Chiral Principal Series Categories

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Samuel David Raskin

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Abstract

This thesis begins a study of principal series categories in geometric representation theory using the Beilinson-Drinfeld theory of chiral algebras. We study Whittaker objects in the unramified principal series category. This provides an alternative approach to the Arkhipov-Bezrukavnikov theory of Iwahori-Whittaker sheaves that exploits the geometry of the Feigin-Frenkel semi-infinite flag manifold.

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1. Introduction

1.1. The goal of this thesis is initiate the *chiral* study of the spectral theory of Whittaker sheaves. The precise meaning of these words will be given below, but roughly: we will give a version of the work [AB09] of Arkhipov-Bezrukavnikov — considered to be such a description "over a point in a smooth curve" — that allows "points in the curve to collide."

In this thesis, we will give a new construction of the functor in Arkhipov-Bezrukavnikov theory by very different methods. We intend to show in a future publication that this functor coincides with the functor (inverse to) to the functor of [AB09].

1.2. The motivation for this work comes from problems in the geometric theory of unramified automorphic forms. Chiral methods are useful for moving from local to global in this theory. The Iwahori-Whittaker theory of [AB09] is the starting point for much of the progress in the local geometric Langlands program.

In the forthcoming work [Ras] we will explain an application along these lines of our theory to the problem of the spectral decomposition of geometric Eisenstein series in the global unramified setting.

1.3. This introduction is structured as follows. In §1.5-1.8 we will review the Arkhipov-Bezrukavnikov theory of Iwahori-Whittaker sheaves over a point. In §1.9-1.17 we will recall what the word "chiral" entails. Then in §1.26 we begin to describe the main construction.

The subject of this thesis is technical, and it is not the intention of this introduction to emphasize the technical points. Where it is possible to communicate the sense of a definition rather than giving the definition itself, we prefer to do that, leaving the proper treatment to the body of the text and hoping that the reader will not find this informality too unsettling.

1.4. We fix an algebraically closed field k of characteristic zero throughout this thesis.

1.5. We need the following notations from Lie theory.

Let G be a (necessarily split) reductive group over k, let B be a Borel subgroup of G with unipotent radical N and let T be the Cartan B/N. Let B^- be a Borel opposite to B, i.e., $B^- \cap B \xrightarrow{\simeq} T$. Let N^- denote the unipotent radical of B^-.

Let $\check{G}$ denote the corresponding Langlands dual group with corresponding Borel $\check{B}$, who in turn has unipotent radical $\check{N}$ and torus $\check{T} = \check{B}/\check{N}$, and similarly for $\check{B}^-$ and $\check{N}^-$.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{t}$, $\mathfrak{b}^-$, $\mathfrak{n}^-$, $\check{\mathfrak{g}}$, $\check{\mathfrak{b}}$, $\check{\mathfrak{n}}$, $\check{\mathfrak{t}}$, $\check{\mathfrak{b}}^-$ and $\check{\mathfrak{n}}^-$ denote the corresponding Lie algebras.

Let Λ denote the lattice of weights of T and let $\check{\Lambda}$ denote the lattice of coweights. Let $\mathcal{I}_G$ be the set of vertices in the Dynkin diagram of G. We recall that $\mathcal{I}_G$ is canonically identified with the set of simple positive roots and coroots of G. For $i \in \mathcal{I}_G$, we let $\alpha_i \in \Lambda$ (resp. $\check{\alpha}_i \in \check{\Lambda}$) denote the corresponding root (resp. coroot).

1.6. Let $K = k((t))$ be the local field of Laurent series with k-coefficients, and let $O \subseteq K$ consist of the subring of Taylor series. Let $G(K)$ denote the *loop group*: the group indscheme (over k) of maps $\overset{o}{\mathcal{D}} = \mathrm{Spec}(K) \to G$. Let $G(O) \subseteq G(K)$ denote the group scheme of maps from the disc $\mathcal{D} = \mathrm{Spec}(O)$ to G, and similarly for the other groups.

Let $\mathrm{ev} : G(O) \to G$ be the map given by evaluation on the closed point of $\mathcal{D}$ and let I be the Iwahori subgroup $\mathrm{ev}^{-1}(B)$. The choice of B^- gives rise to the opposite Iwahori subgroup $I^- = \mathrm{ev}^{-1}(B^-)$.

Let $\overset{o}{I} = \mathrm{ev}^{-1}(N)$ and $\overset{o}{I}{}^- = \mathrm{ev}^{-1}(I^-)$ denote the unipotent radicals of these Iwahori subgroups.

1.7. Choose a character $\psi'_{\overset{o}{I}{}^-} : \mathrm{Lie}(\overset{o}{I}{}^-) \to k$ non-degenerate in the sense that $\psi'_{\overset{o}{I}{}^-}|_{\mathfrak{n}^-(O)}$ factors through $\mathfrak{n}^-(O) \to \mathfrak{n}^-$ and $\psi'_{\overset{o}{I}{}^-}(f_i) \neq 0$ for every $i \in \mathcal{I}_G$ and $0 \neq f_i \in \mathfrak{g}$ in the root space $-\alpha_i$ of the negative simple root corresponding to $i \in \mathcal{I}_G$.

By unipotence of $\overset{o}{I}{}^-$, this character integrates to a character $\overset{o}{I}{}^- \to \mathbb{G}_a$ of $\overset{o}{I}{}^-$. Let $\psi_{\overset{o}{I}{}^-}$ denote the multiplicative D-module on $\overset{o}{I}{}^-$ induced by pullback from the exponential D-module on $\overset{o}{I}{}^-$.

1.8. Let Gr_G and $\mathrm{Fl}_G^{\mathrm{aff}}$ denote the indschemes $G(K)/G(O)$ and $G(K)/I$ respectively.

For our purposes, the principal result of [AB09] is the following.

Theorem 1.8.1. *There is a canonical equivalence of categories:*

$$D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}} \simeq \mathsf{QCoh}(\check{\mathfrak{n}}/\check{B}).$$

Here $D(\mathrm{Fl}_G^{\mathrm{aff}})$ is the derived category of D-modules on $\mathrm{Fl}_G^{\mathrm{aff}}$, $D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ denotes the full subcategory of objects satisfying $(\overset{o}{I}^-,\psi_{\overset{o}{I}_-})$-equivariance, $\check{\mathfrak{n}}/\check{B}$ denotes the stack quotient, and QCoh indicates the derived category of quasi-coherent sheaves on this stack.

By comparison, we have the following variant of the results of [FGV01] (see also [ABB+05] Corollary 2.2.3):

$$D(\mathrm{Gr}_G)^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}} \simeq \mathsf{Rep}(\check{G}) := \mathsf{QCoh}(\mathbb{B}\check{G}).$$

Here $\mathbb{B}\check{G}$ is the stack quotient $\mathrm{Spec}(k)/\check{G}$.

Remark 1.8.2. In truth, the cited references use the language of perverse sheaves and the Artin-Shreier sheaf in positive characteristic. One can translate as follows: first, the arguments are purely sheaf-theoretic, and therefore apply verbatim to the setting of holonomic D-modules. Then, as in [BD] §5.3.4, one sees that the relevant DG categories of D-modules are compactly generated, with compact objects exactly the holonomic objects.

1.9. **Factorization.** Next, we recall the meaning of the almost synonymous words *chiral* and *factorization.*

The subject begins with the Beilinson-Drinfeld theory of chiral algebras from [BD04], whose features we recall below.

Remark 1.9.1. We will give a somewhat leisurely introduction to the theory of chiral algebras below. We offer two justifications for this decision.

First, a substantial portion of the present thesis is to develop the chiral theory further in the derived setting.

Furthermore, the subject of chiral algebras carries a reputation of being very technical and for lacking applications, or at least, lacking applications in which the role played by the chiral structure is straightforward and easy to isolate from the arguments. However, there is a rich folklore around this subject, only partially written down, which explains what these things are good for. We hope that in presenting the general aspects of this material, the strategy of the present series of works will be made more transparent to the reader.

1.10. The Beilinson-Drinfeld theory of chiral algebras on a smooth curve X has the following salient features:

(1) Chiral algebras are of local origin on the curve X. Many of their invariants (e.g., modules at a point) are closely related to the geometry of the formal punctured disc, especially algebraic loop spaces and de Rham local systems on the formal punctured disc.

    Moreover, chiral algebras tend to "decrease the complexity" in the following sense. A chiral algebra whose fibers involve only the disc will have invariants associated with the whole of the formal punctured disc. For instances, the chiral geometry of an arc space tends to encode the usual geometry of the associated formal loop space. As another example, the chiral geometry of the Beilinson-Drinfeld affine Grassmannian, recalled below, tends to encode information about the whole of the algebraic loop group, and in particular its group structure.

(2) If X is a proper curve then chiral algebras give rise to interesting global invariants (e.g., through chiral homology).

(3) Chiral algebras appear naturally in much of the geometric representation theory involving the curve X. For example, see [KL 4], [BFS98], [Gai08], [BD] and [BG08]. Note that chiral algebras naturally arise through both algebraic and geometric constructions.

The combination of the above techniques makes the theory of chiral algebras especially relevant to geometric Langlands. Recall that the local geometric Langlands program seeks to decompose representations of the algebraic loop group of a reductive group G, with spectral parameters de Rham local systems on the formal punctured disc with structure group $\check{G}$ the dual reductive group to G.

The geometric and spectral sides each appear in (1) as arising from chiral algebras, and it is therefore natural to expect that local geometric Langlands admits a chiral avatar (c.f. the introduction to [Bei06]). Moreover, this should make the subject *easier*: in certain nice settings, we can move from the simple geometry of the disc to the much more complicated geometry of the formal punctured disc.

Then the local-to-global techniques can be brought to bear to give global applications as well.

Example 1.10.1. A primordial example of the above procedure is implicit in [BD], where the Feigin-Frenkel identification of critical-level chiral $\mathcal{W}$-algebras for Langlands dual Lie algebras is used to construct Hecke eigensheaves for regular opers.

1.11. A wonderful discovery of Beilinson-Drinfeld, explained in [BD04], is the two guises of chiral algebras: as chiral Lie algebras and as factorization algebras.

Chiral Lie algebras, a coordinate-free variant of the more classical notion of vertex algebra (see [Bor86] and [BF04]) are technically convenient in providing an algebraic perspective on chiral algebras. For example, the construction of a chiral Lie algebra from a Lie-* algebra (in the vertex language: vertex Lie algebras) is realized more naturally as an induction functor analogous to the usual enveloping algebra of a Lie algebra.

Factorization algebras, invented by Beilinson-Drinfeld, provide a much more geometric perspective. This is the perspective on which we will presently focus.

1.12. **The factorizable Grassmannian.** To motivate the definition of factorization algebra, it is convenient to recall the definition and features of the Beilinson-Drinfeld affine Grassmannian.

Let X be a smooth curve over k and let $x \in X$ be a closed point.

Let K_x denote the field of Laurent series at x and let $O_x \subseteq K_x$ denote its subring of integral elements. Let Γ be an affine algebraic group over k.

By fpqc descent, the affine Grassmannian $\mathrm{Gr}_{\Gamma,x} := \Gamma(K_x)/\Gamma(O_x)$ at x is the moduli space of Γ-bundles on X with a trivialization on the open $X \backslash x \subseteq X$.

For a positive integer n, the Beilinson-Drinfeld affine Grassmannian Gr_{Γ,X^n} is the moduli space of an n-tuple of points $x_1, \ldots, x_n$ of X, a Γ-bundle on X and a trivialization of the Γ-bundle away from $x_1, \ldots, x_n$.

The spaces Gr_{Γ,X^n} satisfy the following *factorization* property, say for $n = 2$:

$$\begin{gathered} \mathrm{Gr}_{\Gamma,X^2}|_{(X\times X)\backslash X} \simeq \mathrm{Gr}_{\Gamma,X} \times \mathrm{Gr}_{\Gamma,X}|_{(X\times X)\backslash X} \\ \mathrm{Gr}_{\Gamma,X^2}|_X \simeq \mathrm{Gr}_{\Gamma,X} \end{gathered} \tag{1.12.1}$$

where $X \to X \times X$ is the diagonal embedding.

1.13. **Factorization algebras.** Let X be a k-scheme of finite type.

