \mathcal{O}_{X \times S}$ of $X^{(n_1)}$, and an S-point $L_2 \hookrightarrow \mathcal{O}_{X \times S}$ of $X^{(n_2)}$, such that the subsheaf $L_1 \otimes L_2 \hookrightarrow \mathcal{O}_{X \times S}$ coincides with the subsheaf $L \hookrightarrow \mathcal{O}_{X \times S}$ obtained from the S-point of Y^n. Let

$$d := \det(\varphi) \in \Gamma(X \times S, \mathcal{O}_{X \times S}) = \Gamma(S, \mathcal{O}_S),$$

and let U, U_1, U_2 denote the open subschemes of $X \times S$ on which the maps

$$L \longrightarrow \mathcal{O}_{X \times S}, \quad L_1 \longrightarrow \mathcal{O}_{X \times S}, \quad L_2 \longrightarrow \mathcal{O}_{X \times S}$$

are isomorphisms. Then by definition of the right hand side we have

$$U_1 \cap U_2 = U \quad \text{and} \quad U_1 \cup U_2 = X \times S.$$

We now define an S-point of Y^{n_1} by gluing together the required data on U_1 and U_2. Namely, on the one hand we restrict the datum

$$L \overset{\alpha}{\longleftrightarrow} E_1 \overset{\varphi}{\longrightarrow} E_2 \longrightarrow\!\!\!\!\rightarrow \mathcal{O}_{X \times S}$$

to the open subscheme U_2, and on the other hand we consider the datum

$$\mathcal{O}_{U_1} \overset{i_1}{\longleftrightarrow} \mathcal{O}_{U_1} \oplus \mathcal{O}_{U_1} \overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & d|_{U_1} \end{smallmatrix} \right)}{\longrightarrow} \mathcal{O}_{U_1} \oplus \mathcal{O}_{U_1} \overset{pr_1}{\longrightarrow\!\!\!\!\rightarrow} \mathcal{O}_{U_1}$$

over the open subscheme U_1. By Lemma 5.1.3, these two data agree on the intersection $U_1 \cap U_2 = U$, and thus can be glued to form an S-point of $Y^{(n_1)}$. We construct an S-point of $Y^{(n_2)}$ analogously, and by construction they together form an S-point of the left hand side as desired.

We define a map from the left hand side to the right hand side in a similar fashion: Consider the S-points of Y^{n_1} and Y^{n_2} arising from a given S-point of the left hand side, and let $d \in \Gamma(S, \mathcal{O}_S)$ be their common image in $\mathbb{A}^1(S)$. These S-points of Y^{n_1} and Y^{n_2} give rise to open subschemes U_1 and U_2 of $X \times S$ defined exactly as in Lemma 5.1.3, and by definition of the left hand side we have that $U_1 \cup U_2 = X \times S$. Furthermore, by Lemma 5.1.3 the restriction of the data on $X \times S$ comprising the S-point of Y^{n_1} to the intersection $U_1 \cap U_2$ agrees with the restriction of the data comprising the S-point of Y^{n_2}. Thus the two data can be glued to form an S-point of Y^n, and we have constructed the converse map. Finally, it is immediate from the constructions that the two maps are inverse to each other and respect the forgetful maps to $X^{(n_1)} \overset{\circ}{\times} X^{(n_2)}$ and to $\mathbb{A}^1$. $\qquad\square$

5.1.4. *Factorization of the fibers.* For a scalar $c \in \mathbb{A}^1$ let Y_c^n denote the fiber of the map $Y^n \to \mathbb{A}^1$ over c. Thus $Y_{c=0}^n$ is equal to the B-locus Y_B^n of Y^n, and $Y_{c=1}^n$ is equal to the open Zastava space $_0Z^n$ from Section 4.1.5 above. Furthermore, since the top horizontal arrow in Proposition 5.1.2 commutes with the natural maps to $\mathbb{A}^1$, we find:

Corollary 5.1.5. *The spaces Y_c^n are factorizable in the usual sense, i.e., the addition of effective divisors induces a cartesian square*

$$
\begin{array}{ccc}
Y_c^{n_1} \overset{\circ}{\times} Y_c^{n_2} & \longrightarrow & Y_c^n \\
{\scriptstyle \pi_{n_1} \times \pi_{n_2}} \downarrow & & \downarrow {\scriptstyle \pi_n} \\
X^{(n_1)} \overset{\circ}{\times} X^{(n_2)} & \longrightarrow & X^{(n)}
\end{array}
$$

In particular, the B-locus Y_B^n is factorizable in the usual sense.

5.2. **Embedding, section, and contraction.**

In this section we construct a $\mathbb{G}_m$-action on Y^n which contracts Y^n onto a section of the projection map $\pi : Y^n \to X^{(n)}$. This action can be constructed in various ways; here we construct it via a specific embedding of Y^n into a product of Beilinson-Drinfeld affine Grassmannians which we discuss first. This embedding will also be used in Section 5.3 below to derive explicit equations for the local models Y^n.

5.2.1. *Embeddings for Zastava spaces.* Let $\mathrm{Gr}_G^n \longrightarrow X^{(n)}$ denote the Beilinson-Drinfeld affine Grassmannian for $G = \mathrm{SL}_2$, which parametrizes SL_2-bundles on the curve X together with a trivialization away from an effective divisor of degree n. Recall from [BFGM] that the absolute Zastava space Z^n from Section 4.1.5 affords a natural locally closed embedding

$$Z^n \longhookrightarrow \mathrm{Gr}_G^n$$

which commutes with the natural projections to $X^{(n)}$. On k-points, this embedding associates to a point

$$L \longhookrightarrow E \twoheadrightarrow \mathcal{O}_X$$

of Z^n the SL_2-bundle E together with the trivialization of E obtained by splitting the surjection $E \twoheadrightarrow \mathcal{O}_X$ away from the zero locus of the composite map $L \to \mathcal{O}_X$.

Next consider the Zastava space parametrizing the data

$$L' \longhookrightarrow E \longrightarrow \mathcal{O}_X$$

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with notation as in previous sections. Note that here the map on the right is allowed to have zeroes, the line bundle on the right is fixed to be $\mathcal{O}_X$, while the "background" line bundle L' on the left is allowed to vary. Unlike the absolute Zastava space from Section 4.1.5 above, this Zastava space is obtained from the relative Zastava space $_{\mathrm{Bun}_{T,-n}}Z^{-,n}$ by forcing the line bundle L "on the right" to be equal to $\mathcal{O}_X$; we denote this Zastava space by $\tilde{Z}^{-,n}$ for simplicity. An embedding of $\tilde{Z}^{-,n}$ into Gr_G^n is defined exactly as for Z^n.

5.2.2. *Sections for Zastava spaces.* Next we briefly review some constructions for the Zastava space Z^n from [BFGM]; we will use these constructions in Sections 5.2.8 and 5.2.11 below to make similar constructions for the local models Y^n. First, recall that the projection map

$$\pi_Z : \ Z^n \ \longrightarrow \ X^{(n)}$$

admits a natural section s_Z which on k-points sends an effective divisor D to the point

$$\mathcal{O}_X(-D) \xhookrightarrow{i_1} \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{pr_1} \mathcal{O}_X$$

of the Zastava space Z^n.

The case of $\tilde{Z}^{-,n}$ is analogous: The projection

$$\pi_{Z^-} : \ \tilde{Z}^{-,n} \ \longrightarrow \ X^{(n)}$$

admits a natural section s_{Z^-} defined by sending an effective divisor $D \in X^{(n)}$ to the point

$$\mathcal{O}_X(-D) \xrightarrow{i_1} \mathcal{O}_X(-D) \oplus \mathcal{O}_X(D) \xrightarrow{pr_1} \mathcal{O}_X$$

of $\tilde{Z}^{-,n}$.

5.2.3. *Contractions for Zastava spaces.* Next recall from [MV] that any cocharacter $\check{\lambda} : \mathbb{G}_m \to T$ naturally gives rise to an action of $\mathbb{G}_m$ on the Beilinson-Drinfeld affine Grassmannian Gr_G^n which leaves the forgetful map $\mathrm{Gr}_G^n \to X^{(n)}$ invariant. It is shown in [BFGM] that the $(-2\check{\rho})$-action of $\mathbb{G}_m$ preserves the subspace Z^n; moreover, it contracts Z^n onto the section s_Z, i.e., the action map extends to a map

$$\mathbb{A}^1 \times Z^n \longrightarrow Z^n$$

such that the composition

$$Z^n = \{0\} \times Z^n \hookrightarrow \mathbb{A}^1 \times Z^n \longrightarrow Z^n$$

is equal to the composition of projection and section

$$Z^n \xrightarrow{\pi_Z} X^{(n)} \xrightarrow{s_Z} Z^n .$$

The next lemma provides a modular interpretation of the $(-2\check{\rho})$-action of $\mathbb{G}_m$ on Z^n, which will be used below; it can be proven by chasing through the definitions.

Lemma 5.2.4. *The action of an element $a \in \mathbb{G}_m(S) = \Gamma(S, \mathcal{O}_S)^\times$ on an S-point*

$$L \xhookrightarrow{i} E \xrightarrow{p} \mathcal{O}_X$$

of the Zastava space Z^n via the $(-2\check{\rho})$-action of $\mathbb{G}_m$ yields the point

$$L \xhookrightarrow{a \cdot i} E \xrightarrow{\frac{1}{a} \cdot p} \mathcal{O}_X .$$

Similarly, the $(2\check{\rho})$-action of $\mathbb{G}_m$ on Gr_G^n contracts $\tilde{Z}^{-,n}$ onto the section s_{Z^-} from Section 5.2.2 above. Just as for Z^n we have the following modular interpretation:

Lemma 5.2.5. *The action of an element $a \in \mathbb{G}_m(S) = \Gamma(S, \mathcal{O}_S)^{\times}$ on an S-point*

$$L \overset{i}{\hookrightarrow} E \overset{p}{\longrightarrow} \mathcal{O}_X$$

of the Zastava space $\tilde{Z}^{-,n}$ via the $(2\check{\rho})$-action of $\mathbb{G}_m$ yields the point

$$L \overset{\frac{1}{a} \cdot i}{\hookrightarrow} E \overset{a \cdot p}{\longrightarrow} \mathcal{O}_X .$$

5.2.6. *Embeddings for the local models Y^n.* Combining the embeddings of Z^n and $\tilde{Z}^{-,n}$ from Section 5.2.1 above, we obtain a locally closed embedding

$$\tilde{Z}^{-,n} \underset{X^{(n)}}{\times} Z^n \hookrightarrow \mathrm{Gr}_G^n \underset{X^{(n)}}{\times} \mathrm{Gr}_G^n .$$

We now construct the embedding of Y^n mentioned above by in turn constructing a closed immersion

$$\tau : Y^n \hookrightarrow \tilde{Z}^{-,n} \underset{X^{(n)}}{\times} Z^n .$$

Namely, if

$$L \hookleftarrow E_1 \longrightarrow E_2 \twoheadrightarrow \mathcal{O}_{X \times S}$$

is an S-point of Y^n, we can on the one hand compose the middle map φ with the surjection on the right and obtain the S-point

$$L \hookleftarrow E_1 \longrightarrow \mathcal{O}_{X \times S}$$

of $\tilde{Z}^{-,n}$. On the other hand, composing φ with the subbundle map on the left yields an S-point

$$L \longrightarrow E_2 \twoheadrightarrow \mathcal{O}_{X \times S}$$

of Z^n, and by construction the two points in fact lie in the fiber product over $X^{(n)}$ above; we have thus defined the map τ.

Lemma 5.2.7. *The map τ is a closed immersion.*

Proof. Given an S-point of $\tilde{Z}^{-,n} \times_{X^{(n)}} Z^n$, represented by the outer rhombus in the next diagram, we show that there is at most one dotted arrow φ making both triangles commute.

$$
\begin{array}{ccc}
 & E_1 & \\
{}^{g_1}\nearrow & \Big\downarrow{\scriptstyle\varphi} & \searrow^{h_1} \\
L & & \mathcal{O}_{X\times S} \\
{}_{g_2}\searrow & \Big\downarrow & \nearrow_{h_2} \\
 & E_2 &
\end{array}
$$

Since the existence of such an arrow is a closed condition, this will prove the lemma. To prove the uniqueness of the dotted arrow, we form the difference $\delta : E_1 \to E_2$ of any given two such dotted arrows, and show that $\delta = 0$. Namely, by the commutativity assumptions for each dotted arrow, the map δ descends to a map $\bar{\delta} : E_1/L \to E_2$ whose composite with h_2 is 0. Thus the map $\bar{\delta}$ factors through the kernel of h_2, which itself is the trivial line bundle, and we need to show that the resulting map $E/L \to \mathcal{O}_{X\times S}$ is zero.