A *factorization algebra* $\mathcal{A}$ on X is a rule that assigns to each positive integer n a D-module[1] $\mathcal{A}_{X^n}$ on X^n equivariant for the symmetric group S_n and satisfying a linearized version of (1.12.1) that says e.g. for $n = 2$ that we have S_2-equivariant equivalences:

$$\begin{gathered} \mathcal{A}_{X^2}|_{(X\times X)\backslash X} \simeq \mathcal{A}_X \boxtimes \mathcal{A}_X|_{(X\times X)\backslash X} \\ \mathcal{A}_{X^2}|_X \simeq \mathcal{A}_X. \end{gathered} \tag{1.13.1}$$

[1]We only take D-modules as a sheaf-theoretic context for concreteness. One can take quasi-coherent sheaves or ℓ-adic sheaves just as well.

In our setting of D-modules, the latter restriction should be understood in !-sense.

For example, we have the trivial example ω defined by the dualizing D-modules $n \mapsto \omega_{X^n}$.

Remark 1.13.1. Factorization spaces in geometry such as $n \mapsto \mathrm{Gr}_{G,X^n}$ are a rich source of factorization algebras. For example, taking the (quasi-coherent) global sections of the distributional D-module on the unit $X^n \subseteq \mathrm{Gr}_{G,X^n}$ one obtains a factorization algebra encoding the loop algebra $\mathfrak{g}(K_x) := \mathfrak{g} \underset{k}{\otimes} K_x$ for varying points x. One obtains the so-called chiral algebra of differential operators for the loop group of G by a similar procedure, c.f. [AG02].

More generally, correspondences between factorization spaces are very fruitful for producing factorization algebras by means of D-module operations.

1.14. **$\mathcal{E}_n$-algebras.** There is a close analogy between factorization algebras on a curve X and algebras over the homotopy theorist's little 2-discs operad, or more generally, factorization algebras on a smooth scheme X of dimension n are in analogy with operads over the little $2n$-discs operad. The reader may safely skip this analogy, as it will play no role in the text below.

Among classical — that is, non-derived — algebras, there are associative algebras and commutative algebras. The $\mathcal{E}_n$-algebras appear as intermediates in settings of more homotopical complexity, where $\mathcal{E}_1$-algebras are associative algebras and $\mathcal{E}_\infty$-algebras are commutative algebras.

In a traditional setting, namely, in a symmetric monoidal $(1, 1)$-category, an $\mathcal{E}_n$-algebra struture for $n \geqslant 2$ is equivalent to an $\mathcal{E}_\infty$-algebra structure. However, when there is greater homotopical flexibility, this is no longer the case.

For example, in the 2-category of (1,1)-categories, a $\mathcal{E}_2$-algebras is a braided monoidal category, which appeared in the 1980's as an intermediate between monoidal categories and symmetric monoidal categories. Similarly, n-fold loop spaces in topology carry an $\mathcal{E}_n$-algebra structure that cannot generally be upgraded to an $\mathcal{E}_{n+1}$-algebra structure.

Remark 1.14.1. Under this analogy, the factorization structure of the affine Grassmannian appears because the double loop space $\Omega^2(\mathbb{B}G)$ may be realized as the space of continuous maps:

$$D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\} \to \mathbb{B}G$$

sending the boundary $\partial D = S^1$ to the base-point. In words, this is the moduli of G-bundles on the disc with trivialization on the boundary S^1, which functions here as an analogue to the punctured disc.

Perhaps the simplest characterization of $\mathcal{E}_2$-algebra in a symmetric monoidal (higher) category $\mathcal{C}$ is the following: the category $\mathsf{Alg}(\mathcal{C}) = \mathcal{E}_1\text{–}\mathsf{alg}(\mathcal{C})$ forms a symmetric monoidal category itself, under the usual tensor product of associative algebras. Therefore, we can ask for associative algebras in $\mathsf{Alg}(\mathcal{C})$, i.e., $\mathsf{Alg}(\mathsf{Alg}(\mathcal{C}))$. In other words, we have an algebra $A \in \mathcal{C}$ with defining multiplication $m_1 : A \otimes A \to A$, and a second multiplication $m_2 : A \otimes A \to A$ such that m_2 is a morphism of algebras where $A \otimes A$ and A are regarded as algebras with respect to m_1.

Similarly, one may define an $\mathcal{E}_n$-algebra by asking for n-compatible multiplications.

We refer to [Lur12] for a greater discussion of this analogy, where it is explained how to relate $\mathcal{E}_n$-algebras and a topological analogue of factorization algebras.

1.15. **Factorization categories.** The analogy above suggests that not only the notion of factorization algebra is of relevance to representation theory, but also of factorization *category* as well. Indeed, a factorization category on a smooth curve is analogous to a braided monoidal category, which is well-known to be of great importance in representation theory.

Remark 1.15.1. The mathematical physicist's *fusion* procedure can be implemented mathematically in several different ways to draw a closer connection between braided monoidal categories and factorization categories.

In the case $X = \mathbb{P}^1$, [KL 4] used analysis to pass from the *algebraically* defined structure of factorization category on Kac-Moody representations, to obtain a braided monoidal category structure defined. Following physicists, Kazhdan and Lusztig referred to the resulting tensor product as *fusion*.

In fact, in some circumstances the fusion product can be constructed algebraically as well, as in [Gai01]. A general theory of fusion by means of nearby cycles, which is as yet undeveloped but still highly plausible, would be needed for the comparison between our functor and the Arkhipov-Bezrukavnikov functor.

A theory of factorization categories has been anticipated for some time now (c.f. [Gai08]), but has not appeared in the literature at this point due to the technical difficulties foundational in the subject. Such a theory will be provided in detail in Part 2 of this text.

1.16. A difficulty that one must grapple with in the theory of factorization algebras is the fact that the equivalences (1.13.1) must be understood in the derived category (already in the case of the dualizing sheaf!), and the equivalences must be then be required to be *homotopy compatible* in some appropriate sense.

Beilinson and Drinfeld circumvent this problem in [BD04] by working only with smooth curves and sheaves $\mathcal{A}_{X^I}$ such that $\mathcal{A}_{X^I}[-|I|]$ lies in the heart of the usual (alias: perverse) t-structure on the category of D-modules on X^I (this t-structure is referred to in *loc. cit.* as the t-structure for left D-modules); favorable arithmetic then provides a supply of examples of factorization algebras for which only abelian categories are necessary.

1.17. The recent advances in homotopical algebra, notably in [Lur09] and [Lur12], provide an easy language of higher categories in which the notion of homotopy compatibility may be used in a systematic way, unburdened by the construction of clever resolutions and model category structures.[2] This language allows for a different approach, working directly with collections of complexes of sheaves with homotopy compatible equivalences (1.13.1).

This approach is pursued in [FG12], where the theory of higher categories is shown to provide adequate foundations to develop the theory of factorization algebras on arbitrary schemes of finite type, allowing for schemes more general than smooth curves and for complexes of sheaves unfettered by any t-structure.

Moreover, many of the factorization algebras constructed in geometric representation theory by means of Remark 1.13.1 are inherently derived: they are constructed by sheaf-theoretic operations that only under limited and special circumstances preserve the heart of any t-structures. That is, they fall under the purview of the theory of [FG12] exclusively.

Remark 1.17.1. Even in the case of a smooth curve, the Francis-Gaitsgory approach provides a conceptually simpler and more unified theory than overlapping material in [BD04].

1.18. It is desirable to have a version of Theorem 1.8.1 that holds for factorization categories.

There are several difficulties here:

(1) The left hand side does not factorize. Indeed, unlike the maximal parahoric subgroup $G(O)$, the Iwahori subgroup I itself is not compatible with the factorization structure on $G(K)$.

    Indeed, let us attempt to define a factorization version of the Iwahori subgroup that lives over X^2: a point should be a pair of points x_1, x_2 in X, G-bundle on X with a trivialization away from x_1 and x_2, and with a reduction to the Borel B at the points x_1 and x_2.

[2]Note that model categories appear inadequate to the problem at hand: compare to [BD04] §0.12.

However, to formulate this scheme-theoretically, we need to ask for a reduction to B at the *scheme-theoretic union* of the points x_1 and x_2. Therefore, over a point x in the diagonal $X \subseteq X^2$, we are asking for a reduction to B on the first infinitesimal neighborhood of x, which corresponds to a rather smaller subgroup than the Iwahori group.

(2) The right hand side does factorize, but it feels incorrect. Indeed, as in [BD04], any algebraic stack gives rise to a factorization stack.[3]

However, as in §1.10, one expects the spectral theory of Whittaker sheaves to relate the geometry of de Rham local systems on the punctured disc, which are incompatible with this description.

We will explain the necessary modifications to (1) in §1.20 and to (2) in §1.21-1.22 below.

1.19. **Group actions on categories.** Before proceeding, it is useful to have some of the language of actions of the loop group on categories available. This theory, due to unpublished work of Gaitsgory and realized in the literature in [Ber] (and to a lesser and only implicit extent, in the present thesis) gives rise to the following language.

Remark 1.19.1. Let us clarify some potentially confusing language at this point: a group scheme is a scheme (possibly of infinite type) equipped with a group structure. A typical example is $\Gamma(O)$ for Γ an affine algebraic group. Recall that any affine group scheme is a filtered projective limit under dominant structure morphisms of affine algebraic groups, i.e., affine group schemes of finite type.

A group indscheme is an indscheme equipped with a group structure, where we use the appropriate product of indschemes in the definition (e.g., we can take the product of underlying prestacks here). A typical example is $\Gamma(K)$.

[3]In the setting of §1.14, this is analogous to the procedure of restriction from $\mathcal{E}_\infty$-algebras to $\mathcal{E}_2$-algebras.

An ind-group scheme is a group indscheme that can be written as a union of closed group subschemes. A typical example is $\Gamma(K)$ for Γ a unipotent group, or some variants, such as $N(K)T(O)$.

Note that for G a non-trivial reductive group, $G(K)$ is never an ind-group scheme. Note that this aspect is evident already for $G = \mathbb{G}_m$.

Remark 1.19.2. One obtains an analogy with the theory of groups over a local field by replacing k with a finite field and passing to k-points. Then algebraic groups are analogous to finite groups, group schemes are analogous to compact totally disconnected groups, and group indschemes are analogous to group objects in the category of ind-profinite sets.

We work in the "linear" setting of cocomplete (i.e., admitting all direct sums) DG categories $\mathcal{C}$ equipped with continuous functors. For a group indscheme $\mathcal{G}$, there is a notion of category (more precisely: cocomplete DG category) acted on by $\mathcal{G}$. This notion functions as an analogue of the notion of complex representation of a p-adic group.

There is a well-behaved theory of invariants and coinvariants for group schemes $\mathcal{G}$. Moreover, "Maschke's theorem" holds in this setting — we have an equivalence:

$$\mathcal{C}_{\mathcal{G}} \xrightarrow{\simeq} \mathcal{C}^{\mathcal{G}}$$

induced by the *averaging* functor $\mathcal{C} \to \mathcal{C}^{\mathcal{G}}$, which by definition is the right adjoint to the structure functor $\mathcal{C}^{\mathcal{G}} \to \mathcal{C}$. This averaging functor should be regarded as a categorical analogue of the norm map from usual representation theory.

Duality of cocomplete DG categories, in the sense of [Gai12a] or §19, canonically intertwines invariants and coinvariants.

This gives rise to a manageable theory of invariants and coinvariants for ind-group schemes. Indeed, for $\mathcal{G} = \cup \mathcal{G}_i$ we can take:

$$\mathcal{C}^{\mathcal{G}} := \lim_i \mathcal{C}^{\mathcal{G}_i} \text{ and } \mathcal{C}_{\mathcal{G}} := \operatorname{colim}_i \mathcal{C}_{\mathcal{G}_i}$$

where the limits and colimits here understood in the homotopy sense and are taken in the world of cocomplete DG categories. However, as one would expect by analogy with the group-theoretic context, Maschke's theorem fails in this setting.

Remark 1.19.3. There is a good theory of D-modules on spaces such as $G(K)$. It has been developed in the abelian categorical setting in [KV04], and in the specific case of the loop group, in [AG02]. In the derived setting, this theory was developed in some form in [BD] §7 and [FG06], and has recently been improved using modern homotopical algebra following ideas of Gaitsgory. Gaitsgory's theory has recently been developed by Beraldo in [Ber] and in the present thesis in the extended appendix §16.

This theory interacts well with regard to the theory of loop group actions. The group $G(K)$ acts on its category of D-modules $D(G(K))$.[4] Moreover, for a *compact-open* subgroup $\mathcal{K}$ of $G(K)$, i.e., a group subscheme, the quotient $G(K)/\mathcal{K}$ exists as an indscheme of ind-finite type, and we have a canonical identification:

$$D(G(K)/\mathcal{K}) \simeq D(G(K))^{\mathcal{K}}$$

where the functor from left hand side to right is given by pullback.

1.20. In the language of group actions on categories, the Arkhipov-Bezrukavinkov category $D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}{}^-,\psi_{\overset{o}{I}{}^-}}$ is obtained from the factorization category $D(G(K))$ by imposing two Iwahori-type conditions: Iwahori-equivariance on the right and $\psi_{\overset{o}{I}{}^-}$-twisted $\overset{o}{I}{}^-$-equivariance on the right.

First, we replace $\overset{o}{I}{}^-$ and its character $\psi_{\overset{o}{I}{}^-}$ by the group $N^-(K)$ and $\psi_{N^-(K)}$, where $\psi_{N^-(K)}$ is given on the level of Lie algebras as the composition:

[4]For reasons explained in §16, it would be better if we wrote either $D^!(G(K))$ here.

$$\mathfrak{n}^-(K) \to \Big(\mathfrak{n}^-/[\mathfrak{n}^-,\mathfrak{n}^-]\Big)(K) = \bigoplus_{i\in\mathcal{I}_G} K \xrightarrow{(f_i)_{i\in\mathcal{I}_G}\mapsto\sum_{i\in\mathcal{I}_G}\mathrm{Res}(f_i dt)} k \tag{1.20.1}$$

for t a coordinate and Res the residue map.

Indeed, [AB09] already acknowledges that the use of $\overset{o}{I}{}^-$ in place of $N^-(K)$ is somewhat unsatisfactory, and that they make this choice only to avoid group indschemes (or Drinfeld's compactification: c.f. [FGV01]).

Remark 1.20.1. For factorization purposes, it is better to incorporate a twist by 1-forms into the definition of the group $N^-(K)$ so that we do not need to choose a coordinate t. We postpone this issue to the body of the text.

One can show that the categories of $(N^-(K), \psi_{N^-(K)})$ and $(\overset{o}{I}{}^-, \psi_{\overset{o}{I}{}^-})$-equivariant D-modules on the affine flag variety are canonically equivalent: we include this result in the appendix §18 for the reader's convenience.

One has the following general result (modeled on a standard result of p-adic representation theory):

Proposition 1.20.2. *If $\mathcal{C}$ is a DG category acted on by $G(K)$ that is compactly generated such that every compact object $X \in \mathcal{C}$ is equivariant for some sufficiently small (depending on X) compact open subgroup of $\mathcal{C}$. Then the functors:*

$$\mathcal{C}_I \to \mathcal{C}_{B(O)} \to \mathcal{C}_{N(K)T(O)}$$

$$\mathcal{C}^{N(K)T(O)} \to \mathcal{C}^{B(O)} \xrightarrow{\mathrm{Av}_{B(O)\to I,*}} \mathcal{C}^I$$

are equivalences, where $\mathcal{C}_I \to \mathcal{C}_{B(O)}$ is the left adjoint to the tautological functor $\mathcal{C}_{B(O)} \to \mathcal{C}_I$, and similarly for $\mathrm{Av}_{B(O)\to I,*}$.

Remark 1.20.3. Under the "norm" equivalences $\mathcal{C}_I \xrightarrow{\simeq} \mathcal{C}^I$ and $\mathcal{C}_{B(O)} \to \mathcal{C}^{B(O)}$, the functor $\mathcal{C}_{B(O)} \to \mathcal{C}_I$ identifies with $\mathrm{Av}_{B(O)\to I,*}$.

Remark 1.20.4. Under the above hypotheses, one obtains a somewhat complicated equivalence between $\mathcal{C}^{N(K)T(O)}$ and $\mathcal{C}_{N(K)T(O)}$.

We include a proof of this result in §17.

The category of D-modules on $G(K)$, or Whittaker D-modules on $G(K)$, both satisfy this hypothesis. Therefore, we can replace $D(\mathrm{Fl}_G^{\mathrm{aff}})$ with either $D(G(K))^{N(K)T(O)}$ or $D(G(K))_{N(K)T(O)}$.

It is convenient (for reasons we do not presently explain) to choose $D(G(K))_{N(K)T(O)}$. We denote the category by $D(\mathfrak{Fl}^{\frac{\infty}{2}})$ and consider as a category of D-modules on the non-existant *semi-infinite flag manifold* $G(K)/N(K)T(O)$: see [FF90] for more discussion on this point.

Therefore, we obtain our geometric category: we take $(N^-(K), \psi_{N^-(K)})$-invariants and $N(K)T(O)$-coinvariants on the left and on the right of $D(G(K))$. We denote this category by $\mathsf{Whit}^{\frac{\infty}{2}}$.

This category factorizes: we provide a detailed discussion of this structure in §6.

Remark 1.20.5. Working with $N(K)T(O)$ in place of Iwahori introduces new technical difficulties of various kinds. To single out one, the Iwahori subgroup is *parahoric*, so $\mathrm{Fl}_G^{\mathrm{aff}}$ is and ind-proper indscheme. Not only is $G(K)/N(K)T(O)$ not an indscheme, but this parahoric feature of Iwahori bears no obvious counterpart for the semi-infinite flag variety. This is especially troublesome in the factorization setting.

1.21. Replacing the category $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$ is somewhat more direct.

For a point $x \in X$ and an affine algebraic group Γ, let $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ denote the prestack of de Rham local systems on $\overset{o}{\mathcal{D}}_x$.

Formally: we have the indscheme Conn_Γ of $\mathrm{Lie}(\Gamma)$-valued 1-forms (i.e., connection forms) and this is equipped with the usual gauge action of $\Gamma(K_x)$. We form the quotient and stackify for the étale topology on AffSch and denote this by $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$.

Remark 1.21.1. $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is not an algebraic stack of any kind because we quotient by the loop group $\Gamma(K_x)$, an indscheme of ind-infinite type. It could be considered as a prototypical *semi-infinite Artin stack*, the theory of which has not been developed.

The assignment $x \mapsto \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ obviously factorizes.

For $\Gamma = \mathbb{G}_a$, one easily shows that $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is canonically isomorphic to the affine line crossed with $\mathbb{B}\mathbb{G}_a$ by showing that every connection is gauge equivalent to one with regular singularities and then taking the residue of the resulting form.

More generally, for Γ unipotent we have a canonical identification:

$$\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x) \xrightarrow{\simeq} \mathrm{Lie}(\Gamma)/\Gamma$$

by the same construction.

However, this identification does not at all factorize: as in the discussion of the obstruction to factorizing the Iwahori subgroup, the notion of connection with regular singularities is not compatible with factorization.

Similarly, we let $\mathrm{LocSys}_\Gamma(\mathcal{D}_x)$ denote the category of local systems on the disc, defined as above but where we take the group $\Gamma(O_x)$ and the group scheme of 1-forms without poles. This is (compatibly with factorization) identified with the stack $\mathbb{B}\Gamma$: every local system is trivial, and trivializations are equivalent to trivializations of the underlying G-bundle on a point.

We therefore replace $\check{\mathfrak{n}}/\check{B}$ with the equivalent space:

$$\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)$$

of $\check{B}$-local systems on the punctured disc whose underlying $\check{T}$-local system has been extended to the non-punctured disc.

As in the discussion above, this space is point-wise over the curve equivalent to $\check{\mathfrak{n}}/\check{B}$, but carries a different factorization structure.

1.22. Recall that for a finite type scheme (or stack) Z, [GR14] has defined a DG category $\mathsf{IndCoh}(Z)$ of *ind-coherent sheaves* on Z. We recall simply that for Z smooth, we have a canonical identification of $\mathsf{IndCoh}(Z)$ with $\mathsf{QCoh}(Z)$, and we recall that in the general setting the compact generation properties of IndCoh are much simpler than of QCoh.

We would like to replace the category $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B}) = \mathsf{IndCoh}(\check{\mathfrak{n}}/\check{B})$ by the factorization category:

$$x \mapsto \mathsf{IndCoh}\big(\operatorname{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\operatorname{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \operatorname{LocSys}_{\check{T}}(\mathcal{D}_x)\big).$$

However, IndCoh has not been defined in this setting: the spaces of local systems on the punctured disc are defined as the quotient of an indscheme of ind-infinite type by a group of ind-infinite type.

Remark 1.22.1. The choice of notation IndCoh in place of QCoh is because we anticipate that IndCoh should be much more manageable for "semi-infinite" types of spaces, due to its better functoriality and categorical properties. Moreover, we expect that in the factorization setting, there is a meaningful difference between IndCoh and QCoh for the spaces we are considering.

Ignoring these issues, we formulate the following rough conjecture:

Main Conjecture. *There is an equivalence of factorization categories:*

$$\mathsf{Whit}^{\frac{\infty}{2}} \xrightarrow{\simeq} \Big(x \mapsto \mathsf{IndCoh}\big(\operatorname{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\operatorname{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \operatorname{LocSys}_{\check{T}}(\mathcal{D}_x)\big)\Big). \tag{1.22.1}$$

1.23. The main achievement of this thesis is a functor very close to the functor (1.22.1). However, since the right hand side of (1.22.1) is not defined, we need to explain the substitute that we use. We will address this point in §1.25.

1.24. We briefly recall Lurie's approach to deformation theory [Lur11a].

Suppose that $\mathcal{X}$ is a "nice enough" stack and $x \in \mathcal{X}$ is a k-point. Then the shifted tangent complex $T_{\mathcal{X},x}[-1]$ identifies with the Lie algebra $\mathrm{Lie}(\mathrm{Aut}_{\mathcal{X}}(x))$ of the (derived) automorphism group of $\mathcal{X}$ at x, and there is an identification of the DG category $\mathsf{IndCoh}(\mathcal{X}_x^\wedge)$ of ind-coherent sheaves on the formal completion of $\mathcal{X}$ at x with $T_{\mathcal{X},x}[-1]$-modules.

1.25. The stack $\mathrm{LocSys}_{\check{N}}(\overset{o}{\mathcal{D}}_x)$ has shifted tangent complex $H^*_{dR}(\overset{o}{\mathcal{D}}_x, \check{\mathfrak{n}} \otimes k)$ as a (derived) Lie algebra. Ignoring the slight problem of defining this de Rham cohomology, the philosophy of [BD04] indicates that modules for this Lie algebra should be equivalent to chiral modules for the chiral envelope of the Lie-$*$ algebra $\check{\mathfrak{n}} \otimes k_X$ on X.

A slight variant: consider $D(\mathrm{Gr}_T)$ as a commutative chiral category. This chiral category is an avatar of the symmetric monoidal category of $\check{\Lambda}$-graded vector spaces. The grading on $\check{\mathfrak{n}}$ makes it a Lie-$*$ algebra in this commutative chiral category, and chiral modules for its chiral envelope model $\check{\Lambda}$-graded modules for the graded Lie algebra $H^*_{dR}(\overset{o}{\mathcal{D}}_x, \check{\mathfrak{n}} \otimes k)$. We denote the corresponding chiral algebra in $D(\mathrm{Gr}_T)$ by $\Upsilon_{\check{\mathfrak{n}}}$, following notation introduced in [BG08].

Therefore, chiral modules in $D(\mathrm{Gr}_T)$ for $\Upsilon_{\check{\mathfrak{n}}}$ model the category of ind-coherent sheaves on:

$$\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^\wedge \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)$$

where $\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^\wedge$ is the formal completion at the trivial local system.

1.26. We now can state our main construction is a reasonably precise form:

We construct a functor $\mathsf{Whit}^{\frac{\infty}{2}}$ to $\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T))$ to the category of (unital) chiral modules for $\Upsilon_{\check{\mathfrak{n}}}$.

This functor is constructed by the following natural technique. We have a functor $\mathsf{Whit}^{\frac{\infty}{2}} \to D(\mathrm{Gr}_T)$ constructed by forgetting the Whittaker condition and then applying the !-restriction along the map $\mathrm{Gr}_T \to \mathfrak{Fl}^{\frac{\infty}{2}}$.