To do so, observe first that since g_1 is a subbundle map, the quotient E/L is itself a line bundle; its restriction to any $X \times \bar{s}$ has degree $n \geqslant 1$. We prove the above vanishing by showing that in fact the vector space of maps

$$
\mathrm{Hom}_{\mathcal{O}_{X\times S}}(E/L, \mathcal{O}_{X\times S}) \;=\; H^0(X \times S, (E/L)^*)
$$

vanishes, where $(E/L)^*$ denotes the dual line bundle of E/L. For the latter, it suffices to show that the sheaf pushforward $R^0 p_*((E/L)^*)$ along the projection map $p : X \times S \to S$ vanishes. By the theorem on cohomology and base change, this in

turn can be checked on the geometric fibers of the projection p, where it holds for degree reasons. $\square$

5.2.8. *The section for the local models Y^n.* Next we construct a section of the projection map

$$\pi : \ Y^n \ \longrightarrow \ X^{(n)}.$$

First recall that an S-point of $X^{(n)}$ consists of a line bundle L on $X \times S$ together with a map of coherent sheaves $L \to \mathcal{O}_{X \times S}$ which is injective of relative degree n whenever restricted to $X \times \bar{s}$ for every geometric point $\bar{s} \to S$. The latter condition automatically forces the map $L \to \mathcal{O}_{X \times S}$ to be injective. Furthermore, let L^* denote the dual line bundle of L on $X \times S$. Then we define the section

$$s : \ X^{(n)} \ \longrightarrow \ Y^n$$

by associating to an S-point $L \to \mathcal{O}_{X \times S}$ of $X^{(n)}$ the S-point

$$L \ \xrightarrow{i_1} \ L \oplus L^* \ \xrightarrow{\varphi} \ \mathcal{O}_{X \times S} \oplus \mathcal{O}_{X \times S} \ \xrightarrow{pr_1} \ \mathcal{O}_{X \times S}$$

of Y^n, where the map φ in the middle is defined as the composition

$$L \oplus L^* \ \xrightarrow{pr_1} \ L \ \lhook\joinrel\longrightarrow \ \mathcal{O}_{X \times S} \ \xrightarrow{i_1} \ \mathcal{O}_{X \times S} \oplus \mathcal{O}_{X \times S}.$$

It is clear from the definitions that the map s is indeed a section of π. Furthermore, by construction the section s factors through the B-locus Y_B^n of Y^n. In fact we have the following two lemmas, both of which follow easily from the definitions:

Lemma 5.2.9. *The section s induces an isomorphism of $X^{(n)}$ with the stratum of maximal defect ${}_nY_B^n$.*

Lemma 5.2.10. *The section $s_{Z^-} \times s_Z$ of the projection*

$$\pi_{Z^-} \times \pi_Z : \quad \tilde{Z}^{-,n} \underset{X^{(n)}}{\times} Z^n \longrightarrow X^{(n)}$$

induced by the sections s_Z and s_{Z^-} from Section 5.2.2 factors through the closed subspace Y^n, and in fact agrees with the section s.

5.2.11. *The contraction for the local models Y^n.* We now construct a $\mathbb{G}_m$-action on Y^n which contracts it onto the section s, in the sense of Section 5.2.3 above. One can construct this action in various ways; here we construct it using the embedding from Section 5.2.6. Namely, let us define a $\mathbb{G}_m$-action on the fiber product $\mathrm{Gr}_G^n \times_{X^{(n)}} \mathrm{Gr}_G^n$ by acting on the first factor via the cocharacter $2\check{\rho}$ of $G = \mathrm{SL}_2$ and on the second factor via the cocharacter $-2\check{\rho}$.

Lemma 5.2.12. *This $\mathbb{G}_m$-action preserves the locally closed subspace Y^n and contracts it onto the section s.*

Proof. We first show that the action indeed preserves Y^n. By Section 5.2.3, we need to show that an S-point of $\tilde{Z}^{-,n} \times_{X^{(n)}} Z^n$ which lies in Y^n still lies in Y^n after acting by an element $a \in \mathbb{G}_m(S) = \Gamma(S, \mathcal{O}_S)^\times$. In view of the embedding of Lemma 5.2.7 and the modular descriptions of Lemma 5.2.4 and Lemma 5.2.5, we have to show that if the outer rhombus of the diagram

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admits a dotted arrow φ as shown, then the same holds for the following rhombus:

$$
\begin{array}{ccc}
 & E_1 & \\
 \nearrow & \vdots & \searrow \\
\tfrac{1}{a}\cdot g_1 & & a\cdot h_1 \\
L & & \mathcal{O}_{X\times S} \\
a\cdot g_2 \searrow & \vdots & \nearrow \tfrac{1}{a}\cdot h_2 \\
 & E_2 &
\end{array}
$$

This can indeed be achieved by defining the dotted arrow as $a^2 \cdot \varphi$, and hence we have shown that Y^n is preserved by the $\mathbb{G}_m$-action. The second statement follows from the construction together with Lemma 5.2.10. $\qquad\square$

5.2.13. *The contraction principle and preservation of weights.* Having constructed a $\mathbb{G}_m$-action on Y^n which contracts Y^n onto the section s of the projection map π in the sense of Section 5.2.3 above, we arrive at the following consequences for the restriction along the section s. First, the well-known *contraction principle* (see for example [Br, Sec. 3] or [BFGM, Sec. 5]) for contracting $\mathbb{G}_m$-actions states:

Lemma 5.2.14. *For any $\mathbb{G}_m$-monodromic object $F \in D(Y^n)$ there exists a natural isomorphism*

$$ s^*F \;\cong\; \pi_* F . $$

Since by [BBD, Sec. 5] the $*$-pullback does not increase the weights and the $*$-pushforward does not decrease the weights, we obtain:

Corollary 5.2.15. *Let $F \in D(Y^n)$ be $\mathbb{G}_m$-monodromic, and assume in addition that F is pure of some weight w. Then the complex $s^*F = \pi_* F$ is again pure of weight w.*

5.3. **Explicit equations and generalized Picard-Lefschetz families.**

In this section we use the embedding τ from Section 5.2.6 to find explicit equations for the fibers of the projection $\pi : Y^n \to X^{(n)}$. We will primarily be concerned with the fiber of π over the point $nx \in X^{(n)}$. The case of a general point $\sum n_k x_k \in X^{(n)}$ follows from this case via factorization.

5.3.1. *Fibers of Zastava spaces.* Following for example [MV], we use the following notation for the semi-infinite orbits in the affine Grassmannian $\mathrm{Gr}_G = SL_2(k((t)))/SL_2(k[[t]])$. Given any integer $i \in \mathbb{Z}$ the $N(k((t)))$-orbit of the point

$$\begin{pmatrix} t^i & 0 \\ 0 & t^{-i} \end{pmatrix}$$

in Gr_G will be denoted by S^i, and its $N^-(k((t)))$-orbit by T^i. Using the modular interpretation of these orbits it is not hard to show (see for example [BFGM]):

Lemma 5.3.2. *By passing to the fibers over the point $nx \in X^{(n)}$, the embeddings of Z^n and $\tilde{Z}^{-,n}$ into Gr_G^n from Section 5.2.1 above induce identifications*

$$Z^n\big|_{nx} \quad \cong \quad \overline{S^n} \cap T^0$$

and

$$\tilde{Z}^{-,n}\big|_{nx} \quad \cong \quad S^n \cap \overline{T^0}\,.$$

To make the above intersections of semi-infinite orbits more explicit, we will write matrix representatives for elements of $\mathrm{Gr}_G = SL_2(k((t)))/SL_2(k[[t]])$. We have the following well-known lemma:

Lemma 5.3.3. *The following two maps are isomorphisms:*

(a)

$$\mathbb{A}^n \;\longrightarrow\; \overline{S^n} \cap T^0$$

$$(a_{-n}, \ldots, a_{-1}) \;\longmapsto\; \begin{pmatrix} 1 & 0 \\ \sum a_i t^i & 1 \end{pmatrix}$$

(b)

$$\mathbb{A}^n \;\longrightarrow\; S^n \cap \overline{T^0}$$

$$(b_0, \ldots, b_{n-1}) \;\longmapsto\; \begin{pmatrix} t^n & \sum b_i t^i \\ 0 & t^{-n} \end{pmatrix}$$

5.3.4. *The fibers of the local models Y^n.* Let $\mathbb{S}^n$ denote the fiber of the projection $\pi : Y^n \to X^{(n)}$ over the point $nx \in X^{(n)}$. Let $_0\mathbb{S}^n$ denote the *defect-free* open subscheme of $\mathbb{S}^n$, i.e., the open subscheme obtained by intersecting $\mathbb{S}^n$ with the defect-free locus $_0Y^n$ of Y^n.

We will now use the closed embedding τ from Section 5.2.6 to find equations for $\mathbb{S}^n$ and $_0\mathbb{S}^n$. In fact, when describing the embedding on the level of fibers, the exposition seems to be clearer if one at first uses a slight variant $\tilde{\tau}$ of the embedding τ where one slightly enlarges the target; we will remove this "redundancy" afterwards (see Corollary 5.3.7 below).

Namely, instead of τ we will at first use the closed embedding into the larger target

$$\tilde{\tau} : \quad Y^n \;\longhookrightarrow\; \tilde{Z}^{-,n} \underset{X^{(n)}}{\times} Z^n \times \mathbb{A}^1 \,,$$

where the map to the last factor $\mathbb{A}^1 = T_{adj}^+$ is the usual map $Y^n \to \mathbb{A}^1$. Thus over the point $nx \in X^{(n)}$ we obtain a closed embedding

$$\mathbb{S}^n \;\longhookrightarrow\; (S^n \cap \overline{T^0}) \times (\overline{S^n} \cap T^0) \times \mathbb{A}^1 \,.$$

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Denote by $\text{Mat}_{2 \times 2}$ the affine space of 2×2 matrices over k, i.e., the Vinberg semigroup of $G = \text{SL}_2$. Then one can verify directly from the modular interpretation of Y^n:

Lemma 5.3.5.

(a) The above embedding identifies $\mathbb{S}^n$ with the closed subscheme of the product

$$(S^n \cap \overline{T^0}) \times (\overline{S^n} \cap T^0) \times \mathbb{A}^1$$

consisting of those elements (M_1, M_2, d) which satisfy that

$$M_1^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix} \right) M_2 \ \in \ \text{Mat}_{2 \times 2}(k[\![t]\!]).$$

Note that this condition is indeed independent of the choice of representatives for M_1 and M_2.

(b) The open subscheme $_0\mathbb{S}^n$ of $\mathbb{S}^n$ is obtained by additionally requiring that evaluation of the matrix

$$M_1^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix} \right) M_2 \ \in \ \text{Mat}_{2 \times 2}(k[\![t]\!])$$

at $t = 0$ does not yield the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)$. Note that this condition is again independent of the choice of representatives for M_1 and M_2.

Using the isomorphisms of Lemma 5.3.3 above we obtain:

Lemma 5.3.6. *Via Lemma 5.3.3, consider $\mathbb{S}^n$ as a closed subscheme of the affine space $\mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^1$ with coordinates $(b_0, \ldots, b_{n-1}, a_{-n}, \ldots, a_{-1}, d)$. Then $\mathbb{S}^n$ is defined*

by the following n equations:

$$a_{-n}b_0 = d$$

$$a_{-n}b_1 + a_{-n+1}b_0 = 0$$

$$a_{-n}b_2 + a_{-n+1}b_1 + a_{-n+2}b_0 = 0$$

$$\vdots$$

$$a_{-n}b_{n-1} + a_{-n+1}b_{n-2} + \cdots + a_{-1}b_0 = 0$$

Proof. In the notation of Lemma 5.3.3 and Lemma 5.3.5, let

$$g = a_{-n}t^{-n} + \ldots + a_{-1}t^{-1},$$

$$f = b_0t^0 + \ldots + b_{n-1}t^{n-1},$$

$$M_1 = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} t^n & f \\ 0 & t^{-n} \end{pmatrix}.$$

We then have

$$M_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} M_2 = \begin{pmatrix} t^n & f \\ -gt^n & -gf+dt^{-n} \end{pmatrix}.$$

Observe that all matrix entries except the one in the lower right corner already lie in $k[\![t]\!]$ automatically. The entry in the lower right corner equals

$$-gf + dt^{-n} = dt^{-n} - \sum_{k=-n}^{n-2} \left(\sum_{i+j=k} a_i b_j \right) t^k$$

Thus the integrality condition of Lemma 5.3.5 (a) translates to the asserted equations.