The main theorem in our construction is the following.

Theorem 1.26.1. *Under this functor, the unit object in the unital factorization category* $\mathsf{Whit}^{\frac{\infty}{2}}$ *maps to the factorization algebra* $\Upsilon_{\check{\mathfrak{n}}} \in D(\mathrm{Gr}_T)$.

The formalism of chiral categories then produces the desired functor.[5]

2. Conventions

2.1. We fix an algebraically closed field k of characteristic zero throughout the thesis. All schemes, etc, are understood to be defined over k.

2.2. **Lie theory.** We understand *reductive group* to be a connected reductive group over k. We consider Langlands dual reductive groups as also being defined over k.

We fix a (connected) reductive group G through the thesis, and use the accompanying notations from §1.5. Moreover, we fix a choice of Chevalley generators $\{f_i\}_{i \in \mathcal{I}_G}$ of $\mathfrak{n}^-$.

Finally, we use the notation ρ for the half-sum of the positive roots of $\mathfrak{g}$.

We let Λ and $\check{\Lambda}$ denote the weights and coweights of G. We let e.g. Λ^+ denote the dominant weights, and let $\check{\Lambda}^{pos}$ denote the $\mathbb{Z}^{\geqslant 0}$-span of the simple coroots (and similarly for Λ^{pos} and $\check{\Lambda}^+$).

2.3. Let X be a smooth projective curve.

We let Bun_G denote the moduli stack of G-bundles on X. Recall that Bun_G is a smooth Artin stack locally of finite type (though not quasi-compact).

Similarly, we let Bun_B, Bun_N, and Bun_T denote the corresponding moduli stacks of bundles on X. However, we note that we will abuse notation in dealing specifically with bundles of structure group N^-: we will systematically incorporate a twist discussed in detail in §3.7.

[5] A toy model: Given a monoidal category $\mathcal{C}$ with unit $\mathbb{1}_{\mathcal{C}}$ and a lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$, the functor F naturally upgrades to a functor $\mathcal{C} \to F(\mathbb{1}_{\mathcal{C}})\text{–}\mathsf{mod}(\mathcal{D})$. We remark that in the analogy between chiral categories and monoidal categories, the role of chiral functor is played by that of *lax* monoidal functor.

2.4. **Higher categories.** We rely heavily on the theory of higher categories, whose existence is due to the work of many mathematicians. This theory was developed systematically in Lurie's [Lur09] and [Lur12], and we use these as our preferred reference where appropriate.

We assume that the reader is comfortable with higher category theory and derived algebraic geometry. However, we will carefully establish notation and conventions below, highlighting the points where our terminology differs from [Lur09] and [Lur12].

Unlike [Lur09], our use of the theory is *model independent*: there are different[6] models of $(\infty, 1)$-categories[7] (quasicategories, Segal sets, etc.), each with its own intrinsic notion of, say, homotopy colimit. We use the theory only in as much as it can be implemented in each of these different models, that is, we allow ourselves to use the language of homotopy colimits, but not to use the language of, say, quasicategories.[8]

We use terms such as isomorphism and equivalence interchangeably.

2.5. We find it convenient to assume higher category theory as the basic assumption in our language. That is, we will understand "category" and "1-category" to mean "$(\infty, 1)$-category," "colimit" to (necessarily) mean "homotopy colimit," "groupoid" to mean "∞-groupoid" (aliases: homotopy type, space, etc.), "2-category" to mean $(\infty, 2)$-category, etc. "Morphism" means 1-morphism. We use the phrase "set" interchangeably with "discrete groupoid," i.e., a groupoid whose higher homotopy groups at any basepoint vanish.

[6]However, these theories are mutually Quillen equivalent; see [Toë05].

[7]We recall for the reader's convenience that (n, m)-category ($0 \leqslant m \leqslant n \leqslant \infty$) refers to a higher category with possibly non-trivial k-morphisms for $k \leqslant n$, and in which k-morphisms are assumed invertible for $k \geqslant m$. E.g., a $(1, 1)$-category is a usual category, a $(2, 2)$-category is a usual 2-category, etc.

[8]The reader uncomfortable with this approach may happily understand everything to be implemented in quasicategories as in [Lur09], though our language will differ from *loc. cit.* at some places; the translation should always be clear.

When we need to refer to the more traditional notion of category, we use the term $(1,1)$-category.

In particular, we refer to the notion of "stable ∞-category" from [Lur12] as a stable category.

When we say that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, we mean that there is a functor $\mathcal{D} \to \mathcal{C}$ given inducing equivalences on the groupoids of morphisms.

2.6. **Aside: on new foundations.** We draw the reader's attention to Voevodsky's program [V$^+$13]. This program, not yet fully implemented, offers a different perspective, and one that we implicitly take up in our use of higher category theory. Namely, that the idea of set theory as a foundation for mathematics is inadequate, and should be replaced by more categorical foundations.

In set theory, the predicate is equality of elements of a set. This is inadequate to standard mathematical practice: for example, it allows one to speak of different sets with one element, even though there is no "test" using practical mathematics that could distinguish such sets. This problem is also visible in the difference between isomorphism and equivalence of (usual) categories, reflecting that the usual definition of category as founded on set theory is an inadequate notion.

By contrast, in Voevodsky's vision, the basic predicate is that of having specified an identification between two different objects. Immediately, the atomic sets are replaced by the more fluid *homotopy sets*, i.e., ∞-groupoids: indeed, here we see objects, ways of identifying two objects, ways of saying that two identifications of two objects are the same, and so on.

This is the perspective that we implicitly take, anticipating that proper foundations based on groupoids and not on sets will be completed. Still, as emphasized above, there are various frameworks (such as [Lur09]) in the set-theoretic paradigm that are perfectly adequate for our needs.

2.7. **Conventions regarding 2-categories.** The theory of (unital) chiral categories is most naturally developed using the theory of 2-categories. Recall that Segal categories provide an adequate model for 2-categories, granted a theory of 1-categories (this approach is developed in detail e.g. in [GR14]).

Every 2-category has an underlying 1-category in which we forget all non-invertible 2-morphisms. For many purposes (such as computing limits and colimits), this underlying category is perfectly adequate, and where it is irrelevant, we do not pay particular attention to the distinction, hoping that this makes it easier for the reader.

For $\mathcal{C}$ a 2-category, we use the notation $\mathsf{Hom}_{\mathcal{C}}(X,Y)$ (as opposed to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$) to indicate that we take the category of maps $X \to Y$, not the groupoid of maps.

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ of 2-categories is *1-fully faithful* if the induced maps:

$$\mathsf{Hom}_{\mathcal{C}}(X,Y) \to \mathsf{Hom}_{\mathcal{D}}(F(X),F(Y))$$

are fully-faithful functors. A *1-full subcategory* means the essential image of such a functor. If in practice "full subcategory" means that we impose some conditions on a class of objects, then "1-full" means that we impose conditions on both objects and morphisms.

2.8. **Accessibility.** We will typically ignore cardinality issues that arise in category theory. The standard way to do this is through the use of accessible categories (we recall that this condition is satisfied for essentially small categories and for compactly generated categories). The author's opinion is that focusing too much on accessibility issues distracts the reader who is not familiar with the ideas, while omitting these points will not create confusion for the reader who is.

But we will enforce the following conventions:

- Categories are assumed to be locally small, i.e., Hom groupoids are essentially small.

- We use the term “indexing category” synonymously with “essentially small category.” A category seen indexing a colimit or limit is assumed to be essentially small. If we use e.g. the term “all colimits” (as in: “such and such functor commutes with all colimits”), this certainly means “all small colimits.”
- All functors between accessible categories are assumed to be accessible.
- DG categories are always assumed to be accessible.
- The term “groupoid” nearly always refers to an essentially small groupoid.

2.9. **Notation.** Let Cat denote the (2-)category of essentially small categories and let Gpd denote the category of essentially small groupoids.

Let Cat_{pres} denote the category of presentable (i.e., cocomplete and accessible) categories under functors that commute with arbitrary colimits. We consider Cat_{pres} as a symmetric monoidal category equipped with the tensor product $\otimes$ of [Lur12] §6.3.

For $\mathcal{C}$ and $\mathcal{D}$ categories, we let $\mathsf{Hom}(\mathcal{C}, \mathcal{D})$ denote the category of functors between $\mathcal{C}$ and $\mathcal{D}$.

For $\mathcal{C}$ an essentially small category, we let $\mathsf{Ind}(\mathcal{C})$ denote the category of its ind-objects, as in [Lur09].

2.10. **Grothendieck construction.** For $F : \mathcal{I} \to \mathsf{Cat}$ a functor, we let $\mathsf{Groth}(F) \to \mathcal{I}$ denote the corresponding coCartesian fibration attached by the (higher-categorical) Grothendieck construction, and we let $\mathsf{coGroth}(F) \to \mathcal{I}^{op}$ denote the corresponding Cartesian fibration.

For $\alpha : i \to j$ a morphism in $\mathcal{I}$ and $Y \in F(i) = \mathsf{Groth}(F) \times_{\mathcal{I}} \{i\}$, we will often use the notation $\alpha(Y)$ for the induced object of $F(j) = \mathsf{Groth}(F) \times_{\mathcal{I}} \{j\}$.

2.11. **DG categories.** By DG category, we mean an (accessible) stable category enriched over k-vector spaces. We denote the category of DG categories under k-linear exact functors by DGCat and the category of cocomplete DG categories under continuous[9] k-linear functors by DGCat_{cont}. As with Cat_{pres}, we consider DGCat_{cont} as equipped with the symmetric monoidal structure $\otimes$ from [Lur12] §6.3.

Recall that from the higher categorical perspective, the cone is equivalently a cokernel. Therefore, we use the notation Coker where others might use Cone.

For $\mathcal{C}$ a DG category equipped with a t-structure, we let $\mathcal{C}^{\geqslant 0}$ denote the subcategory of *coconnective* objects, and $\mathcal{C}^{\leqslant 0}$ the subcategory of *connective* objects (i.e., the notation is the standard notation relative to the cohomological grading convention). We let $\mathcal{C}^{\heartsuit}$ denote the heart of the t-structure.

We let Vect denote the DG category of k-vector spaces: this DG category has a t-structure with heart $\mathsf{Vect}^{\heartsuit}$ the abelian category of k-vector spaces. Similarly, for A a k-algebra (i.e., an algebra in Vect), we let $A\text{–}\mathsf{mod}$ denote the DG category of its left modules.

We use the material of the short note [Gai12a] freely, taking for granted the reader's comfort with the ideas of *loc. cit.*

2.12. **Monoidal categories.** We assume the reader is throughly familiar with this theory.

We will use the following conventions.

We use the term *colored operad* in place of the term of ∞-*operad* from [Lur12], preferring to use *operad* for a "colored operad with one color." We assume the presence of units according to standard conventions, so e.g. "commutative operad," we understand the operad controlling *unital*[10] commutative algebras. Symmetric monoidal functors between symmetric monoidal categories are assumed to be unital, though we allow ourselves

[9]There is some disagreement in the literature of the meaning of this word. By *continuous functor*, we mean a functor commuting with filtered colimits. Similarly, by a *cocomplete* category, we mean one admitting all colimits.

[10]To not be misleading: the phrase "commutative algebra" appearing in isolation indicates a unital commutative algebra.

to speak of e.g. symmetric monoidal functors between non-unital symmetric monoidal categories, obviously meaning the non-unital version.

Next, we use the term *lax symmetric monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories to refer to a morphism of the underlying colored operads. We recall that such an F is equipped with functorial associative maps:

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

for $X, Y \in \mathcal{C}$. We use the term *colax monoidal functor* for the dual notion, in which we have functorial morphisms:

$$F(X \otimes Y) \to F(X) \otimes F(Y).$$

2.13. **Cofinality.** There is some disagreement in the literature over the meaning of *cofinal* (typically due to trying to avoid confusion with the word "final," which ought not to take disparate meanings in category theory). We say that a functor $F : \mathcal{I} \to \mathcal{J}$ of indexing categories is *cofinal* if for every category $\mathcal{C}$, a functor $G : \mathcal{J} \to \mathcal{C}$ admits a colimit if and only its restriction to $\mathcal{I}$ does, and the induced map:

$$\operatorname{colim} G \circ F \to \operatorname{colim} G$$

is an equivalence. We use the term *op-cofinal* to mean that $F^{op} : \mathcal{I}^{op} \to \mathcal{J}^{op}$ is cofinal, i.e., that the above conditions are satisfied for limits instead of colimits.