□

Using the first equation in Lemma 5.3.6 one eliminates the last coordinate and obtains finally:

Corollary 5.3.7. *The scheme $\mathbb{S}^n$ is equal to the closed subscheme of the affine space $\mathbb{A}^n \times \mathbb{A}^n$ defined by the following $(n-1)$ equations:*

$$a_{-n}b_1 + a_{-n+1}b_0 = 0$$

$$a_{-n}b_2 + a_{-n+1}b_1 + a_{-n+2}b_0 = 0$$

$$\vdots$$

$$a_{-n}b_{n-1} + a_{-n+1}b_{n-2} + \cdots + a_{-1}b_0 = 0$$

For the open subscheme ${}_0\mathbb{S}^n$ of $\mathbb{S}^n$ we have:

Lemma 5.3.8. *The open subscheme ${}_0\mathbb{S}^n$ of $\mathbb{S}^n$ is obtained by removing from $\mathbb{S}^n$ the closed subscheme defined by additionally requiring that*

$$a_{-n} = 0 = b_0$$

and

$$a_{-n+1}b_{n-1} + \cdots + a_{-2}b_2 + a_{-1}b_1 = 0.$$

Proof. We continue to use the notation from the proof of Lemma 5.3.6, and simply write out the condition stated in Lemma 5.3.5 (b). Namely, evaluating the matrix

$$\begin{pmatrix} t^n & f \\ -gt^n & -gf + dt^{-n} \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(k[\![t]\!])$$

at $t = 0$ yields the matrix

$$\begin{pmatrix} 0 & b_0 \\ -a_{-n} & -\sum_i a_i b_i \end{pmatrix},$$

and the assertion follows. $\qquad\square$

In a similar fashion one easily checks:

Lemma 5.3.9. *In terms of the coordinates of Corollary 5.3.7, the composite map*

$$\mathbb{S}^n \longrightarrow Y^n \longrightarrow \mathbb{A}^1$$

sends a point $(a_i, b_j)_{i,j}$ to the scalar $a_{-n}b_0$. Furthermore, the contracting $\mathbb{G}_m$-action from Lemma 5.2.12 acts quadratically on each coordinate:

$$c \cdot (a_i, b_j)_{i,j} \;=\; (c^2 a_i, c^2 b_j)_{i,j}$$

5.3.10. *Equations for other fibers.* Let $\sum_{k=1}^m n_k x_k$ be a point of $X^{(n)}$, with the x_k distinct. Then since the space Y^n factorizes in families in the sense of Proposition 5.1.2, the fiber of Y^n over the point $\sum n_k x_k$ is equal to the iterated fiber product

$$\mathbb{S}^{n_1} \underset{\mathbb{A}^1}{\times} \mathbb{S}^{n_2} \underset{\mathbb{A}^1}{\times} \cdots \underset{\mathbb{A}^1}{\times} \mathbb{S}^{n_m} \,,$$

and one can write explicit equations for the latter space using Corollary 5.3.7 and Lemma 5.3.9.

5.3.11. *The classical Picard-Lefschetz situation for $\mathbb{S}^1$.* Specializing to $n = 1$ in Corollary 5.3.7, Lemma 5.3.8, and Lemma 5.3.9, we see that the family $\mathbb{S}^1 \to \mathbb{A}^1$ recovers the classical Picard-Lefschetz family of hyperbolas degenerating to a node: The space $\mathbb{S}^1$ is isomorphic to the affine plane $\mathbb{A}^2$ with the two coordinates (a_{-1}, b_0), and the map $\mathbb{S}^1 \to \mathbb{A}^1$ sends a point (a_{-1}, b_0) to the product $a_{-1} \cdot b_0 \in \mathbb{A}^1$. The B-locus $\mathbb{S}^1_B$ consists of the union of the two coordinate axes.

Next we make explicit the stratification of the B-locus $\mathbb{S}^1_B$ induced by the defect stratification of Y^1_B, using the analogous notation for the strata. By Corollary 4.1.9

above the B-locus $\mathbb{S}_B^1$ is stratified by the three strata

$$_{(1,0,0)}\mathbb{S}_B^1\,, \quad _{(0,1,0)}\mathbb{S}_B^1\,, \quad \text{and} \quad _{(0,0,1)}\mathbb{S}_B^1\,.$$

In terms of the Picard-Lefschetz family, the strata $_{(1,0,0)}\mathbb{S}_B^1$ and $_{(0,0,1)}\mathbb{S}_B^1$ form the two axes of the node $\mathbb{S}_B^1$, with the point of their intersection removed from both. Similarly, the stratum of maximal defect $_{(0,1,0)}\mathbb{S}_B^1$ corresponds to the point in which the axes meet. As prescribed by Corollary 4.1.9 above the closure of either of the strata $_{(1,0,0)}\mathbb{S}_B^1$ and $_{(0,0,1)}\mathbb{S}_B^1$ is obtained by adding the stratum of maximal defect $_{(0,1,0)}\mathbb{S}_B^1$.

6. Nearby cycles

In this section we prove the main theorem about nearby cycles for the local models, Theorem 4.2.1, and hence also Theorem 3.3.3 for VinBun_G, as explained in Section 4.2 above. The general structure of the argument occupies Sections 6.1 through 6.3 below. However, we postpone two key statements to separate sections later in the text, hoping that this might bring out the structure of the argument better than a logically linear proof. Moreover, each of the two statements requires proof techniques that are somewhat different from the present section, possibly justifying the separate treatment.

We will prove Theorem 4.2.1 by induction on the integer n appearing in its formulation. We remark that our inductive procedure will implicitly also show that the

full nearby cycles are in fact unipotent; this is needed to invoke the factorization of the nearby cycles (see for example [BB, Sec. 5]) during the induction step.

6.1. The base case $n = 1$.

We begin by establishing the base case $n = 1$ of the induction by an explicit calculation involving the geometry of the local models and the equations from Section 5.3 above.

Proposition 6.1.1. *Theorem 4.2.1 holds for the case $n = 1$.*

Proof. Since $n = 1$ the computation of $\mathrm{Gr}\Psi(\mathrm{IC}_{Y_G^1})$ reduces to the computation of $\mathrm{Gr}\Psi(\mathrm{IC}_{\mathbb{S}_G^1})$, where $\mathbb{S}_G^1$ denotes the G-locus of the fiber $\mathbb{S}^1$ of the projection map π studied in Section 5.3 above. We use the same notation as in Section 5.3.11 above for the stratification of $\mathbb{S}_B^1$ induced by the defect stratification of Y_B^1. We then have to show that on the B-locus $\mathbb{S}_B^1$ there exists an isomorphism

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{\mathbb{S}_G^1}) \quad \cong \quad \overline{\mathbb{Q}}_{\ell}{}_{(1,0,0)\overline{\mathbb{S}_B^1}}[1](\tfrac{1}{2}) \ \oplus \ \left(V \otimes \overline{\mathbb{Q}}_{\ell}{}_{(0,1,0)\mathbb{S}_B^1}\right) \ \oplus \ \overline{\mathbb{Q}}_{\ell}{}_{(0,0,1)\overline{\mathbb{S}_B^1}}[1](\tfrac{1}{2})$$

which respects the action of the Lefschetz-$\mathfrak{sl}_2$. To see this, recall first that the explicit description of $\mathbb{S}_B^1$ in coordinates in Section 5.3.11 above shows that the strata closures $_{(1,0,0)}\overline{\mathbb{S}_B^1}$ and $_{(0,0,1)}\overline{\mathbb{S}_B^1}$ form the two axes of the reducible node $\mathbb{S}_B^1$; similarly, the stratum $_{(0,1,0)}\mathbb{S}_B^1$ corresponds to the point in which the axes meet. Furthermore, by Section 5.3.11 above the family $\mathbb{S}^1 \to \mathbb{A}^1$ is precisely the Picard-Lefschetz family of hyperbolas, and hence the required calculation is precisely the assertion of Lemma 3.1.9 above. $\qquad\square$

6.2. **Reduction to the stratum of maximal defect.**

In the present and the next section we deal with the induction step from $n-1$ to n. The present section reduces the assertion of Theorem 4.2.1 to a simpler version which takes place entirely on the stratum of maximal defect $_nY_B^n$; the latter case will be established in the next section, modulo the two separate statements referred to earlier.

From now on we abbreviate the right hand side of Theorem 4.2.1 by

$$C_n \quad := \quad \bigoplus_{(n_1,k,n_2)} \bar{f}_{n_1,k,n_2,*} \, \widetilde{\mathcal{P}}_{n_1,k,n_2} \,.$$

We first record:

Lemma 6.2.1. *The perverse sheaf C_n is semisimple. The perverse sheaf $\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})$ becomes semisimple after forgetting the Weil structure.*

Proof. For C_n it suffices to show that the perverse sheaves $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ are semisimple since the compactified maps $\bar{f}_{n_1,k,n_2,*}$ are finite. This follows from Lemma 3.2.5 and the definition of $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ as a shifted and twisted pullback of the Picard-Lefschetz oscillator $\mathcal{P}_k$ in Section 3.3.1. The assertion about $\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})$ follows from Gabber's theorem (Proposition 3.1.4 above), together with the decomposition theorem of Beilinson, Berstein, and Deligne from [BBD] for pure perverse sheaves. $\qquad\square$

Over the course of proving Theorem 4.2.1 we will establish that $\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})$ is in fact semisimple also as a perverse Weil sheaf; this is of course a posteriori also a consequence of Theorem 4.2.1.

6.2.2. *Splitting according to loci of support.* Lemma 6.2.1 is already sufficient to split the perverse sheaves $\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})$ and C_n into direct sums according to where their simple constituents are supported: Namely, we can decompose

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G}) \;\;=\;\; \left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})\right)_{\mathrm{on}\,_nY^n_B} \bigoplus \left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})\right)_{\mathrm{not\,on}\,_nY^n_B}$$

where all simple constituents of the first summand are supported on the stratum $_nY^n_B$ and where all simple constituents of the second summand are not supported on $_nY^n_B$. Analogously we write

$$C_n \;\;=\;\; \left(C_n\right)_{\mathrm{on}\,_nY^n_B} \bigoplus \left(C_n\right)_{\mathrm{not\,on}\,_nY^n_B}.$$

By definition both direct sum decompositions are respected by the action of the Lefschetz-$\mathfrak{sl}_2$. Thus to prove Theorem 4.2.1, it suffices to construct

(a) *the isomorphism on the stratum of maximal defect*

$$\left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})\right)_{\mathrm{on}\,_nY^n_B} \;\cong\; \left(C_n\right)_{\mathrm{on}\,_nY^n_B}$$

(b) *the isomorphism away from the stratum of maximal defect*

$$\left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})\right)_{\mathrm{not\,on}\,_nY^n_B} \;\cong\; \left(C_n\right)_{\mathrm{not\,on}\,_nY^n_B}$$

where both isomorphisms need to respect the action of the Lefschetz-$\mathfrak{sl}_2$. The existence of the isomorphism (b) away from the stratum of maximal defect follows directly from the induction hypothesis, as we explain in the next paragraph. For the remainder of Section 6 we will then be concerned with establishing the isomorphism (a) on the stratum of maximal defect.

6.2.3. *The isomorphism away from the stratum of maximal defect.* Recall from Corollary 4.1.9 the open subscheme $_{\leqslant(n-1)}Y^n_B$ of the B-locus Y^n_B defined by allowing the

defect degree to be at most $n-1$. Then as in Section 4.2 the induction hypothesis implies the validity of Theorem 4.2.1 after restricting to the open subscheme $_{\leqslant(n-1)}Y_B^n$, i.e.:

Lemma 6.2.4. *Assume the main theorem about nearby cycles for the local models, Theorem 4.2.1, holds for the integer $n-1$. Then on the open subscheme $_{\leqslant(n-1)}Y_B^n$ of Y_B^n there exists an isomorphism of perverse sheaves*

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})\Big|_{_{\leqslant(n-1)}Y_B^n}^* \quad \cong \quad \bigoplus_{(n_1,k,n_2)} \left(\bar{f}_{n_1,k,n_2,*}\,\widetilde{\mathcal{P}}_{n_1,k,n_2}\right)\Big|_{_{\leqslant(n-1)}Y_B^n}^*$$

which is compatible with the action of the Lefschetz-$\mathfrak{sl}_2$.

Combining Lemma 6.2.4 and Lemma 6.2.1 above, we already know that the restriction of $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{not\ on}\ _nY_B^n}$ to $_{\leqslant(n-1)}Y_B^n$ is semisimple. Since by definition none of the simple constituents of $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{not\ on}\ _nY_B^n}$ are supported on the complement of $_{\leqslant(n-1)}Y_B^n$, it must in fact be equal to the intermediate extension of its restriction to $_{\leqslant(n-1)}Y_B^n$. Thus applying the intermediate extension functor to the isomorphism in Lemma 6.2.4 yields the desired isomorphism (b) above.

6.3. The isomorphism on the stratum of maximal defect.

Recall from Corollary 4.1.9 that the stratum of maximal defect $_nY_B^n$ is canonically identified with the symmetric power $X^{(n)}$ of the curve X via the natural projection map $Y^n \to X^{(n)}$. Throughout this section we will identify $_nY_B^n$ and $X^{(n)}$ without further mention. Observe furthermore that by definition of C_n we have

$$\left(C_n\right)_{\mathrm{on}\ _nY_B^n} \quad = \quad \mathcal{P}_n$$

where $\mathcal{P}_n$ denotes the n-th Picard-Lefschetz oscillator as in Section 3.2.4 above.

Lemma 6.3.1. *The objects* $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\,{_n}Y_B^n}$ *and* $(C_n)_{\mathrm{on}\,{_n}Y_B^n}$ *admit natural factorization structures which respect the action of the Lefschetz-$\mathfrak{sl}_2$.*

Proof. For $(C_n)_{\mathrm{on}\,{_n}Y_B^n} = \mathcal{P}_n$ this was already dealt with in Lemma 3.2.3 and Lemma 3.2.5 above. We now prove the assertion for $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\,{_n}Y_B^n}$. First, the factorization-in-families of the local models Y^n from Proposition 5.1.2 above, together with the compatibility of $\mathrm{Gr}\Psi$ with fiber products (see for example [BB, Sec. 5]), shows that on $Y_B^{n_1} \overset{\circ}{\times} Y_B^{n_2}$ we have:

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})\Big|^*_{Y_B^{n_1}\overset{\circ}{\times}Y_B^{n_2}} \quad = \quad \mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^{n_1}}) \overset{\circ}{\boxtimes} \mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^{n_2}})$$

Here the left hand side denotes the $*$-pullback of $\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})$ along the etale factorization map

$$Y_B^{n_1} \overset{\circ}{\times} Y_B^{n_2} \longrightarrow Y_B^n\,.$$

The above identification respects the action of the Lefschetz-$\mathfrak{sl}_2$ due to the comment in Proposition 5.1.2 about compatibility with respect to maps to $\mathbb{A}^1$.