Remark 2.13.1. Our use of *cofinal* is in accordance with [Lur09]. In [Lur12], Lurie uses the terminology *left cofinal* for our cofinal, and *right cofinal* for our op-cofinal.

2.14. **Derived algebraic geometry.** Following our "always-derived" conventions, our default assumption is that algebraic geometry means derived algebraic geometry.

Roughly, the development goes as follows: the category AffSch is defined to be the opposite category to the category of commutative k-algebras that are connective as vector spaces, i.e., commutative k-algebras in $\mathsf{Vect}^{\leqslant 0}$. We then define the category PreStk of *prestacks* as the the category of (accessible) functors $\mathsf{AffSch}^{op} \to \mathsf{Gpd}$. We have Yoneda embedding $\mathsf{AffSch} \hookrightarrow \mathsf{PreStk}$, and schemes are defined so that this extends to an embedding $\mathsf{AffSch} \hookrightarrow \mathsf{Sch} \hookrightarrow \mathsf{PreStk}$.

We say that an affine scheme is *classical* if it is of the form $\mathrm{Spec}(A)$ with $H^i(A) = 0$ for $i \neq 0$, i.e., if it is a "usual" affine scheme. More generally, we say that a prestack is *classical* if it lies in the subcategory of functors $\mathsf{AffSch}^{op} \to \mathsf{Gpd}$ that are left Kan extensions of their restriction to the (1,1)-category of classical affine schemes.

For X a prestack, we let $\mathsf{QCoh}(X)$ denote the symmetric monoidal DG category of its quasi-coherent sheaves, defined by right Kan extension from the functor $\mathrm{Spec}(A) \mapsto A\text{–}\mathsf{mod}$. A crucial point of derived algebraic geometry (that is not true in classical algebraic geometry) is that for $X \to Z \leftarrow Y$ schemes, the map:

$$\mathsf{QCoh}(X) \underset{\mathsf{QCoh}(Z)}{\otimes} \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X \underset{Z}{\times} Y)$$

is an equivalence.

For G a group stack, we let $\mathbb{B}G = \mathbb{B}(G)$ denote the classifying stack of G, i.e., the étale sheafification of the functor:

$$(S \in \mathsf{AffSch}^{op}) \mapsto \mathbb{B}(G(S))$$

where in this equation, $\mathbb{B}$ is also denoting the delooping functor for group-like monoids in Gpd.

For X a scheme, we let $\Omega^1_X \in \mathsf{QCoh}(X)^{\leqslant 0}$ denote the cotangent complex, and let $\Omega^{1,cl}_X := H^0(\Omega^1_X) \in \mathsf{QCoh}(X)^{\heartsuit}$ denote the classical cotangent sheaf.

To avoid overburdening the terminology, we use "finite type" for a morphism in derived algebraic geometry where others use "almost finite type." When we say a scheme X is finite type, this certainly means relative to the structure map $X \to \mathrm{Spec}(k)$.

2.15. **Non-derived algebraic geometry.** In fact, the heart of this thesis is about geometric computations with D-modules, which are immune to the distinction between derived and classical schemes (or even classical and reduced schemes). Therefore, in Part 1 and in §16, we impose the convention that schemes and prestacks are supposed to be classical, since it would be overly burdensome to write "classical" everywhere. We alert the reader's attention to this point here, though we reiterate in *loc. cit.*

2.16. D**-modules.** We use the D-module formalism in the format developed in [GR14].

For S a scheme of finite type, we let $D(S)$ denote the DG category of D-modules on S. Recall that the prestack S_{dR} is defined by $S_{dR}(T) := S(T^{cl,red})$ for an affine scheme T, where $T^{cl,red}$ is the reduced classical scheme underlying T; then we have:

$$D(S) := \mathsf{QCoh}(S_{dR}) \overset{-\otimes\omega_{S_{dR}}}{\simeq} \mathsf{IndCoh}(S_{dR})$$

for ω the dualizing sheaf of the ind-coherent theory.

For $f : S \to T$ a morphism, we let $f^! : D(T) \to D(S)$ denote the corresponding map. Recall that this functor is the *-pullback in the QCoh picture and the !-pullback in the IndCoh picture. Let $f_{*,dR} : D(S) \to D(T)$ denote the de Rham pushforward functor constructed in [GR14]. We let $f_!$ and $f^{*,dR}$ denote the corresponding partially-defined left adjoints.

For S a scheme with structure map $p : S \to \mathrm{Spec}(k)$, we let $\omega_S := p^!(k) \in D(S)$ and $k_S := p^{*,dR}(k) \in D(S)$ denote the dualizing sheaf and the constant sheaf respectively. Let $\mathrm{IC}_S \in D(S)$ denote the intersection cohomology D-module. Recall that for S smooth, $\mathrm{IC}_S = k_S[\dim_S] = \omega_S[-\dim_S]$.

We consider $D(S)$ as equipped with the t-structure called the "right t-structure" in [GR14]. We note that for S smooth, this is the t-structure considered in the usual D-module theory, and for general S it corresponds to the perverse t-structure under the Riemann-Hilbert correspondence; in particular, we have $\mathrm{IC}_S \in D(S)^\heartsuit$. We therefore refer to this t-structure as the *perverse t-structure* where such clarification is necessary.

We will also use the *constructible t-structure* on the regular holonomic subcategory, the t-structure whose heart corresponds to constructible sheaves under Riemann-Hilbert.

We use $\overset{!}{\otimes}$ to denote the standard tensor product of D-modules, for which $\mathcal{F} \overset{!}{\otimes} \mathcal{G} = \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ for Δ the diagonal, and the "partially-defined tensor-product" $\overset{*}{\otimes}$, for which $\mathcal{F} \overset{*}{\otimes} \mathcal{G}$ is $\Delta^{*,dR}(\mathcal{F} \boxtimes \mathcal{G})$ if it is defined (which is the case e.g. if $\mathcal{F}$ and $\mathcal{G}$ are holonomic, or if one of them is lisse).

Part 1. The Chevalley complex

We remind the reader that throughout this part, all schemes are assumed to be classical (meaning: non-derived) schemes, and similarly, all (pre)stacks are assumed to be classical.

We assume for convenience that the derived group of G is simply-connected. However, one may remove this assumption following [Sch12] §7, and accordingly noting that [Sch12] also allows us to remove the corresponding hypothesis from [BFGM02].

3. Review of Zastava spaces

3.1. In this section, we review the geometry of Zastava spaces, introduced in [FM99] and [BFGM02].

3.2. **The basic affine space.** Recall that the map:

$$G/N \to \overline{G/N} := \mathrm{Spec}(H^0(\Gamma(G/N, \mathcal{O}_{G/N}))) = \mathrm{Spec}(\mathrm{Fun}(G)^N)$$

is an open embedding. We call G/N the *basic affine space* $\overline{G/N}$ the *affine closure of the basic affine space.*

The following result is direct from the Peter-Weyl theorem.

Lemma 3.2.1. *A map $\varphi : S \to \overline{G/N}$ with $\varphi^{-1}(G/N)$ dense in S is equivalent to a "Drinfeld structure" on the trivial G-bundle $G \times S \to S$, i.e., a sequence of maps for $\lambda \in \Lambda^+$.*

$$\sigma^\lambda : \ell^\lambda \underset{k}{\otimes} \mathcal{O}_S \to V^\lambda \underset{k}{\otimes} \mathcal{O}_S$$

that are monomorphisms of quasi-coherent sheaves.

Remark 3.2.2. By *dense*, we mean scheme-theoretically, not topologically (e.g., for Noetherian S, the difference here is only apparent in the presence of associated points).

Example 3.2.3. For $G = \mathrm{SL}_2$, $\overline{G/N}$ identifies equivariantly with $\mathbb{A}^2$. The corresponding map $\mathrm{SL}_2 \to \mathbb{A}^2$ here is (necessarily) given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c) \in \mathbb{A}^2.$$

3.3. Let $\overline{T}$ be the closure of $T = B/N \subseteq G/N$ in $\overline{G/N}$.

Lemma 3.3.1. *(1) $\overline{T}$ is the toric variety* $\mathrm{Spec}(k[\Lambda^+])$ *(here $k[\Lambda^+]$ is the monoid algebra defined by the monoid Λ^+). Here the map $T = \mathrm{Spec}(k[\Lambda]) \to \overline{T}$ corresponds to the embedding $\Lambda^+ \to \Lambda$ and the map $\mathrm{Fun}(G)^N \to k[\Lambda^+]$ realizes the latter as N-coinvariants of the former.*

(2) The action of T on $\overline{G/N}$ extends to an action of the monoid $\overline{T}$ on $\overline{G/N}$ (where the coalgebra structure on $\mathrm{Fun}(\overline{T}) = k[\Lambda^+]$ is the canonical one, that is, defined by the diagonal map for the monoid Λ^+).

Here (1) follows again from the Peter-Weyl theorem and (2) follows similarly, noting that $V^\lambda \otimes \ell^{\lambda,\vee} \subseteq \mathrm{Fun}(G)^N = \mathrm{Fun}(\overline{G/N})$ has Λ-grading (relative to the right action of T on $\overline{G/N}$) equal to $\lambda \in \Lambda^+$.

3.4. Note that (after the choice of opposite Borel) $\overline{T}$ is canonically a retract of $\overline{G/N}$, i.e., the embedding $\overline{T} \hookrightarrow \overline{G/N}$ admits a canonical splitting:

$$\overline{G/N} \to \overline{T}. \tag{3.4.1}$$

Indeed, the retract corresponds to the map $k[\Lambda^+] \to \mathrm{Fun}(G)^N$ sending λ to the canonical element in:

$$\ell^\lambda \otimes \ell^{\lambda,\vee} \subseteq V^\lambda \otimes V^{\lambda,\vee} \subseteq \mathrm{Fun}(G)$$

(note that the embedding $\ell^{\lambda,\vee} \hookrightarrow V^{\lambda,\vee}$ uses the opposite Borel).

By construction, this map factors as $\overline{G/N} \to N^-\backslash(\overline{G/N}) \to \overline{T}$.

Let T act on $\overline{G/N}$ through the action induced by the adjoint action of T on G. Choosing a regular dominant coweight $\lambda_0 \in \check{\Lambda}^+$ we obtain a $\mathbb{G}_m$-action on $\overline{G/N}$ that contracts[11] onto $\overline{T}$. The induced map $\overline{G/N} \to \overline{T}$ coincides with the one constructed above.

Warning 3.4.1. The induced map $G/N \to \overline{T}$ does *not* factor through T. The inverse image in $\overline{G/N}$ of $T \subseteq \overline{T}$ is the open Bruhat cell B^-N/N.

3.5. Define the stack $\overline{\mathbb{B}B}$ as $G\backslash\overline{G/N}/T$. Note that $\overline{\mathbb{B}B}$ has canonical maps to $\mathbb{B}G$ and $\mathbb{B}T$.

3.6. **Local Zastava stacks.** Let $\overset{o}{\zeta}$ denote the stack $B^-\backslash G/B = \mathbb{B}B^- \times_{\mathbb{B}G} \mathbb{B}B$ and and let ζ denote the stack $B^-\backslash(\overline{G/N})/T = \mathbb{B}B^- \times_{\mathbb{B}G} \overline{\mathbb{B}B}$. We have the sequence of open embeddings:

$$\mathbb{B}T \hookrightarrow \overset{o}{\zeta} \hookrightarrow \zeta$$

where $\mathbb{B}T$ embeds as the open Bruhat cell.

The map $\mathbb{B}T \hookrightarrow \zeta$ factors as:

$$\mathbb{B}T = T\backslash(T/T) \hookrightarrow T\backslash(\overline{T}/T) = \mathbb{B}T \times \overline{T}/T \hookrightarrow \zeta. \tag{3.6.1}$$

[11]We recall that a *contracting* $\mathbb{G}_m$*-action* on an algebraic stack $\mathcal{Y}$ is an action of the multiplicative monoid $\mathbb{A}^1$ on $\mathcal{Y}$. For schemes, this is a property of the underlying $\mathbb{G}_m$ action, but for stacks it is not. Therefore, by the phrase "that contracts," we rather mean that it canonically admits the structure of contracting $\mathbb{G}_m$-action. See [DG13] for further discussion of these points.

One immediately verifies that the retraction $\overline{G/N} \to \overline{T}$ of (3.4.1) is $B^- \times T$-equivariant, where B^- acts on the left on $\overline{G/N}$ and T acts on the right, and the action on $\overline{T}$ is similar but is induced by the $T \times T$-action and the homomorphism $B^- \times T \to T \times T$. Therefore, we obtain a canonical map:

$$\zeta = B^- \backslash \overline{G/N}/T \to B^- \backslash \overline{T}/T \to T \backslash \overline{T}/T.$$

Moreover, up to the choice of λ_0 from *loc. cit.* this retraction realizes $\mathbb{B}T \times \overline{T}/T$ as a "deformation retract" of ζ.

We will identify $T \backslash \overline{T}/T$ with $\mathbb{B}T \times \overline{T}/T$ in what follows by writing the former as $T \backslash (\overline{T}/T)$ and noting that T acts trivially here on $\overline{T}/T$.

In particular, we obtain a canonical map:

$$\zeta \to \overline{T}/T. \tag{3.6.2}$$

By Lemma 3.3.1 (2) we have an action of the monoid stack $\overline{T}/T$ on ζ. The morphism $\zeta \xrightarrow{r} \mathbb{B}T \times \overline{T}/T \xrightarrow{p_2} \overline{T}/T$ is $\overline{T}/T$-equivariant.

Lemma 3.6.1. *A map $\varphi : S \to \overline{T}/T$ with $\varphi^{-1}(\mathrm{Spec}(k))$ dense (where $\mathrm{Spec}(k)$ is realized as the open point T/T) is canonically equivalent to a $\check{\Lambda}^{neg}$-valued Cartier divisor on S.*

First, we recall the following standard result.

Lemma 3.6.2. *A map $S \to \mathbb{G}_m \backslash \mathbb{A}^1$ with inverse image of the open point dense is equivalent to the data of an effective Cartier divisor on S.*

Proof. Tautologically, a map $S \to \mathbb{G}_m \backslash \mathbb{A}^1$ is equivalent to a line bundle $\mathcal{L}$ on S with a section $s \in \Gamma(S, \mathcal{L})$.

We need to check that the morphism $\mathcal{O}_S \xrightarrow{s} \mathcal{L}$ is injective as a morphism of quasi-coherent sheaves under the density hypothesis. This is a local statement, so we can trivialize $\mathcal{L}$. Now s is a function f whose locus of non-vanishing is dense, and it is easy to see that this is equivalent to f being a non-zero divisor.

$\square$

Proof of Lemma 3.6.1. Let $G' \subseteq G$ denote the derived subgroup $[G,G]$ of G and let $T' = T \cap G'$ and $N' = N \cap G'$. Then with $\overline{T}'$ defined as the closure of T' in the affine closure of G'/N', the induced map:

$$\overline{T}'/T' \to \overline{T}/T$$

is an isomorphism, reducing to the case $G = G'$.

Because the derived group (assumed to be equal to G now) is assumed simply-connected, we have have canonical fundamental weights $\{\vartheta_i\}_{i\in\mathcal{I}_G}$, $\vartheta_i \in \Lambda^+$. The map $\prod_{i\in\mathcal{I}_G}\vartheta_i : T \to \prod_{i\in\mathcal{I}_G}\mathbb{G}_m$ extends to a map $\overline{T} \to \prod_{i\in\mathcal{I}_G}\mathbb{A}^1$ inducing an isomorphism:

$$\overline{T}/T \xrightarrow{\simeq} (\mathbb{A}^1/\mathbb{G}_m)^{\mathcal{I}_G}.$$

Because we use the right action of T on $\overline{T}$, the functions on $\overline{T}$ are graded negatively, and therefore we obtain the desired result.

$\square$

3.7. **Twists.** Fix an irreducible smooth projective curve X. We digress for a minute to normalize certain twists.

First, for an integer n, we will sometimes use the notation Ω_X^n for $\Omega_X^{\otimes n}$, there being no risk for confusion with n-forms because X is a curve.

We fix $\Omega_X^{\frac{1}{2}}$ a square root of Ω_X. This choice extends the definition of Ω_X^n to $n \in \frac{1}{2}\mathbb{Z}$. We obtain the T-bundle:

$$\mathcal{P}_T^{can} := \check{\rho}(\Omega_X^{-1}) := 2\check{\rho}(\Omega_X^{-\frac{1}{2}}). \tag{3.7.1}$$

We use the following notation:

$$\mathrm{Bun}_{N^-} := \mathrm{Bun}_{B^-} \underset{\mathrm{Bun}_T}{\times} \{\mathcal{P}_T^{can}\}$$

$$\mathrm{Bun}_{\mathbb{G}_a^-} := \mathrm{Bun}_{\mathbb{G}_m \ltimes \mathbb{G}_a} \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \{\Omega_X\}.$$

Here $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by homotheties, i.e., $\mathbb{G}_m \ltimes \mathbb{G}_a$ is the "positive" Borel of PGL_2.

Note that $\mathrm{Bun}_{\mathbb{G}_a^-}$ classifies extensions of $\mathcal{O}_X$ by Ω_X and therefore there is a canonical map:

$$\mathrm{can}_{\mathbb{G}_a^-} : \mathrm{Bun}_{\mathbb{G}_a^-} \to H^1(X, \Omega_X) = \mathbb{G}_a.$$

The choice of Chevalley generators $\{f_i\}_{i \in \mathcal{I}_G}$ of $\mathfrak{n}^-$ defines a map:

$$B^-/[N^-, N^-] \to \prod_{i \in \mathcal{I}_G} (\mathbb{G}_m \ltimes \mathbb{G}_a).$$

By definition of $\mathcal{P}_T^{can}$, this induces a map:

$$\prod_{i \in \mathcal{I}_G} \mathfrak{r}_i : \mathrm{Bun}_{N^-} \to \prod_{i \in \mathcal{I}_G} \mathrm{Bun}_{\mathbb{G}_a^-}.$$

We form the sequence:

$$\mathrm{Bun}_{N^-} \to \prod_{i \in \mathcal{I}_G} \mathrm{Bun}_{\mathbb{G}_a^-} \xrightarrow{\prod_{i \in \mathcal{I}_G} \mathrm{can}_{\mathbb{G}_a^-}} \prod_{i \in \mathcal{I}_G} \mathbb{G}_a \to \mathbb{G}_a$$

and denote the composition by:

$$\mathrm{can} : \mathrm{Bun}_{N^-} \to \mathbb{G}_a.$$

3.8. For a pointed stack $(\mathcal{Y}, y \in \mathcal{Y}(k))$ and a test scheme S, we say that $X \times S \to \mathcal{Y}$ is *non-degenerate* if there exists $U \subseteq X \times S$ universally schematically dense relative to S in the sense of [GAB+66] Exp. XVIII, and such that the induced map $U \to \mathcal{Y}$ admits a factorization as $U \to \mathrm{Spec}(k) \xrightarrow{y} \mathcal{Y}$ (so this is a property for a map, not a structure).

We let $\mathrm{Maps}_{non-degen.}(X, \mathcal{Y})$ denote the open substack of $\mathrm{Maps}(X, \mathcal{Y})$ consisting of non-degenerate maps $X \to \mathcal{Y}$.

We consider $\overset{o}{\zeta}$, ζ, and $\overline{T}/T$ as openly pointed stacks in the obvious ways.

3.9. **Zastava spaces.** Observe that there is a canonical map:

$$\zeta \to \mathbb{B}T \tag{3.9.1}$$

given as the composition:

$$\zeta = \mathbb{B}B^- \underset{\mathbb{B}G}{\times} \overline{\mathbb{B}B} \to \mathbb{B}B^- \to \mathbb{B}T.$$

Let $\mathcal{Z}$ be the stack of $\mathcal{P}_T^{can}$-twisted non-degenerate maps $X \to \zeta$, i.e., the fiber product:

$$\mathrm{Maps}_{non-degen.}(X, \zeta) \underset{\mathrm{Bun}_T}{\times} \{\mathcal{P}_T^{can}\}$$

where the map $\mathrm{Maps}_{non-degen.}(X, \zeta) \to \mathrm{Bun}_T$ is given by (3.9.1).

Let $\overset{o}{\mathcal{Z}} \subseteq \mathcal{Z}$ be the open substack of $\mathcal{P}_T^{can}$-twisted non-degenerate maps $X \to \overset{o}{\zeta}$. Note that $\mathcal{Z}$ and $\overset{o}{\mathcal{Z}}$ lie in $\mathsf{Sch} \subseteq \mathsf{PreStk}$. We call $\mathcal{Z}$ the *Zastava space* and $\overset{o}{\mathcal{Z}}$ the *open Zastava space*. We let $\jmath : \overset{o}{\mathcal{Z}} \to \mathcal{Z}$ denote the corresponding open embedding.

We have a Cartesian square where all maps are open embeddings:

$$\begin{tikzcd} \overset{o}{\mathcal{Z}} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathrm{Bun}_{N^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B \arrow[r] & \mathrm{Bun}_{N^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B \end{tikzcd}$$

The horizontal arrows realize the source as the subscheme of the target where the two reductions are generically transverse.

3.10. Let $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} = \mathrm{Maps}_{non-degen.}(X, \overline{T}/T)$ denote the scheme of $\check{\Lambda}^{pos}$-divisors on X (we include the subscript "eff" for emphasis that we are not taking $\check{\Lambda}$-valued divisors).

We have the canonical map:

$$\deg : \pi_0(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}) \to \check{\Lambda}^{pos}.$$

For $\check{\lambda} \in \check{\Lambda}^{pos}$ let $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}^{pos}}$ denote the corresponding connected component of $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$.

Remark 3.10.1. Writing $\check{\lambda} = \sum_{i \in \mathcal{I}_G} n_i \check{\alpha}_i$ as a sum of simple coroots, we see that $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ is a product $\prod_{i \in \mathcal{I}_G} \mathrm{Sym}^{n_i} X$ of the corresponding symmetric powers of the curve.

Recall that we have the canonical map $r : \zeta \to \mathbb{B}T \times \overline{T}/T$. For any non-degenerate map $X \times S \to \zeta$, Warning 3.4.1 implies that the induced map to $\overline{T}/T$ (given by composing r with the second projection) is non-degenerate as well.

Therefore we obtain the map:

$$\pi : \mathcal{Z} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}.$$

We let $\overset{o}{\pi}$ denote the restriction of π to $\overset{o}{\mathcal{Z}}$. It is well-known that the morphism π is affine.

Let $\mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$) denote the fiber of $\mathcal{Z}$ (resp. $\overset{o}{\mathcal{Z}}$) over $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$. We let $\pi^{\check{\lambda}}$ (resp. $\overset{o}{\pi}{}^{\check{\lambda}}$) denote the restriction of π to $\mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). We let $\jmath^{\check{\lambda}} : \overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \to \mathcal{Z}^{\check{\lambda}}$ denote the restriction of the open embedding $\jmath$.

Note that π admits a canonical section $\mathfrak{s} : \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \to \mathcal{Z}$, whose restriction to each $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ we denote by $\mathfrak{s}^{\check{\lambda}}$. Note that up to a choice of regular dominant coweight, the situation is given by contraction.

Each $\mathcal{Z}^{\check{\lambda}}$ is of finite type (and therefore the same holds for $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). It is known (c.f. [BFGM02] Corollary 3.8) that $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ is a smooth variety.

For $\check{\lambda} = 0$, we have $\overset{o}{\mathcal{Z}}{}^{0} = \mathcal{Z}^0 = \mathrm{Div}_{\mathrm{eff}}^{0} = \mathrm{Spec}(k)$.

We have a canonical (up to choice of Chevalley generators) map $\mathcal{Z} \to \mathbb{G}_a$ defined as the composition $\mathcal{Z} \to \mathrm{Bun}_{N^-} \xrightarrow{\mathrm{can}} \mathbb{G}_a$. For $\check{\alpha}_i$ a positive simple coroot the induced map:

$$\mathcal{Z}^{\check{\alpha}_i} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\alpha}_i} \times \mathbb{G}_a = X \times \mathbb{G}_a \tag{3.10.1}$$

is an isomorphism that identifies $\overset{o}{\mathcal{Z}}{}^{\check{\alpha}_i}$ with $X \times \mathbb{G}_m$.