Next we claim that the above identification in fact induces an identification of the desired summands:

$$(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\,{_n}Y_B^n}\Big|^*_{Y_B^{n_1}\overset{\circ}{\times}Y_B^{n_2}} = (\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^{n_1}}))_{\mathrm{on}\,{_{n_1}}Y_B^{n_1}} \overset{\circ}{\boxtimes} (\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^{n_2}}))_{\mathrm{on}\,{_{n_2}}Y_B^{n_2}}$$

Indeed, using the identification without requirements on the support and the fact that the stratum of maximal defect "factorizes" in the sense that the square

$$
\begin{array}{ccc}
{_{n_1}}Y_B^{n_1} \overset{\circ}{\times} {_{n_2}}Y_B^{n_2} & \longrightarrow & {_n}Y_B^n \\
\downarrow & & \downarrow \\
Y_B^{n_1} \overset{\circ}{\times} Y_B^{n_2} & \longrightarrow & Y_B^n
\end{array}
$$

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is cartesian, one can identify both the left hand side and the right hand side with the direct summand of the perverse sheaf

$$\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})\Big|^*_{Y_B^{n_1}\overset{\circ}{\times}Y_B^{n_2}}$$

consisting of those simple constituents supported on $_{n_1}Y_B^{n_1}\overset{\circ}{\times}{}_{n_2}Y_B^{n_2}$. $\qquad\square$

6.3.2. *Simple constituents supported on the diagonal.* Let Δ_X denote the main diagonal

$$\Delta_X = X \hookrightarrow X^{(n)}$$

of the symmetric power $X^{(n)}$. The following lemma is crucial to our approach to the proof of Theorem 4.2.1:

Lemma 6.3.3. *For any $n \geqslant 2$, none of the simple constituents of the perverse sheaves* $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{on\,_nY_B^n}$ *and* $(C_n)_{on\,_nY_B^n}$ *are supported on the diagonal* Δ_X.

For $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\,_nY_B^n}$ the proof of Lemma 6.3.3 is the topic of Section 6.4 below. The case of $(C_n)_{\mathrm{on}\,_nY_B^n}$ however follows directly from the definitions: Since $(C_n)_{\mathrm{on}\,_nY_B^n} = \mathcal{P}_n$, this follows from the fact that the Picard-Lefschetz oscillator $\mathcal{P}_n$ is the intermediate extension of a local system on the disjoint locus of $X^{(n)}$ by Lemma 3.2.5.

In the next lemma and below *the union of all diagonals in $X^{(n)}$* refers to the natural closed subscheme of $X^{(n)}$ complementary to the disjoint locus of $X^{(n)}$. From the definition of factorizability one verifies:

Lemma 6.3.4. *Let F_n be a factorizable collection of perverse sheaves on $X^{(n)}$, and assume that for each $n \geqslant 2$ none of the simple constituents of F_n is supported on the main diagonal Δ_X. Then for any $n \geqslant 2$ none of the simple constituents of F_n*

is supported on the union of all diagonals in $X^{(n)}$. In particular, none of the simple constituents of $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{on\,{}_nY_B^n}$ and $(C_n)_{on\,{}_nY_B^n}$ is supported on the union of all diagonals in $X^{(n)}$.

6.3.5. *Generic agreement.* Lemma 6.3.4 above shows that it suffices to construct the desired isomorphism (a) above after restricting $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{on\,{}_nY_B^n}$ and $(C_n)_{on\,{}_nY_B^n}$ to the disjoint locus $\mathring{X}^{(n)}$ of the symmetric power. Indeed, by Lemma 6.3.4 both perverse sheaves are the intermediate extensions of their restrictions to the disjoint locus, and hence the desired isomorphism can be obtained by intermediate extension as well. As a first step towards the isomorphism on the disjoint locus, we construct the following weaker version. Let

$$\mathrm{add}_{disj}:\ \mathring{X}^n\ \longrightarrow\ \mathring{X}^{(n)}$$

denote the addition map from the disjoint locus of the cartesian product to the disjoint locus of the symmetric product of the curve X. Then directly from the definition of factorization we obtain:

Lemma 6.3.6. *The isomorphism for $n = 1$ from Section 6.1 and the factorization structure from Lemma 6.3.1 above together yield isomorphisms of the pullbacks*

$$\mathrm{add}_{disj}^*\Big(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n})\Big)_{on\,{}_nY_B^n}\ \cong\ \mathrm{add}_{disj}^*\big(C_n\big)_{on\,{}_nY_B^n}$$

for any $n \geqslant 2$ which respect the action of the Lefschetz-$\mathfrak{sl}_2$.

The addition map add_{disj} being a torsor for the symmetric group S^n, the pullbacks $\mathrm{add}_{disj}^*(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{on\,{}_nY_B^n}$ and $\mathrm{add}_{disj}^*(C_n)_{on\,{}_nY_B^n}$ carry natural S^n-equivariant structures. Thus to construct the desired isomorphism on the stratum of maximal

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defect, it suffices to prove that the isomorphism between the pullbacks constructed in Lemma 6.3.6 above in fact respects the S^n-equivariant structures.

This is one of the key computations of the present article, and is in fact part of how the Picard-Lefschetz oscillators were found in the first place. Fortunately, in the present situation it suffices to verify the case $n = 2$:

Lemma 6.3.7. *To show that the isomorphisms*

$$\mathrm{add}^*_{disj}\left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^n_G})\right)_{on\,_nY^n_B} \quad \cong \quad \mathrm{add}^*_{disj}\left(C_n\right)_{on\,_nY^n_B}$$

constructed in Lemma 6.3.6 above respect the natural S^n-equivariant structures for all $n \geqslant 2$, it in fact suffices to verify the case $n = 2$.

Proof. Since the symmetric group S^n is generated by transpositions, it suffices to verify that the above isomorphisms respect the equivariant structure for any transposition in S^n. Without loss of generality we may assume the transposition under consideration interchanges the elements $1, 2 \in \{1, \ldots, n\}$. Factoring the map add_{disj} as the composition

$$\mathring{X}^n \longrightarrow \mathring{X}^{(2)} \times \mathring{X}^{n-2} \longrightarrow \mathring{X}^{(n)}$$

and using that the isomorphisms in Lemma 6.3.6 are constructed via the factorization structures then reduces the assertion to the case $n = 2$. $\qquad\square$

The required assertion in the case $n = 2$ will be dealt with in Section 6.5 and Section 6.6 below. Namely, we will give two different proofs, one via an abstract calculation in the Grothendieck group, and another one via a direct computation of the IC-sheaf of the local model Y^2 based on the explicit geometry available from Section 5 above. This completes the proof of the main theorem about nearby cycles

for the local models, Theorem 4.2.1, and hence also Theorem 3.3.3 for VinBun_G, modulo Sections 6.4 through 6.6 below.

6.4. Fighting simples on the main diagonal.

In this section we prove Lemma 6.3.3 for $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\,_n Y_B^n}$. As the lemma is part of the induction step of the inductive proof of the main theorem, Theorem 4.2.1, we are allowed to assume the validity of Theorem 4.2.1 for the integer $n-1$ in the course of the proof of Lemma 6.3.3. Using the inductive hypothesis, the question about simples on the diagonal can be translated into a similar question about the cohomology of the Zastava spaces. In the next two subsections we first discuss the latter as well as its implications for the local models, and only then proceed to the actual proof.

6.4.1. *Compactly supported cohomology of open Zastava spaces.* As in Section 4.1.5 let $_0Z^n$ denote the open Zastava space, and as before let

$$\pi_Z : \ _0Z^n \ \longrightarrow \ X^{(n)}$$

denote the projection map. We now record some information about the object

$$\widetilde{\Omega}_n \ := \ \pi_{Z,!}\big(\mathrm{IC}_{_0Z^n}\big) \ = \ \pi_{Z,!}\big((\overline{\mathbb{Q}}_\ell)_{_0Z^n}[\dim_{_0Z^n}](\tfrac{1}{2}\dim_{_0Z^n})\big)$$

and will then apply it to the proof of Lemma 6.3.3. In fact, we will only need to understand $\widetilde{\Omega}_n$ on a fairly coarse level, namely on the level of the Grothendieck group. An expression of $\widetilde{\Omega}_n$ on the level of the Grothendieck group can fortunately be extracted from the work [BG2] of Braverman and Gaitsgory. The study of $\widetilde{\Omega}_n$ as

an object of the derived category is much more involved, and has been carried out by Sam Raskin in the forthcoming article [R].

To state the description of $\widetilde{\Omega}_n$ in the Grothendieck group, let

$$\text{add}: \ X^{(i)} \times X^{(j)} \ \longrightarrow \ X^{(n)}$$

denote the addition map of effective divisors as before; however, unlike before, here we do not restrict to the disjoint locus of the product $X^{(i)} \times X^{(j)}$. Furthermore, using the notation from Section 3.2.2 we denote by

$$\Lambda^{(j)}(\overline{\mathbb{Q}}_{\ell X})[j](j)$$

the i-th external exterior power of the constant local system on the curve X, shifted and twisted as indicated. The following description of $\widetilde{\Omega}_n$ in the Grothendieck group then follows directly from Corollary 4.5 of [BG2]:

Lemma 6.4.2. *In the Grothendieck group on $X^{(n)}$ we have:*

$$\widetilde{\Omega}_n \ = \ \sum_{i+j=n} add_* \Big(\overline{\mathbb{Q}}_{\ell X^{(i)}} \ \boxtimes \ \Lambda^{(j)}(\overline{\mathbb{Q}}_{\ell X})[j](j) \Big)$$

Here the sum runs over all pairs of integers (i, j) with $0 \leqslant i, j, \leqslant n$ and $i + j = n$.

6.4.3. *Stalks of the extension of the constant sheaf.* We now explain how the object $\widetilde{\Omega}_n$ arises in a sheaf-theoretic computation on the local models Y^n. To do so, let

$$j_G: \ Y_G^n \ \hookrightarrow \ Y^n$$

denote the open inclusion of the G-locus of Y^n, and recall from Section 5.2.8 that the inclusion of the stratum of maximal defect $_n Y_B^n \hookrightarrow Y^n$ agrees with the section $s: X^{(n)} \hookrightarrow Y^n$ under the identification $_n Y_B^n = X^{(n)}$. Furthermore let $H_c^*(\mathbb{A}^1 \smallsetminus \{0\})$

denote the compactly supported cohomology of $\mathbb{A}^1 \smallsetminus \{0\}$. Then, using the geometry of the local models Y^n discussed in Section 5 above, we can now prove:

Proposition 6.4.4.

$$s^! \, j_{G,!} \, \mathrm{IC}_{Y_G^n} \;\; = \;\; \widetilde{\Omega}_n \otimes H_c^*(\mathbb{A}^1 \smallsetminus \{0\})[1](\tfrac{1}{2})$$

Proof. Let π_G denote the restriction of the projection map $\pi : Y^n \to X^{(n)}$ to the G-locus Y_G^n. Then $j_{G,!} \, \mathrm{IC}_{Y_G^n}$ is $\mathbb{G}_m$-equivariant for the $\mathbb{G}_m$-action from Section 5.2.11 above, and hence the contraction principle from Lemma 5.2.14 above yields:

$$s^! \, j_{G,!} \, \mathrm{IC}_{Y_G^n} \;\; = \;\; \pi_! \, j_{G,!} \, \mathrm{IC}_{Y_G^n} \;\; = \;\; \pi_{G,!} \, \mathrm{IC}_{Y_G^n}$$

On the other hand, in terms of the product decomposition of Y_G^n from Lemma 4.1.7 above we have

$$\mathrm{IC}_{Y_G^n} \;\; = \;\; \mathrm{IC}_{0Z^n} \boxtimes \mathrm{IC}_{\mathbb{A}^1 \smallsetminus \{0\}} \; .$$

Thus the compatibility of the projection maps in Lemma 4.1.7 implies that

$$\pi_{G,!} \, \mathrm{IC}_{Y_G^n} \;\; = \;\; \pi_{Z,!}(\mathrm{IC}_{0Z^n}) \otimes H_c^*(\mathbb{A}^1 \smallsetminus \{0\})[1](\tfrac{1}{2}) \;\; = \;\; \widetilde{\Omega}_n \otimes H_c^*(\mathbb{A}^1 \smallsetminus \{0\})[1](\tfrac{1}{2}),$$

as desired. $\qquad\square$

6.4.5. *The proof.* We can now proceed to the proof of Lemma 6.3.3 for the perverse sheaf $(\mathrm{Gr} \, \Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on} \, _n Y_B^n}$. In line with the appearance of the lemma in the induction step of the inductive proof of the main theorem, Theorem 4.2.1, we are allowed to assume the validity of Theorem 4.2.1 for the integer $n - 1$ in the course of the proof.