Remark 3.10.2.

The dimension of $\mathcal{Z}^{\check{\lambda}}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ is $(2\rho, \check{\lambda}) = (\rho, \check{\lambda}) + \dim \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ (this follows e.g. from the factorization property discussed in §3.11 below and then by the realization discussed in §3.12 of the central fiber as an intersection of semi-infinite orbits in the Grassmannian, that are known by [BFGM02] §6 to be equidimensional with dimension $(\rho, \check{\lambda})$).

Example 3.10.3. Let us explain in more detail the case of $G = \mathrm{SL}_2$. In this case, tensoring with the bundle $\Omega_X^{\frac{1}{2}}$ identifies $\mathcal{Z}$ with the moduli of commutative diagrams:

$$\begin{tikzcd}
 & & \mathcal{L} \arrow[d] \arrow[dr, "\varphi"] & & \\
0 \arrow[r] & \Omega_X \arrow[r] \arrow[dr, "\varphi^\vee \otimes \mathrm{id}_{\Omega_X}"'] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] & 0 \\
 & & \mathcal{L}^\vee \underset{\mathcal{O}_X}{\otimes} \Omega_X & &
\end{tikzcd}$$

in which the composition $\mathcal{L} \to \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \Omega_X$ is zero and the morphism φ is non-zero. The subscheme $\overset{o}{\mathcal{Z}}$ is the moduli where the induced map $\mathrm{Coker}(\mathcal{L} \to \mathcal{E}) \to \mathcal{L}^\vee \underset{\mathcal{O}_X}{\otimes} \Omega_X$ is an isomorphism. The associated divisor of such a datum is defined by the injection $\mathcal{L} \hookrightarrow \mathcal{O}_X$.

Because we have removed a twist above by tensoring with $\Omega_X^{\frac{1}{2}}$, the forgetful map $\mathcal{Z} \to \mathrm{Bun}_{GL_2}$ is given by mapping the above to $\mathcal{E} \otimes \Omega_X^{-\frac{1}{2}}$, and similarly for the forgetful map to $\overline{\mathrm{Bun}}_B$.

Over a point $x \in X$, we have an identification of the fiber $\overset{o}{\mathcal{Z}}{}^1_x$ of $\overset{o}{\mathcal{Z}}{}^1$ over $x \in X$ (considering $1 \in \mathbb{Z} = \check{\Lambda}_{\mathrm{SL}_2}$ as the unique positive simple coroot) with $\mathbb{G}_m$. The point $1 \in \mathbb{G}_m$ corresponds to a canonical extension of $\mathcal{O}_X$ by Ω^1_X associated to the point x, that can be constructed explicitly using the *Atiyah sequence* of the line bundle $\mathcal{O}_X(x)$.

Recall that for a vector bundle $\mathcal{E}$, the Atiyah sequence (c.f. [Ati57]) is a canonical short exact sequence:

$$0 \to \mathrm{End}(\mathcal{E}) \to \mathrm{At}(\mathcal{E}) \to T_X \to 0$$

whose splittings correspond to connections on $\mathcal{E}$. For a line bundle $\mathcal{L}$, we obtain a canonical extension $\mathrm{At}(\mathcal{L}) \otimes \Omega^1_X$ of $\mathcal{O}_X$ by Ω^1_X. Taking $\mathcal{L} = \mathcal{O}_X(x)$, we obtain the extension underlying the canonical point of $\overset{o}{\mathcal{Z}}{}^1_x$.

Note that we have a canonical map $\mathcal{L} = \mathcal{O}_X(x) \to \mathrm{At}(\mathcal{O}_X(x)) \otimes \Omega^1_X$ that may be thought of as a splitting of the Atiyah sequence with a pole of order 1, and this splitting corresponds to the obvious connection on $\mathcal{O}_X(x)$ with a pole of order 1. This defines the corresponding point of $\overset{o}{\mathcal{Z}}{}^1$ completely.

3.11. **Factorization.** Now we recall the crucial factorization property of $\mathcal{Z}$.

Let $\mathrm{add} : \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ denote the addition map for the commutative monoid structure defined by addition of divisors. For $\check{\lambda}$ and $\check{\mu}$ fixed, we let $\mathrm{add}^{\check{\lambda},\check{\mu}}$ denote the induced map $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\mu}} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}+\check{\mu}}$.

Define:

$$[\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$$

as the moduli of pairs of disjoint $\check{\Lambda}^{pos}$-divisors. Note that the restriction of add to this locus is étale.

Then we have canonical "factorization" isomorphisms:

$$\mathcal{Z} \underset{\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}}{\times} [\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj} \xrightarrow{\simeq} (\mathcal{Z} \times \mathcal{Z}) \underset{\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}}{\times} [\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj}$$

that are associative in the natural sense.

The morphisms π and $\mathfrak{s}$ are compatible with the factorization structure.

3.12. **The central fiber.** By definition, the *central fiber* $\mathfrak{Z}^{\check{\lambda}}$ of the Zastava space $\mathcal{Z}^{\check{\lambda}}$ is the fiber product:

$$\mathfrak{Z}^{\check{\lambda}} := \mathcal{Z}^{\check{\lambda}} \underset{\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}}{\times} X$$

where $X \to \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ is the closed "diagonal" embedding, i.e., it is the closed subscheme where the divisor is concentrated at a single point. We let $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}$ denote the open in $\mathfrak{Z}^{\check{\lambda}}$ corresponding to $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$. Similarly, we let $\mathfrak{Z} \subseteq \mathcal{Z}$ be the closed corresponding to the union of the $\mathfrak{Z}^{\check{\lambda}}$.

We let $\beta^{\check{\lambda}}$ (resp. $\curlyvee^{\check{\lambda}}$) denote the closed embedding $\mathfrak{Z}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$).

3.13. **Twisted affine Grassmannian.** Let $\mathcal{P}^{can}_G, \mathcal{P}^{can}_B$ and $\mathcal{P}^{can}_{B^-}$ be the torsors induced by the T-torsor $\mathcal{P}^{can}_T$ under the embeddings of T into each of these groups.

We let $\mathrm{Gr}_{G,X}$ denote the "$\mathcal{P}^{can}_G$-twisted Beilinson-Drinfeld affine Grassmannian" classifying a point $x \in X$, a G-bundle $\mathcal{P}_G$ on X, and an isomorphism $\mathcal{P}^{can}_G|_{X\setminus x} \simeq \mathcal{P}_G|_{X\setminus x}$. More precisely, the S-points are:

$$S \mapsto \left\{ \begin{array}{l} x : S \to X, \ \mathcal{P}_G \text{ a } G\text{-bundle on } X \times S, \\ \alpha \text{ an isomorphism } \mathcal{P}_G|_{X\times S\setminus\Gamma_x} \simeq \mathcal{P}^{can}_G|_{X\times S\setminus\Gamma_x} \end{array} \right\}.$$

Similarly for $\mathrm{Gr}_{B,X}$, etc. We define $\mathrm{Gr}_{N^-,X} := \mathrm{Gr}_{B^-,X} \times_{\mathrm{Gr}_{T,X}} X$ the map $X \to \mathrm{Gr}_{T,X}$ being the tautological section.

Let $\overline{\mathrm{Gr}}_{B,X}$ denote the "union of closures of semi-infinite orbits," i.e., the indscheme:

$$\overline{\mathrm{Gr}}_{B,X} : S \mapsto \left\{ \begin{array}{l} x : S \to X, \ \varphi : X \times S \to G\backslash(\overline{G/N})/T, \\ \alpha \text{ a factorization of } \varphi|_{(X\times S)\setminus\Gamma_x} \text{ through the} \\ \text{canonical map } \mathrm{Spec}(k) \to G\backslash(\overline{G/N})/T. \end{array} \right\}.$$

Here Γ_x denotes the graph of the map x.

3.14. In the above notation, we have a canonical isomorphism:

$$\mathfrak{Z} \xrightarrow{\simeq} \mathrm{Gr}_{N^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}_{B,X}.$$

Indeed, this is immediate from the definitions.

Note that $\mathrm{Gr}_{B,X}$ has a canonical map to $\mathrm{Gr}_{T,X} = \coprod_{\check{\lambda}\in\check{\Lambda}} \mathrm{Gr}^{\check{\lambda}}_{T,X}$. Letting $\mathrm{Gr}^{\check{\lambda}}_{B,X}$ be the fiber over the corresponding connected component of $\mathrm{Gr}_{T,X}$, we obtain:

$$\mathfrak{Z}^{\check{\lambda}} \xrightarrow{\simeq} \mathrm{Gr}_{N^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}}_{B,X}.$$

3.15. By §3.6, we have an action of $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}}$ on $\mathcal{Z}$ so that the morphism π is $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}}$-equivariant. We let $\mathrm{act}_{\mathcal{Z}}$ denote the action map $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}} \times \mathcal{Z} \to \mathcal{Z}$. We abuse notation in denoting by $\mathrm{act}_{\overset{o}{\mathcal{Z}}}$ the induced map $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}} \times \overset{o}{\mathcal{Z}} \to \mathcal{Z}$ (that does not define an action on $\overset{o}{\mathcal{Z}}$, i.e., this map does not factor through $\overset{o}{\mathcal{Z}}$).

For $\check{\lambda} \in \check{\Lambda}$ acting on $\mathcal{Z}^{\check{\lambda}}$ defines the map:

$$\mathrm{act}^{\check{\lambda}}_{\mathcal{Z}} : \mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}} \times \mathcal{Z}^{\check{\lambda}} \to \mathcal{Z}.$$

For $\check{\eta} \in \check{\Lambda}^{pos}$ we use the notation $\mathrm{act}^{\check{\eta},\check{\lambda}}_{\mathcal{Z}}$ for the induced map:

$$\mathrm{act}^{\check{\eta},\check{\lambda}}_{\mathcal{Z}} : \mathrm{Div}^{\check{\eta}}_{\text{eff}} \times \mathcal{Z}^{\check{\lambda}} \to \mathcal{Z}^{\check{\lambda}+\check{\eta}}.$$

Similarly, we have the maps $\mathrm{act}^{\check{\lambda}}_{\overset{o}{\mathcal{Z}}}$ and $\mathrm{act}^{\check{\lambda},\check{\eta}}_{\overset{o}{\mathcal{Z}}}$.

The following lemma is well-known (see e.g. [BFGM02]).

Lemma 3.15.1. *Each map $\mathrm{act}^{\check{\lambda},\check{\eta}}_{\mathcal{Z}}$ is a finite morphism and $\mathrm{act}^{\check{\lambda},\check{\eta}}_{\overset{o}{\mathcal{Z}}}$ is a locally closed embedding. For fixed $\check{\lambda}$ the set of locally closed subschemes of $\mathcal{Z}^{\check{\lambda}}$:*

$$\{\mathrm{act}^{\check{\eta},\check{\mu}}_{\overset{o}{\mathcal{Z}}}(\mathrm{Div}^{\check{\eta}}_{\text{eff}} \times \overset{o}{\mathcal{Z}}^{\check{\mu}})\}_{\substack{\check{\mu}+\check{\eta}=\check{\lambda} \\ \check{\mu},\check{\eta}\in\check{\Lambda}^{pos}}}$$

forms a partition by locally closed subschemes.

3.16. **Intersection cohomology of Zastava.** For $\check{\lambda} \in \check{\Lambda}^{pos}$ we now review the description from [BFGM02] of the fibers of the intersection cohomology D-module $\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}$ along the strata described above, i.e., the D-modules:

$$\mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}) \in D(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}}), \quad \check{\eta}, \check{\mu} \in \check{\Lambda}^{pos}, \check{\mu} + \check{\eta} = \check{\lambda}.$$

Theorem 3.16.1. *With notation as above, the D-module:*

$$\mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})[-\dim \mathcal{Z}^{\check{\mu}}] \in D(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}})$$

is a constructible sheaf, i.e., it lies in the heart of the constructible t-structure on the category of regular holonomic D-modules.

Remark 3.16.2. As above, $\mathcal{Z}^{\check{\mu}}$ is equidimensional with $\dim \mathcal{Z}^{\check{\mu}} = 2(\rho, \check{\mu})$.