Proof of Lemma 6.3.3 for $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{on\,_nY_B^n}$.

Let i_B denote the inclusion of the B-locus Y_B^n into Y^n, and as before let j_G denote the open immersion of the G-locus Y_G^n into Y^n. Then on Y_B^n the usual triangle for the map can : $\Psi \to \Phi$ applied to the object $j_{G,*}\,\mathrm{IC}_{Y_G^n}$ takes the form

$$i_B^*\,j_{G,*}\,\mathrm{IC}_{Y_G^n}[-1](-\tfrac{1}{2}) \longrightarrow \Psi(\mathrm{IC}_{Y_G^n}) \xrightarrow{\ N\ } \Psi(\mathrm{IC}_{Y_G^n})(-1) \xrightarrow{\ +1\ }.$$

To better understand the object $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^n}))_{\text{on }_nY_B^n}$ we will use the $*$-pullback of the above triangle along s, i.e., the triangle

$$s^*\,j_{G,*}\,\mathrm{IC}_{Y_G^n}[-1](-\tfrac{1}{2}) \longrightarrow s^*\,\Psi(\mathrm{IC}_{Y_G^n}) \xrightarrow{\ N\ } s^*\,\Psi(\mathrm{IC}_{Y_G^n})(-1) \xrightarrow{\ +1\ }.$$

More precisely, we will exploit the relation the latter triangle induces in the Grothendieck group of perverse sheaves on $X^{(n)}$. Namely, let the image of $s^*\,\Psi(\mathrm{IC}_{Y_G^n})$ in the Grothendieck group on $X^{(n)}$ be expressed uniquely as a minimal $\mathbb{Z}$-linear combination of simple perverse sheaves. Then we claim that none of the simple perverse sheaves occurring in this expression is supported on the main diagonal Δ_X of $X^{(n)}$. Indeed, since the third term of the last triangle is a non-trivial twist of the middle term $s^*\,\Psi(\mathrm{IC}_{Y_G^n})$, it suffices to prove the analogous claim for the first term. However, by Proposition 6.4.4 above, the first term is Verdier dual to the object

$$\widetilde{\Omega}_n \otimes H_c^*(\mathbb{A}^1 \smallsetminus \{0\})[2](1).$$

Thus it in turn suffices to show the analogous claim for $\widetilde{\Omega}_n$: If the image of $\widetilde{\Omega}_n$ in the Grothendieck group of perverse sheaves on $X^{(n)}$ is expressed as a minimal $\mathbb{Z}$-linear combination of simple perverse sheaves, then none of the simples occurring in this expression is supported on the main diagonal Δ_X. This last claim however follows directly from the explicit description of $\widetilde{\Omega}_n$ on the level of the Grothendieck group in

Lemma 6.4.2 above: Each summand in this description is equal to the intermediate extension to $X^{(n)}$ of a local system on the disjoint locus of $X^{(n)}$.

We have now established that none of the simple perverse sheaves occurring in the minimal description of the image of $s^* \Psi(\mathrm{IC}_{Y_G^n})$ in the Grothendieck group on $X^{(n)}$ is supported on the main diagonal Δ_X. From this we now deduce that the same holds for the image of $(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\, {}_n Y_B^n}$ in the Grothendieck group; since $(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n}))_{\mathrm{on}\, {}_n Y_B^n}$ is a perverse sheaf, this completes the proof.

To make the required deduction, note first that on the level of the Grothen-dieck group the objects $\Psi(\mathrm{IC}_{Y_G^n})$ and $\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n})$ coincide. Thus any simple perverse sheaf occurring in the minimal description of the image of the direct sum

$$s^* \mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n}) \;\; = \;\; \left(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n})\right)_{\mathrm{on}\, {}_n Y_B^n} \;\; \bigoplus \;\; s^*\left(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n})\right)_{\mathrm{not\, on}\, {}_n Y_B^n}$$

in the Grothendieck group of $X^{(n)}$ cannot be supported on the main diagonal Δ_X. Hence to establish the desired claim for the first summand, it suffices to establish the analogous claim for the second summand. The second summand can however be dealt with via the induction hypothesis: Since we are allowed to assume the validity of Theorem 4.2.1 for the integer $n - 1$, we may apply Lemma 6.2.4 above and may hence make use of the identification

$$\left(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^n})\right)_{\mathrm{not\, on}\, {}_n Y_B^n} \;\; \cong \;\; \left(C_n\right)_{\mathrm{not\, on}\, {}_n Y_B^n}$$

from Section 6.2.3 above. This in turn reduces the assertion about the second summand to the analogous assertion for the object $s^*\left(C_n\right)_{\mathrm{not\, on}\, {}_n Y_B^n}$. The next lemma however provides a strengthening of this last assertion, and therefore completes the proof of Lemma 6.3.3. □

Lemma 6.4.6. *For each triple (n_1, k, n_2) as in Theorem 4.2.1 the object $s^* \bar{f}_{n_1,k,n_2,*} \widetilde{\mathcal{P}}_{n_1,k,n_2}$ is a direct sum of cohomologically shifted simple perverse sheaves on $X^{(n)}$, none of which is supported on the main diagonal Δ_X.*

Proof. First observe that the square

$$
\begin{array}{ccc}
X^{(n_1)} \times X^{(k)} \times X^{(n_2)} & \xrightarrow{\text{section}} & Z^{-,n_1}_{(\mathrm{Bun}_{T,-n})} \underset{\mathrm{Bun}_T}{\times} \left(X^{(k)} \times Z^{n_2} \right) \\
\Big\downarrow {\scriptstyle \text{add}} & & \Big\downarrow {\scriptstyle \bar{f}_{n_1,k,n_2}} \\
X^{(n)} & \xrightarrow{\ \ s\ \ } & Y_B^n
\end{array}
$$

is cartesian, where the top arrow is the natural map formed by combining the three section maps s_{Z^-}, s, and s_Z from Section 5.2 above, and where the left horizontal map is the addition map of effective divisors. Next, from the definition of $\widetilde{\mathcal{P}}_{n_1,k,n_2}$ and the properness of $\bar{f}_{n_1,k,n_2}$ we obtain that

$$
s^* \bar{f}_{n_1,k,n_2,*} \widetilde{\mathcal{P}}_{n_1,k,n_2} \ = \ \mathrm{add}_* \left(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{P}_k \boxtimes \overline{\mathbb{Q}}_\ell \right) [2n - 2k](n - k).
$$

Then the finiteness of the map add and the properties of $\mathcal{P}_k$ stated in Lemma 3.2.5 together yield the assertion. $\square$

6.5. **Finding the Picard-Lefschetz oscillators.**

In this section we give the first proof that the isomorphism

$$
\mathrm{add}^*_{disj} \left(\mathrm{Gr}\, \Psi(\mathrm{IC}_{Y_G^2}) \right)_{\text{on } {}_2 Y_B^2} \ \cong \ \mathrm{add}^*_{disj} \left(C_2 \right)_{\text{on } {}_2 Y_B^2}
$$

constructed in Lemma 6.3.6 indeed respects the natural S_2-equivariant structure; this completes the proof of Theorem 4.2.1. In Section 6.6 below we give a second proof.

The first proof is an abstract calculation in the Grothendieck group and essentially a refinement of the arguments of Section 6.4, exploiting the specific expression for $\widetilde{\Omega}$ given in Lemma 6.4.2 above. The second proof does not require this specific expression, but instead deduces the required assertion from an intersection cohomology computation for the space Y^2 which relies on the geometry of the local models developed in Section 5 above; this is in fact how the Picard-Lefschetz oscillators were found originally. We include this second proof as it also provides an example of a direct IC-sheaf computation without passing through the nearby cycles, and might illuminate how one can work with the local models in very explicit terms.

6.5.1. *The first proof of the compatibility.* We will show that the images of $(\operatorname{Gr}\Psi(\operatorname{IC}_{Y_G^2}))_{\text{on } 2Y_B^2}$ and $(C_2)_{\text{on } 2Y_B^2} = \mathcal{P}_2$ in the Grothendieck group of perverse sheaves on $X^{(2)}$ agree; since both are in fact semisimple perverse sheaves, this will prove the claim. When writing expressions in the Grothendieck group on $X^{(2)}$, we will for notational simplicity denote by $\overline{\mathbb{Q}}_\ell$ and by sign the IC-extensions to $X^{(2)}$ of the constant and sign local systems on the disjoint locus of $X^{(2)}$. First, from the definition one finds that

$$\mathcal{P}_2 = \operatorname{sign}(1) + \operatorname{sign}(0) + \operatorname{sign}(-1) + \overline{\mathbb{Q}}_\ell(0)$$

in the Grothendieck group.

To compute $(\operatorname{Gr}\Psi(\operatorname{IC}_{Y_G^2}))_{\text{on } 2Y_B^2}$ we exploit the relation in the Grothendieck group induced by the triangle

$$s^* j_{G,*}\, \operatorname{IC}_{Y_G^2}[-1](-\tfrac{1}{2}) \ \longrightarrow\ s^*\,\Psi(\operatorname{IC}_{Y_G^2}) \ \xrightarrow{N}\ s^*\,\Psi(\operatorname{IC}_{Y_G^2})(-1) \ \xrightarrow{+1}$$

from the proof in Section 6.4.5 above in the case $n = 2$. Namely, as a first step we use this triangle to show that in the Grothendieck group the difference

$$(\operatorname{Gr}\Psi(\operatorname{IC}_{Y_G^2}))_{\text{on } 2Y_B^2} \ - \ (\operatorname{Gr}\Psi(\operatorname{IC}_{Y_G^2}))_{\text{on } 2Y_B^2}(-1)$$

is equal to

$$\overline{\mathbb{Q}}_\ell(0) - \overline{\mathbb{Q}}_\ell(-1) + \mathrm{sign}(1) - \mathrm{sign}(-2) \,.$$

To see this, we need to compute the images of the first term of the triangle and of $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{not\ on}\,{}_2Y_B^2}$ in the Grothendieck group: For the image of the first term of the triangle we find the expression

$$\overline{\mathbb{Q}}_\ell(1) - 2\,\overline{\mathbb{Q}}_\ell(0) + \overline{\mathbb{Q}}_\ell(-1) - \mathrm{sign}(0) + 2\,\mathrm{sign}(-1) - \mathrm{sign}(-2)$$

by Lemma 6.4.2, Lemma 6.4.4, and the fact that

$$H_c^*(\mathbb{A}^1 \smallsetminus \{0\}) \;=\; \overline{\mathbb{Q}}_\ell[-2](-1) \oplus \overline{\mathbb{Q}}_\ell[-1](0) \,.$$

For $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{not\ on}\,{}_2Y_B^2}$ we first invoke Lemma 6.2.4 and then compute its image in the Grothendieck group to be

$$\overline{\mathbb{Q}}_\ell(1) - \mathrm{sign}(1) - 2\,\overline{\mathbb{Q}}_\ell(0) - 2\,\mathrm{sign}(0)$$

by using the cartesian square from the proof of Lemma 6.4.6 above. We have now established the above formula for

$$(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{on}\,{}_2Y_B^2} \;-\; (\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{on}\,{}_2Y_B^2}(-1) \,.$$

But since $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{on}\,{}_2Y_B^2}$ is perverse, it can be reconstructed from this difference by induction on the length, starting with a simple of minimal weight; executing this algorithm yields

$$(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_{\tilde{G}}^2}))_{\mathrm{on}\,{}_2Y_B^2} \;=\; \mathrm{sign}(1) + \mathrm{sign}(0) + \mathrm{sign}(-1) + \overline{\mathbb{Q}}_\ell(0)$$

as desired, completing the proof.

6.6. **Picard-Lefschetz oscillators via IC-stalks for small defect.**

In this section we give the aforementioned second proof of the correctness of the S_2-equivariant structure. Namely, we will deduce the assertion from the following explicit computation:

6.6.1. *Intersection cohomology for Y^2.* Our next goal is to prove:

Proposition 6.6.2. *The restriction of the IC-sheaf of Y^2 to the stratum of maximal defect $_2Y_B^2 = X^{(2)}$ equals:*

$$s^* \mathrm{IC}_{Y^2} \quad = \quad \overline{\mathbb{Q}}_{\ell X^{(2)}}[3](\tfrac{3}{2}) \ \oplus \ \overline{\mathbb{Q}}_{\ell X^{(2)}}[5](\tfrac{5}{2})$$

We begin with the following lemma:

Lemma 6.6.3. *None of the simple perverse sheaves occurring in the minimal $\mathbb{Z}$-linear combination of $s^* \mathrm{IC}_{Y^2}$ in the Grothendieck group of perverse sheaves on $X^{(2)}$ is supported on the diagonal Δ_X of $X^{(2)}$.*

Proof. On the level of the Grothendieck group the object $s^* \mathrm{IC}_{Y^2}$ agrees up to twist and sign with the $*$-restriction along s of the associated graded $\mathrm{Gr}\, \mathrm{IC}_{Y^2}|_{Y_B^2}^*[-1](-\tfrac{1}{2})$. The latter associated graded object is however a subobject of the perverse sheaf $\mathrm{Gr}\Psi(\mathrm{IC}_{Y_G^2})$ by Section 3.1; the claim hence follows from the analogous claim for $s^*\mathrm{Gr}\Psi(\mathrm{IC}_{Y_G^2})$. To prove the latter, we split $\mathrm{Gr}\Psi(\mathrm{IC}_{Y_G^2})$ as a direct sum as in Section 6.2.2 above: For the summand $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^2}))_{\text{on }_2Y_B^2}$ the needed assertion is then precisely Lemma 6.3.3 above for $n = 2$. For the summand $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^2}))_{\text{not on }_2Y_B^2}$ the needed assertion follows from the validity of Theorem 4.2.1 for $n = 1$ and Lemma 6.4.6 above. $\qquad \square$

As before we denote by

$$\mathrm{add}: X \overset{\circ}{\times} X \longrightarrow X^{(2)}$$

the addition map of effective divisors. Complementarily to Lemma 6.6.3 we now prove:

Lemma 6.6.4.

$$\mathrm{add}^* s^! \, \mathrm{IC}_{Y^2} \;=\; \overline{\mathbb{Q}}_{\ell \, X \overset{\circ}{\times} X}[1](\tfrac{1}{2}) \;\oplus\; \overline{\mathbb{Q}}_{\ell \, X \overset{\circ}{\times} X}[-1](-\tfrac{1}{2})$$

Proof. By the contraction principle (Lemma 5.2.14 above) and the factorization in families (Proposition 5.1.2) above, we have to show that

$$(\pi_1 \times \pi_1)_! \, \mathrm{IC}_{Y^1 \underset{\mathbb{A}^1}{\overset{\circ}{\times}} Y^1} \;=\; \overline{\mathbb{Q}}_{\ell \, X \overset{\circ}{\times} X}[1](\tfrac{1}{2}) \;\oplus\; \overline{\mathbb{Q}}_{\ell \, X \overset{\circ}{\times} X}[-1](-\tfrac{1}{2})$$

on the disjoint locus of $X \times X$. As in the proof of Proposition 6.1.1 above it suffices to verify this at the level of $*$-stalks. To do this, note that the contracting $\mathbb{G}_m$-action on Y^1 induces a $\mathbb{G}_m$-action on the fiber product $Y^1 \underset{\mathbb{A}^1}{\overset{\circ}{\times}} Y^1$ by Lemma 5.3.9; this action respects the product projection $\pi_1 \times \pi_1$ and contracts the fiber product onto the product section $s_1 \times s_1$. Thus, applying the contraction principle again, we are left to verify that the !-stalk of the IC-sheaf of the subvariety $\mathbb{S}^1 \times_{\mathbb{A}^1} \mathbb{S}^1 \subset \mathbb{A}^4$ at the origin $0 \in \mathbb{A}^4$ is equal to

$$\mathrm{IC}_{\mathbb{S}^1 \underset{\mathbb{A}^1}{\times} \mathbb{S}^1} \Big|_0^! \;=\; \overline{\mathbb{Q}}_\ell[-1](-\tfrac{1}{2}) \;\oplus\; \overline{\mathbb{Q}}_\ell[-3](-\tfrac{3}{2}) \,.$$

But the equations from Section 5.3 show that the subvariety $\mathbb{S}^1 \times_{\mathbb{A}^1} \mathbb{S}^1 \subset \mathbb{A}^4$ is precisely the affine quadric cone defined by the equation $XY = ZW$ in $\mathbb{A}^4$; the standard calculation of the IC-stalk at the vertex of the cone, for example via a resolution of singularities, then yields the result. $\qquad \square$

By Corollary 5.2.15 above we already know that $s^* \mathrm{IC}_{Y^2}$ is pure of weight 0; combining this with Lemma 6.6.3 and Lemma 6.6.4 above, we conclude that

$$s^! \mathrm{IC}_{Y^2} \;=\; L_1[-1](-\tfrac{1}{2}) \oplus L_2[-3](-\tfrac{3}{2})$$

where L_1 and L_2 can be either equal to the shifted constant sheaf $(\overline{\mathbb{Q}}_\ell)_{X^{(2)}}[2](1)$ or to the IC-extension of the sign local system from the disjoint locus in $X^{(2)}$. Thus to prove Proposition 6.6.2 above, we have to prove that both L_1 and L_2 are equal to the constant sheaf, i.e., we have to rule out the appearance of sign local systems. To do so, we will "compute over the diagonal", for which we will utilize our concrete understanding of the space $\mathbb{S}^2$ in coordinates. More precisely, since the stalk of the IC-extension of the sign local system at a point on the diagonal Δ_X of $X^{(2)}$ vanishes, Proposition 6.6.2 will follow from the following lemma:

Lemma 6.6.5. *The $*$-stalk of $s^! \mathrm{IC}_{Y^2}$ at any point on the diagonal Δ_X is equal to*

$$\overline{\mathbb{Q}}_\ell[1](\tfrac{1}{2}) \;\oplus\; \overline{\mathbb{Q}}_\ell[-1](-\tfrac{1}{2}).$$

Proof. Since $s^! \mathrm{IC}_{Y^2} = \pi_! \mathrm{IC}_{Y^2}$ by the contraction principle, Lemma 5.2.14 above, the above $*$-stalk is equal to the compactly supported cohomology

$$H^*_c(\mathbb{S}^2, \mathrm{IC}_{Y^2}\,|^*_{\mathbb{S}^2})$$

of the restriction $\mathrm{IC}_{Y^2}\,|^*_{\mathbb{S}^2}$. We thus have to show that these cohomology groups are 1-dimensional in the relevant degrees 1 and -1; in doing so, the weights are irrelevant, so we suppress them from the notation throughout the proof. To compute these cohomology groups, we will use the long exact sequence in compactly supported

cohomology

$$H_c^*({}_{\leqslant 1}\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{{}_{\leqslant 1}\mathbb{S}^2}^*) \longrightarrow H_c^*(\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{\mathbb{S}^2}^*) \longrightarrow H_c^*({}_2\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{{}_2\mathbb{S}_B^2}^*)$$

associated to the pair $({}_{\leqslant 1}\mathbb{S}^2, {}_2\mathbb{S}^2)$ consisting of the complementary open and closed subvarieties

$$ {}_{\leqslant 1}\mathbb{S}^2 \xrightarrow{\text{open}} \mathbb{S}^2 \xleftarrow{\text{closed}} {}_2\mathbb{S}_B^2 \,.$$

Observe that the open subvariety ${}_{\leqslant 1}\mathbb{S}^2$ complementary to ${}_2\mathbb{S}_B^2$ consists of the G-locus $\mathbb{S}_G^2$ as well as the strata ${}_0\mathbb{S}_B^2$ and ${}_1\mathbb{S}_B^2$ of the B-locus.

To analyze the term $H_c^*({}_2\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{{}_2\mathbb{S}_B^2}^*)$ in this sequence, observe that ${}_2\mathbb{S}^2$ consists of precisely one point, which we will denote by p; hence this term is simply equal to the stalk $\mathrm{IC}_{Y^2}\big|_p^*$. But applying Verdier duality to our preliminary knowledge of $s^! \mathrm{IC}_{Y^2}$ in terms of L_1 and L_2 above, we already know that this stalk must be concentrated in cohomological degrees -3 and -5.

To analyze the term $H_c^*({}_{\leqslant 1}\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{{}_{\leqslant 1}\mathbb{S}^2}^*)$, note first that since $\mathbb{S}^1$ and hence also Y^1 are smooth by Section 5.3, the open locus ${}_{\leqslant 1}Y^2 \subset Y^2$ is smooth as well. Since ${}_{\leqslant 1}\mathbb{S}^2 \subset {}_{\leqslant 1}Y^2$ and since $\dim Y^2 = 5$ we hence conclude that

$$\mathrm{IC}_{Y^2}\big|_{{}_{\leqslant 1}\mathbb{S}^2}^* = \overline{\mathbb{Q}}_{\ell\,{}_{\leqslant 1}\mathbb{S}^2}[5]\,.$$

Combining the last two observations, the long exact sequence shows:

$$H_c^1(\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{\mathbb{S}^2}^*) = H_c^1({}_{\leqslant 1}\mathbb{S}^2, \overline{\mathbb{Q}}_\ell[5])$$

$$H_c^{-1}(\mathbb{S}^2, \mathrm{IC}_{Y^2}\big|_{\mathbb{S}^2}^*) = H_c^{-1}({}_{\leqslant 1}\mathbb{S}^2, \overline{\mathbb{Q}}_\ell[5])$$

Thus the proof of the lemma is completed by the computation of the compactly supported cohomology groups of the variety ${}_{\leqslant 1}\mathbb{S}^2$ on the right hand side in the next lemma. $\qquad\square$

Continuing to suppress the weights from the notation due to their irrelevance for the present question, we conclude the proof of Proposition 6.6.2 by showing:

Lemma 6.6.6.

$$H_c^6({}_{\leqslant 1}\mathbb{S}^2, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$$

$$H_c^4({}_{\leqslant 1}\mathbb{S}^2, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$$

Proof. From Section 5.3 above it follows that the subvariety $\mathbb{S}^2 \subset \mathbb{A}^4$ is the affine quadric cone in $\mathbb{A}^4$ defined by the equation $XY + ZW = 0$; the closed subvariety ${}_2\mathbb{S}_B^2$ corresponds precisely to the vertex of the cone. The open subvariety ${}_{\leqslant 1}\mathbb{S}^2$ thus forms a $\mathbb{G}_m$-bundle over the smooth quadric surface in $\mathbb{P}^3$ and is hence smooth itself. Since ${}_{\leqslant 1}\mathbb{S}^2$ is 3-dimensional and irreducible, the first claim follows. For the second claim, we can by Poincare duality equivalently compute $H^2({}_{\leqslant 1}\mathbb{S}^2, \overline{\mathbb{Q}}_\ell)$. The latter cohomology group can in turn be shown to be isomorphic to $\overline{\mathbb{Q}}_\ell$ using the Gysin sequence for the first Chern class of the $\mathbb{G}_m$-bundle ${}_{\leqslant 1}\mathbb{S}^2$ over the quadric surface in $\mathbb{P}^3$. $\qquad\square$

6.6.7. *The second proof of the compatibility.* We now give the second proof that the isomorphism

$$\mathrm{add}_{disj}^*\left(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^2})\right)_{\mathrm{on}\,_2Y_B^2} \cong \mathrm{add}_{disj}^*\left(C_2\right)_{\mathrm{on}\,_2Y_B^2}$$

constructed in Lemma 6.3.6 is compatible with the equivariant structures on both sides with respect to the symmetric group $S_2 = \mathbb{Z}/2\mathbb{Z}$. More precisely, we will relate the last question to the intersection cohomology computation in Proposition 6.6.2 of the previous section, and play the symmetries coming from the S_2-action and from the action of the Lefschetz-$\mathfrak{sl}_2$ off of each other.

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To make the task more explicit, observe first that the S_2-equivariant structure on the pullback $\mathrm{add}^*_{disj}(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y^2_G}))_{\mathrm{on}\,_2 Y^2_B}$ corresponds to a representation of the symmetric group S_2 on the tensor product $V \otimes V$ of the standard representation

$$V \;=\; \overline{\mathbb{Q}}_\ell(\tfrac{1}{2}) \oplus \overline{\mathbb{Q}}_\ell(-\tfrac{1}{2})$$

of the Lefschetz-$\mathfrak{sl}_2$ with itself, as in Section 3.2.4. In particular this action of S_2 must commute with the action of the Lefschetz-$\mathfrak{sl}_2$. We now have to verify that the action of the non-trivial element $\sigma \in S_2 = \mathbb{Z}/2\mathbb{Z}$ on $V \otimes V$ is given by flipping the two factors and multiplying by -1.

Denote by U_k the irreducible representation of the Lefschetz-$\mathfrak{sl}_2$ of highest weight $k \in \mathbb{Z}_{\geq 0}$. Then since by definition $V = U_1$ the tensor product $V \otimes V$ decomposes as the direct sum

$$V \otimes V \;=\; \Lambda^2 V \oplus \mathrm{Sym}^2 V \;=\; U_0 \oplus U_2\,.$$

Since the action of σ commutes with the action of the Lefschetz-$\mathfrak{sl}_2$, the action of σ respects this direct sum decomposition; as the summands are irreducible as representations of the Lefschetz-$\mathfrak{sl}_2$, the action of σ on each of the summands must then be given by multiplication by either $+1$ or -1. We have to show that σ acts by $+1$ on U_0 and by -1 on U_2. To determine these signs, it is of course enough to know how sigma acts on the lowest weight lines $M_0 = \overline{\mathbb{Q}}_\ell$ of U_0 and $M_2 = \overline{\mathbb{Q}}_\ell(1)$ of U_2. It is precisely these signs on the lowest weight lines that we can access via the intersection cohomology of Y^2, as we discuss next.

Let $\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|^*_{Y^2_B}[-1](-\tfrac{1}{2}))$ denote the associated graded perverse sheaf with respect to the weight filtration on $\mathrm{IC}_{Y^2}\,|^*_{Y^2_B}[-1](-\tfrac{1}{2})$, and let

$$(\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|^*_{Y^2_B}[-1](-\tfrac{1}{2})))_{\mathrm{on}\,_2 Y^2_B}$$

denote its direct summand consisting of those simples which are supported on the stratum of maximal defect $_2Y_B^2$. By Lemma 3.1.5 and Lemma 3.1.7 the latter object is precisely the perverse kernel of the monodromy operator N acting on $(\mathrm{Gr}\,\Psi(\mathrm{IC}_{Y_G^2}))_{\mathrm{on}\,_2Y_B^2}$. Its pullback to the disjoint locus of $X\times X$ hence corresponds to the $\mathbb{Z}/2\mathbb{Z}$-subrepresentation

$$M_0 \oplus M_2 \ \subset \ U_0 \oplus U_2 \ = \ V \otimes V$$

formed by the direct sum of the lowest weight lines M_0 and M_2. Since $M_0 = \overline{\mathbb{Q}}_\ell$ and $M_2 = \overline{\mathbb{Q}}_\ell(1)$ are of different weight, the signs by which σ acts on M_0 and M_2 can thus be read off from the simple summands appearing in

$$(\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|_{Y_B^2}^*[-1](-\tfrac{1}{2})))_{\mathrm{on}\,_2Y_B^2}\,,$$

or even its restriction to the disjoint locus in $X^{(2)}$. Namely, from the Tate twists of the local systems on the right hand side in the next lemma we conclude that σ acts by $+1$ on M_0 and by -1 on M_2, completing the proof.

Lemma 6.6.8. *On the disjoint locus* $\overset{\circ}{X}{}^{(2)}$ *we have*

$$(\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|_{Y_B^2}^*[-1](-\tfrac{1}{2})))_{\mathrm{on}\,_2Y_B^2}\Big|_{\overset{\circ}{X}{}^{(2)}}^* \quad = \quad \left(\overline{\mathbb{Q}}_\ell \oplus \mathrm{sign}(1)\right)_{\overset{\circ}{X}{}^{(2)}}[2](1)\,,$$

where $\mathrm{sign}(1)$ *denotes the sign local system on* $\overset{\circ}{X}{}^{(2)}$ *twisted by 1.*

Proof. For readability we erase from the notation all symbols indicating a restriction to the disjoint locus of $X^{(2)}$, throughout the proof. Since $(\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|_{Y_B^2}^*[-1](-\tfrac{1}{2})))_{\mathrm{on}\,_2Y_B^2}$ is a semisimple perverse sheaf, it suffices to perform the necessary calculation in the Grothendieck group of perverse sheaves on $X^{(2)}$. But in the Grothendieck group the latter object is equal to the difference

$$s^*\,\mathrm{IC}_{Y^2}[-1](-\tfrac{1}{2}) \quad - \quad s^*(\mathrm{Gr}(\mathrm{IC}_{Y^2}\,|_{Y_B^2}^*)[-1](-\tfrac{1}{2}))_{\mathrm{not\,on}\,_2Y_B^2}\,.$$

We first compute the second term: As before we invoke the validity of Theorem 4.2.1 for $n = 1$ and apply Lemma 6.2.4 above; the second term is thus equal to the $*$-pullback along s of the kernel of the action of the monodromy operator N on $(C_2)_{\text{not on } _2Y_B^2}$. Using the exact same cartesian diagram as in the proof of Lemma 6.4.6 above one then computes that this second term is equal to

$$3 \cdot \overline{\mathbb{Q}}_\ell(1) \; + \; \text{sign}(1) \; - \; 2 \cdot \overline{\mathbb{Q}}_\ell(1) \; - \; 2 \cdot \text{sign}(1) \quad = \quad \overline{\mathbb{Q}}_\ell(1) \; - \; \text{sign}(1) \, .$$

Here, for notational brevity, we write $\overline{\mathbb{Q}}_\ell$ and sign for the perverse sheaves $\overline{\mathbb{Q}}_\ell[2](1)$ and $\text{sign}[2](1)$ of weight 0. However, by Proposition 6.6.2 above, the first term is equal to

$$\overline{\mathbb{Q}}_\ell(0) \; + \; \overline{\mathbb{Q}}_\ell(1) \, .$$

Taking the difference of the two terms we find the desired expression

$$\overline{\mathbb{Q}}_\ell(0) \; + \; \text{sign}(1) \, .$$

$\square$

7. Intersection cohomology

In this Section we comment on how Theorem 3.4.1 about the intersection cohomology follows from Theorem 3.3.3, as well as on how Theorem 3.4.1 in turn can be used to compute the IC-stalks.

As Theorem 3.5.2 follows from Proposition 6.4.4 by the same argument as in Section 4.2, the present section concludes the proof of the main theorems stated in Section 3 above.

7.1. Intersection cohomology from nearby cycles.

7.1.1. *Classical Schur-Weyl duality.* Recall that, working over an algebraically closed field of characteristic 0, the irreducible representations of the symmetric group S_k are in one-to-one correspondence with Young diagrams consisting of precisely k boxes. Furthermore, any Young diagram with at most m rows, but an arbitrary number of boxes, gives rise to an irreducible representation of the general linear group GL_m. For a Young diagram D with precisely k boxes and at most m rows we denote ρ_D and by U_D the corresponding irreducible representations of S_k and GL_m.

Let now U_{taut} denote the tautological m-dimensional representation of GL_m. The m-fold tensor product $U_{taut} \otimes \ldots \otimes U_{taut}$ carries the diagonal action of GL_m as well as the permutation action of the symmetric group S_k, and these actions commute. The classical Schur-Weyl duality then states:

Lemma 7.1.2 (Classical Schur-Weyl duality). *As a bi-representation of GL_m and S_k the k-fold tensor product $U_{taut} \otimes \ldots \otimes U_{taut}$ decomposes as*

$$U_{taut} \otimes \ldots \otimes U_{taut} \quad = \quad \bigoplus_D U_D \otimes \rho_D$$

where the sum runs over all Young diagrams D consisting of precisely k boxes and at most m rows.

We now apply this in the following context:

7.1.3. *Decomposing the Picard-Lefschetz oscillators.* As in Section 3.2.4 above let

$$V \;=\; \overline{\mathbb{Q}}_\ell(\tfrac{1}{2}) \oplus \overline{\mathbb{Q}}_\ell(-\tfrac{1}{2})\,.$$

As in Lemma 3.2.5 above let $V \otimes \ldots \otimes V$ denote the k-fold tensor product of V, together with the action of S_k defined by permuting the factors and multiplying with the sign of the permutation. Since V is precisely the tautological 2-dimensional representation of the Lefschetz-$\mathfrak{sl}_2$, the appropriate variant of Lemma 7.1.2 above yields:

Lemma 7.1.4. *As a bi-representation of S_k and the Lefschetz-$\mathfrak{sl}_2$ the k-fold tensor product $V \otimes \ldots \otimes V$ decomposes as*

$$V \otimes \ldots \otimes V \;=\; \bigoplus_{0 \leqslant r \leqslant \frac{k}{2}} U_{k-2r} \otimes \rho_{(k-r,r)}\,.$$

Here we denote by U_{k-2r} the irreducible representation of the Lefschetz-$\mathfrak{sl}_2$ of hightest weight $k - 2r$ and by $\rho_{(k-r,r)}$ the irreducible representation of S_k corresponding to the Young diagram with $k-r$ boxes in the first column and r boxes in the second column.

7.1.5. *Proof of Theorem 3.4.1.* Observe first that Lemma 7.1.4 yields an explicit direct sum decomposition into simple perverse sheaves of the Picard-Lefschetz oscillator $\mathcal{P}_k$ by Lemma 3.2.5 above. We however only need the following consequence:

Lemma 7.1.6. *The perverse kernel $\ker(N)$ of the monodromy operator N acting on the Picard-Lefschetz oscillator $\mathcal{P}_k$ is equal to the IC-extension of the local system on the disjoint locus of $X^{(k)}$ corresponding to the following representation of the symmetric group S_k:*

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$$\bigoplus_{0 \leqslant r \leqslant \frac{k}{2}} \rho_{(k-r,r)} \otimes \overline{\mathbb{Q}}_\ell(\tfrac{k}{2} - r)$$

Here, as before, we denote by $\rho_{(k-r,r)}$ the irreducible representation of S_k corresponding to the Young diagram with $k - r$ boxes in the first column and r boxes in the second column; here the second tensor factor indicates the appropriate Tate twist.

To prove Theorem 3.4.1 it suffices, by Lemma 3.1.5 above, to compute the kernel of the monodromy operator N on $\mathrm{Gr}\Psi(\mathrm{IC}_{\mathrm{VinBun}_{G,G}})$. Using Theorem 3.3.3 and the fact that the maps $\bar{f}_{n_1,k,n_2}$ are finite, the assertion thus follows from Lemma 7.1.6 above.

7.1.7. *Remark.* Theorem 3.4.1 can also be used to compute IC-stalks: As in Section 4.2 above, to compute the IC-stalks of VinBun_G along the strata of defect k we can equivalently compute the IC-stalks of the local model Y^k along the stratum of maximal defect ${}_kY^k$. Thus it suffices to derive an explicit formula for the restriction $s^* \mathrm{IC}_{Y^k}$ of the IC-sheaf of Y^k along the section s. To do so, note that Theorem 3.4.1 yields an explicit formula for $s^* \mathrm{IC}_{Y^k}$ in the Grothendieck group. But by Corollary 5.2.15 the complex $s^* \mathrm{IC}_{Y^k}$ is pure of weight 0, and one can show that this suffices to reconstruct $s^* \mathrm{IC}_{Y^k}$ from its image in the Grothendieck group. We plan to return to this argument and the resulting formulas elsewhere.

8. An application on the level of functions

8.1. **The statement.**

8.1.1. *The question.* This section is separate from the main text; its goal is to answer the following question on the level of functions. Let $\mathbb{F}_q$ be a finite field with q elements, let X be a smooth projective curve over $\mathbb{F}_q$, let $G = \mathrm{SL}_2$ over $\mathbb{F}_q$, and consider Bun_G over $\mathbb{F}_q$. As above let

$$\Delta: \ \mathrm{Bun}_G \ \xrightarrow{\ \Delta\ } \ \mathrm{Bun}_G \times \mathrm{Bun}_G$$

denote the diagonal morphism of Bun_G and let $\overline{\mathbb{Q}}_{\ell\,\mathrm{Bun}_G}$ denote the constant sheaf on Bun_G. Then we will answer the following question:

Question 8.1.2. *Under the sheaf-function correspondence, what is the function on $\mathrm{Bun}_G \times \mathrm{Bun}_G$ corresponding to the pushforward $\Delta_* \overline{\mathbb{Q}}_{\ell\,\mathrm{Bun}_G}$? I.e., what is the trace of the action of the geometric Frobenius on the $*$-stalks of the pushforward $\Delta_* \overline{\mathbb{Q}}_{\ell\,\mathrm{Bun}_G}$ at $\mathbb{F}_q$-points of $\mathrm{Bun}_G \times \mathrm{Bun}_G$?*

In fact, following a suggestion of Drinfeld, we will use the compactification $\overline{\mathrm{Bun}}_G$ of the diagonal Δ: Answering the above question amounts to understanding the $*$-stalks of the pushforward of the constant sheaf $\overline{\mathbb{Q}}_{\ell\,\mathrm{Bun}_G}$ along the natural map

$$b: \ \mathrm{Bun}_G \ \longrightarrow \ \overline{\mathrm{Bun}}_G\,.$$

8.1.3. *Notation.* To state the answer to the above question we need to introduce the following notation. First, given an $\mathbb{F}_q$-point (E_1, E_2) of $\mathrm{Bun}_G \times \mathrm{Bun}_G$, we denote by $\mathrm{Isom}_{\mathrm{SL}_2}(E_1, E_2)(\mathbb{F}_q)$ the set of vector bundle isomorphisms $E_1 \to E_2$ of determinant 1, i.e., the set of isomorphisms as SL_2-bundles. Next, let $\varphi: E_1 \to E_2$ be a non-zero

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morphism of vector bundles which is not an isomorphism. Factoring φ as

$$E_1 \twoheadrightarrow M_1 \hookrightarrow M_2 \hookleftarrow E_2$$

as in Section 2.3.1 above we associate to φ its defect divisor D_φ, which forms an $\mathbb{F}_q$-point of the symmetric power $X^{(n)}$ for some integer n. The defect divisor D_φ can be written as a sum

$$D_\varphi = \sum_k n_{k,\varphi}\, x_{k,\varphi}$$

where the $x_{k,\varphi}$ are distinct closed points of the curve X over $\mathbb{F}_q$. We then denote by $d_{k,\varphi}$ the degree of the residue field extension at the point $x_{k,\varphi}$. We can now state:

8.1.4. *The answer.* With the above notation we have:

Proposition 8.1.5. *Let (E_1, E_2) be an $\mathbb{F}_q$-point of $\mathrm{Bun}_G \times \mathrm{Bun}_G$. Then the trace of the geometric Frobenius on the $*$-stalk at (E_1, E_2) of the pushforward $\Delta_* \overline{\mathbb{Q}}_\ell$ is equal to:*

$$\left|\,\mathrm{Isom}_{\mathrm{SL}_2}(E_1, E_2)(\mathbb{F}_q)\,\right| \quad - \sum_{\substack{\varphi \,\in\, \mathrm{Hom}(E_1, E_2)(\mathbb{F}_q) \\ \varphi \ \text{is not an isomorphism} \\ \varphi \neq 0}} \prod_k (1 - q^{d_{k,\varphi}})$$

8.2. Reduction to a trace computation on $\overline{\mathrm{Bun}}_G$.

In this section we deduce Proposition 8.1.5 above from a computation on $\overline{\mathrm{Bun}}_G$ stated in Proposition 8.2.1 below; in the next section we will then prove Proposition 8.2.1. To state the proposition, recall from Section 2.1.3 the natural map

$$b : \mathrm{Bun}_G \longrightarrow \overline{\mathrm{Bun}}_G ,$$

and let $z = (E_1, E_2, L, \varphi)$ be an $\mathbb{F}_q$-point of the B-locus of $\overline{\mathrm{Bun}}_G$. Exactly as in Section 8.1.3 above we can associate to the point z, via the defect divisor of the map φ, the collection of closed points $x_{k,\varphi}$ and integers $d_{k,\varphi}$. With this notation we have:

Proposition 8.2.1. *The trace of the geometric Frobenius on the $*$-stalk of $b_* \overline{\mathbb{Q}}_\ell$ at z is equal to*

$$(1 - q) \cdot \prod_k (1 - q^{d_{k,\varphi}}) \,.$$

From this Proposition 8.1.5 above follows via the Lefschetz trace formula:

Proof of Proposition 8.1.5. Consider the diagram

$$
\begin{array}{ccc}
 & & \mathrm{Bun}_G \\
 & & \downarrow{\scriptstyle b} \\
\mathbb{P}(\mathrm{Hom}(E_1, E_2)) & \xrightarrow{\ g\ } & \overline{\mathrm{Bun}}_G \qquad \Delta \\
\downarrow & & \downarrow{\scriptstyle \bar{\Delta}} \\
\mathrm{Spec}\,\mathbb{F}_q & \xrightarrow{(E_1, E_2)} & \mathrm{Bun}_G \times \mathrm{Bun}_G
\end{array}
$$

where the square is cartesian. Since $\bar{\Delta}$ is proper, we can compute the desired trace via the Lefschetz trace formula applied to $g^* b_* \overline{\mathbb{Q}}_\ell$ on the projectivization $\mathbb{P}(\mathrm{Hom}(E_1, E_2))$ of $\mathrm{Hom}(E_1, E_2)$: The desired trace is equal to

$$\sum_{z \,\in\, \mathbb{P}(\mathrm{Hom}(E_1,E_2))(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}, z^* b_* \overline{\mathbb{Q}}_\ell) \,.$$

To rewrite this formula, we abuse notation and denote again by z the $\mathbb{F}_q$-point of $\overline{\mathrm{Bun}}_G$ obtained via the map g from the $\mathbb{F}_q$-point z of $\mathbb{P}(\mathrm{Hom}(E_1, E_2))$. We then split the sum according to whether the $\mathbb{F}_q$-point z lies in the G-locus or the B-locus of

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$\overline{\mathrm{Bun}}_G$, i.e., according to whether the corresponding map φ is an isomorphism or not. The summand corresponding to the B-locus is computed by Proposition 8.2.1 above and contributes the second term in the formula in Proposition 8.1.5. To compute the summand corresponding to the G-locus, recall first that the map

$$b : \ \mathrm{Bun}_G \ \longrightarrow \ \overline{\mathrm{Bun}}_G$$

is not an open immersion: It forms an etale cover of degree 2 of the G-locus of $\overline{\mathrm{Bun}}_G$ if the characteristic is not 2, and defines a radicial map onto the G-locus if the characteristic is equal to 2. To avoid having to distinguish these two cases, let

$$\mathbb{P}\,\mathrm{Isom}_{\mathrm{GL}_2}(E_1, E_2) \ \subset \ \mathbb{P}\,\mathrm{Hom}(E_1, E_2)$$

denote the quotient by $\mathbb{G}_m$ of the space of isomorphisms of vector bundles $\mathrm{Isom}_{\mathrm{GL}_2}(E_1, E_2)$, and let

$$r : \ \mathrm{Isom}_{\mathrm{SL}_2}(E_1, E_2) \ \longrightarrow \ \mathbb{P}\,\mathrm{Isom}_{\mathrm{GL}_2}(E_1, E_2)$$

denote the natural map. Then the restriction of $g^* b_* \overline{\mathbb{Q}}_\ell$ to the open subscheme $\mathbb{P}\,\mathrm{Isom}_{\mathrm{GL}_2}(E_1, E_2)$ is equal to $r_* \overline{\mathbb{Q}}_\ell$. Since the map r is finite we conclude that the contribution of the G-locus is equal to

$$\sum_{z \ \in \ \mathbb{P}(\mathrm{Isom}(E_1,E_2))(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}, z^* b_* \overline{\mathbb{Q}}_\ell) \ \ = \ \ |\,\mathrm{Isom}_{\mathrm{SL}_2}(E_1, E_2)(\mathbb{F}_q)|\,,$$

contributing the first term in the formula in Proposition 8.1.5. □

8.3. **Proof of the trace computation via local models.**

We now prove Proposition 8.2.1 above. First, we reduce the assertion to the analogous assertion for VinBun_G, or equivalently for the local models Y^n, stated in Lemma 8.3.1 below. A minor reduction step is necessary since the map b above is not an

open immersion. We then prove Lemma 8.3.1, using the results of Section 6.4 above. To state the lemma let

$$j_G : \ Y_G^n \ \longhookrightarrow \ Y^n$$

denote the open inclusion of the G-locus of Y^n, and let z be an $\mathbb{F}_q$-point of the stratum ${}_n Y_B^n = X^{(n)}$ of maximal defect. Then with the exact same notation as in the previous two sections we have:

Lemma 8.3.1. *The trace of the geometric Frobenius on the $*$-stalk of $j_{G,*} \, \overline{\mathbb{Q}}_\ell$ at the point z is equal to*

$$(1 - q) \cdot \prod_k (1 - q^{d_{k,\varphi}}) \, .$$

8.3.2. *Reduction to the lemma.* As in Section 4.2 above, knowing Lemma 8.3.1 above for all integers $n \geqslant 0$ is equivalent to knowing the analogous assertion for VinBun_G. To deduce Proposition 8.2.1 from the latter, we first assume that the characteristic is not equal to 2.

Denote by sign the sign local system on $\mathbb{A}^1 \smallsetminus \{0\}$, and denote by v^* sign its pullback to the G-locus $\mathrm{VinBun}_{G,G}$ along the natural map

$$v : \ \mathrm{VinBun}_{G,G} \ \longrightarrow \ \mathbb{A}^1 \smallsetminus \{0\} \, .$$

Furthermore, denote by

$$\gamma : \ \mathrm{VinBun}_G \ \longrightarrow \ \overline{\mathrm{Bun}}_G$$

the natural forgetful map, and let $j_{\mathrm{VinBun}_{G,G}}$ denote the open inclusion of the G-locus of VinBun_G. Then chasing through the definitions one finds:

Lemma 8.3.3.

$$\gamma^* b_* \overline{\mathbb{Q}}_{\ell \, \mathrm{Bun}_G} \ = \ j_{\mathrm{VinBun}_{G,G},*} \left(\overline{\mathbb{Q}}_{\ell \, \mathrm{VinBun}_{G,G}} \ \oplus \ v^* \, \mathrm{sign} \right)$$

Since the Frobenius traces on the $*$-stalks of $j_{\mathrm{VinBun}_{G,G},*}\,\overline{\mathbb{Q}}_{\ell\,\mathrm{VinBun}_{G,G}}$ can be computed on the local models Y^n, Proposition 8.2.1 follows from Lemma 8.3.1 above once we show that the second summand in Lemma 8.3.3 does not contribute. More precisely, letting $i_{\mathrm{VinBun}_{G,B}}$ denote the inclusion of the B-locus of VinBun_G, we need to show:

Lemma 8.3.4.

$$i_{\mathrm{VinBun}_{G,B}}^*\,j_{\mathrm{VinBun}_{G,G},*}\,v^*\,\mathrm{sign}\ =\ 0$$

Proof. As before it suffices to prove the analogous statement on the local models Y^n. Thus we have to show that

$$s^*\,j_{G,*}\,v^*\,\mathrm{sign}\ =\ 0\,,$$

where s and j_G are as before and v denotes the natural map

$$Y_G^n\ \longrightarrow\ \mathbb{A}^1\smallsetminus\{0\}\,.$$

To prove this, note first that Lemma 5.3.9 above shows that $j_{G,*}\,v^*\,\mathrm{sign}$ is naturally $\mathbb{G}_m$-equivariant for the contracting $\mathbb{G}_m$-action constructed in Section 5.2 above. Applying the contraction principle (see Lemma 5.2.14) and Lemma 4.1.7 above, the desired vanishing follows from the fact that the sign local system on $\mathbb{A}^1\smallsetminus\{0\}$ has trivial cohomology. $\square$

This concludes the reduction step under the assumption that the characteristic is not equal to 2. If the characteristic is equal to 2, then the map b defines a radicial map from Bun_G to the G-locus $\overline{\mathrm{Bun}}_{G,G}$. Thus the summand $v^*\,\mathrm{sign}$ does not appear in Lemma 8.3.3, and Lemma 8.3.4 is not even needed.

8.3.5. *Proof of Lemma 8.3.1.* We begin by recalling the following trace computation. As before let $\Lambda^{(n)}(\overline{\mathbb{Q}}_{\ell X})$ denote the n-th external exterior power on $X^{(n)}$ of the constant local system $\overline{\mathbb{Q}}_{\ell X}$ on the curve X over $\mathbb{F}_q$. Let D be an $\mathbb{F}_q$-point of $X^{(n)}$. As before we write

$$D = \sum_k n_k x_k$$

for certain distinct closed points x_k of the curve X and all $n_k \geqslant 1$, and we let d_k denote the degree of the residue field extension at x_k. We then have:

Lemma 8.3.6. *The trace of the geometric Frobenius on the $*$-stalk of $\Lambda^{(n)}(\overline{\mathbb{Q}}_{\ell X})$ is 0 unless all n_k are equal to 1. If all n_k are equal to 1, then the trace is equal to*

$$\prod_k (-1)^{d_k+1}.$$

To prove Lemma 8.3.1, we first apply Verdier duality to both sides of the equation in Proposition 6.4.4 above, and then combine the result with Lemma 6.4.2 above to obtain an expression for $s^* j_{Y_G^n *}\, \overline{\mathbb{Q}}_\ell$ in the Grothendieck group. Applying Lemma 8.3.6 above and taking into account that

$$H_c^*(\mathbb{A}^1 \smallsetminus \{0\}) = \overline{\mathbb{Q}}_\ell[-2](-1) \oplus \overline{\mathbb{Q}}_\ell[-1](0)$$

and that

$$\dim Y_G^n = 2n+1,$$

we find the following formula for the trace of the geometric Frobenius on the $*$-stalk of $j_{Y_G^n *}\, \overline{\mathbb{Q}}_\ell$ at a point $D \in X^{(n)}(\mathbb{F}_q) = {}_n Y_B^n(\mathbb{F}_q)$:

$$(1-q) \cdot \sum_{i+j=n} \sum_{D_1+D_2=D} (-1)^j q^j \cdot \prod_{x \,\in\, \mathrm{supp}(D_2)} (-1)^{\deg(x)+1}$$

Here the second sum runs over all pairs (D_1, D_2) of $\mathbb{F}_q$-points $D_1 \in X^{(i)}$, $D_2 \in X^{(j)}$ such that $D_1 + D_2 = D$ and such that the effective divisor D_2 is simple, i.e., each closed point occurring in D_2 appears with multiplicity 1; furthermore, we write $x \in \mathrm{supp}(D_2)$ to denote that a closed point x of the curve X occurs in D_2, and we let $\deg(x)$ denote the degree of the residue field extension at the point x.

To reformulate the above formula, let

$$D = \sum_k n_k x_k$$

for certain distinct closed points x_k of the curve X, as before. Then the datum of a pair (D_1, D_2) with the above properties is equivalent to the datum of a subset S of the set of closed points $\{x_k\}$ occurring in the effective divisor D. We can then rewrite the above formula as

$$(1 - q) \cdot \sum_S (-1)^{|S|} \cdot q^{\sum_{x \in S} \deg(x)}$$

where the sum ranges over all subsets S of the set of closed points occurring in the effective divisor $D \in X^{(n)}(\mathbb{F}_q)$. We do allow the set S to be the empty set, and in this case the corresponding summand is equal to 1.

Finally, to deduce the formula in Lemma 8.3.1 from the above preliminary formula, recall that the elementary symmetric polynomials in the variables $X_1, \ldots, X_m$ are precisely the coefficients appearing in the expansion of the product

$$\prod_{k=1}^{m} (T + X_k)$$

as a polynomial in T. Taking m to be the number of closed points appearing in the effective divisor D, setting $T = 1$, and setting $X_k = -q^{\deg x_k}$ transforms the preliminary formula to the desired one in Lemma 8.3.1.

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