3.17. **Locality.** For X a smooth (possibly affine) curve with choice of $\Omega_X^{\frac{1}{2}}$, we obtain an identical geometric picture. One can either realize this by restriction from a compactification, or by reinterpreting e.g. the map $\mathcal{Z} \to \mathbb{G}_a$ through residues instead of through global cohomology.

4. Limiting case of the Casselman-Shalika formula

4.1. The goal for this section is to prove Theorem 4.3.1, an unpublished result of Gaitsgory regarding the vanishing of certain Whittaker cohomology groups.

4.2. **Artin-Schreier sheaves.** We define the !-Artin-Schreier D-module $\overset{!}{\psi} \in D(\mathbb{G}_a)$ to be the exponential local system normalized cohomologically so that $\overset{!}{\psi}[-1] \in D(\mathbb{G}_a)^{\heartsuit}$. Note that $\overset{!}{\psi}$ is multiplicative with respect to upper-! pullback.

We define the $*$-Artin-Schreier D-module $\overset{*}{\psi} \in D(\mathbb{G}_a)$ to be the Verdier dual to $\overset{!}{\psi}$. Note that $\overset{*}{\psi}$ lies it the heart of the constructible t-structure on $\mathbb{G}_a$ and is multiplicative sheaf with respect to upper-$*$ pullback.

4.3. For $\check{\lambda} \in \check{\Lambda}^{pos}$, let $\overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}} \in D(\mathcal{Z}^{\check{\lambda}})$ denote the $*$-pullback of the Artin-Schreier D-module $\overset{*}{\psi}$ via the composition:

$$\mathcal{Z}^{\check{\lambda}} \to \mathrm{Bun}_{N^-} \xrightarrow{\text{can}} \mathbb{G}_a.$$

Note that $\overset{*}{\psi}[\dim \mathcal{Z}^{\check{\lambda}}] \in D(\mathcal{Z}^{\check{\lambda}})^{\heartsuit}$. Also define:

$$\overset{*}{\psi}_{\overset{\circ}{\mathcal{Z}}{}^{\check{\lambda}}} = \jmath^{\check{\lambda},*}(\overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}}).$$

Theorem 4.3.1. *If $\check{\lambda} \neq 0$, then:*

$$\pi_!^{\check{\lambda}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}}) = 0.$$

The proof will be given in §4.5 below.

This theorem is étale local on X, and therefore we may assume that we have $X = \mathbb{A}^1$. In particular, we have a fixed trivialization of $\Omega_X^{\frac{1}{2}}$.

4.4. **Central fibers via affine Schubert varieties.** In the proof of Theorem 4.3.1 we will use Proposition 4.4.1 below. We note that it is well-known, though we do not know a published reference.

Throughout §4.4, we work only with reduced schemes and indschemes, so all symbols refer to the reduced indscheme underlying the corresponding indscheme. Note that this restriction does not affect D-modules on the corresponding spaces.

Let $\mathfrak{J}ets_X^{mer}(T)$ denote the group scheme of jets into T over X. Because we have chosen an identification $X \simeq \mathbb{A}^1$, we have a canonical homomorphism:

$$\mathrm{Gr}_{T,X} \simeq \mathbb{A}^1 \times \check{\Lambda} \to \mathfrak{J}ets_X^{mer}(T) \simeq \mathbb{A}^1 \times T(K)$$

$$(x, \check{\lambda}) \mapsto (x, \check{\lambda}(t))$$

where t is the uniformizer of $\mathbb{A}^1$. Of course, the formula $\mathrm{Gr}_{T,X} \simeq \mathbb{A}^1 \times \check{\Lambda}$ is only valid at the reduced level. This induces an action of the X-group indscheme $\mathrm{Gr}_{T,X}$ on $\mathrm{Gr}_{B,X}$, $\mathrm{Gr}_{G,X}$ and $\mathrm{Gr}_{N^-,X} = \mathrm{Gr}^0_{B^-,X}$.

Using this action, we obtain a canonical isomorphism:

$$\mathfrak{Z}^{\check{\lambda}} = \mathrm{Gr}^0_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}}_{B,X} \xrightarrow{\simeq} \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,X}$$

of X-schemes for every $\check{\eta} \in \check{\Lambda}$.

Proposition 4.4.1. *For $\check{\eta}$ deep enough[12] in the dominant chamber we have:*

$$\mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,X} = \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,X}}.$$

This equality also identifies:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{B,X} = \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,X}\,.$$

Proof. It suffices to verify the result fiberwise and therefore we fix $x = 0 \in X = \mathbb{A}^1$ (this is really just a notational convenience here). We let $\mathfrak{Z}^{\check{\lambda}}_x$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$) denote the fiber of $\mathfrak{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}$) at x. Let $t \in K_x$ be a coordinate at x.

Because there are only finitely many $0 \leqslant \check{\mu} \leqslant \check{\lambda}$ and because each $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ is finite type, for $\check{\eta}$ deep enough in the dominant chamber we have:

$$\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x = \mathrm{Gr}_{N^-,x} \cap \mathrm{Ad}_{-\check{\eta}(t)}(N(O_x)) \cdot \check{\mu}(t)$$

($\check{\mu}(t)$ being regarded as a point in $\mathrm{Gr}_{G,x}$ here and the intersection symbol is short-hand for fiber product over $\mathrm{Gr}_{G,x}$) for all $0 \leqslant \check{\mu} \leqslant \check{\lambda}$. Choosing $\check{\eta}$ possibly larger, we can also assume that $\check{\eta} + \check{\mu}$ is dominant for all $0 \leqslant \check{\mu} \leqslant \check{\lambda}$. Then we claim that such a choice $\check{\eta}$ suffices for the purposes of the proposition.

[12]This should be understood in a way depending on $\check{\lambda}$.

Observe that for each $0 \leqslant \check{\mu} \leqslant \check{\lambda}$ we have:

$$\mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}} = \check{\eta}(t) \cdot \overset{o}{\mathfrak{Z}}{}_x^{\check{\mu}} \subseteq \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \bigg(N(O_x) \cdot (\check{\mu}+\check{\eta})(t) \bigg) \subseteq \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{G,x}^{\check{\mu}+\check{\eta}}.$$

Recall (c.f. [MV07]) that $\overline{\mathrm{Gr}}_{B,x}^{\check{\lambda}+\check{\eta}}$ is a union of strata:

$$\overline{\mathrm{Gr}}_{B,x}^{\check{\mu}+\check{\eta}}, \check{\mu} \leqslant \check{\lambda}$$

while for $\check{\mu}$:

$$\mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}} = \varnothing$$

unless $\check{\mu} \geqslant 0$. Therefore, $\mathrm{Gr}_{B^-,x}^{\check{\eta}}$ intersects $\overline{\mathrm{Gr}}_{B,x}^{\check{\mu}}$ only in the strata $\mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}}$ for $0 \leqslant \check{\mu} \leqslant \check{\lambda}$.

The above analysis therefore shows that:

$$\mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \overline{\mathrm{Gr}}_{B,x}^{\check{\lambda}+\check{\eta}} \subseteq \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \overline{\mathrm{Gr}_{G,x}^{\check{\lambda}+\check{\eta}}}.$$

Now observe that $B(O_x) \cdot (\check{\lambda}+\check{\eta})(t)$ is open in $\mathrm{Gr}^{\check{\lambda}}$. Therefore, we have:

$$\overline{\mathrm{Gr}_{G,x}^{\check{\lambda}+\check{\eta}}} \subseteq \overline{\mathrm{Gr}}_{B}^{\check{\lambda}+\check{\eta}}$$

giving the opposite inclusion above.

It remains to show that the equality identifies $\overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}}$ in the desired way. We have already shown that:

$$\mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{B,x}^{\check{\lambda}+\check{\eta}} \subseteq \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{G,x}^{\check{\lambda}+\check{\eta}}.$$

so it remains to prove the opposite inclusion. Suppose that y is a geometric point of the right hand side. Then, by the Iwasawa decomposition, $y \in \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}}$ for some (unique) $\check{\mu} \in \check{\Lambda}$ and we wish to show that $\check{\mu} = \check{\lambda}$.

Because:

$$y \in \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}} \cap \overline{\mathrm{Gr}_{G,x}^{\check{\lambda}+\check{\eta}}} \neq \varnothing$$

we have $\check{\mu} \leqslant \check{\lambda}$. We also have:

$$y \in \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}} \cap \mathrm{Gr}_{B^-,x}^{\check{\eta}} \neq \varnothing$$

which implies $\check{\mu} \geqslant 0$. Therefore, by construction of η we have:

$$y \in \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{B,x}^{\check{\mu}+\check{\eta}} \subseteq \mathrm{Gr}_{B^-,x}^{\check{\eta}} \cap \mathrm{Gr}_{G,x}^{\check{\mu}+\check{\eta}} \subseteq \mathrm{Gr}_{G,x}^{\check{\mu}+\check{\eta}}$$

but $\mathrm{Gr}_{G,x}^{\check{\mu}+\check{\eta}} \cap \mathrm{Gr}_{G,x}^{\check{\lambda}+\check{\eta}} = \varnothing$ if $\check{\mu} \neq \check{\lambda}$ (because $\check{\mu}+\check{\eta}$ and $\check{\lambda}+\check{\eta}$ are assumed dominant) and therefore we must have $\check{\mu} = \check{\lambda}$ as desired.

□

We continue to use the notation introduced in the proof of Proposition 4.4.1.

Recall that $\beta^{\check{\lambda}}$ (resp. $\Upsilon^{\check{\lambda}}$) denotes the closed embedding $\mathfrak{Z}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). For $x \in X$, let $\beta_x^{\check{\lambda}}$ (resp. $\Upsilon_x^{\check{\lambda}}$) denote the closed embedding $\mathfrak{Z}_x^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$).

Corollary 4.4.2. *(1) If $0 \neq \check{\lambda} \in \check{\Lambda}^{pos}$ then for every $x \in X$ we have:*

$$H_{dR,c}^{2\dim \overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}}}\Big(\overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}}, \Upsilon_x^{\check{\lambda},*}(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}})\Big) = 0.$$

(2) If $0 \neq \check{\lambda} \in \check{\Lambda}^{pos}$ then we have Euler characteristic vanishing:

$$\chi\bigg(\Gamma_{dR,c}\Big(\mathfrak{Z}_x^{\check{\lambda}}, \beta_x^{\check{\lambda},*}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})\Big)\bigg) = 0.$$

Remark 4.4.3. To orient the reader at this point, we note that e.g. $\Upsilon_x^{\check{\lambda},*}(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}})$ is a local system shifted to lie in the heart of the constructible t-structure on $\overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}}$.

Proof of Corollary 4.4.2. Fix $0 \neq \check{\lambda}$ and then $\check{\eta}$ as in the proof of Proposition 4.4.1. As in *loc. cit.* we use $\check{\eta}$ to identify:

$$\mathfrak{Z}_x \xrightarrow{\simeq} \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}.$$

By Proposition 4.4.1 and the Casselman-Shalika formula [FGV01] Theorem 1, the restriction of $\Upsilon^{\check{\lambda},*}_x(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}})$ to every irreducible component of $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$ is a non-constant rank 1 local system, implying (1).

It remains to show (2). The key step is to establish the following equality:

$$[\beta^{\check{\lambda},*}_x(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})] = [\iota^*(\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}})] \in K_0(D^b_{rh}(\mathfrak{Z}^{\check{\lambda}}_x))$$

in the Grothendieck group of bounded complexes of coherent and regular holonomic D-modules on $\mathfrak{Z}^{\check{\lambda}}_x$. Here the map ι is defined as:

$$\mathfrak{Z}_x \xrightarrow{\simeq} \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}} \to \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}.$$

It suffices to show that for each $0 \leqslant \check{\mu} \leqslant \check{\lambda}$ the $*$-restrictions of these classes coincide in the Grothendieck group of:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}.$$

Indeed, these locally closed subvarieties form a stratification as $\check{\mu}$ varies.

First, note that the $*$-restriction of $\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}}$ to $\mathrm{Gr}^{\check{\mu}+\check{\lambda}}_{G,x}$ has constant cohomologies (by $G(O)$-equivariance). Moreover, by [Lus83] the corresponding class in the Grothendieck group is the dimension of the weight component:

$$\dim V^{\check{\lambda}+\check{\eta}}(\check{\mu}+\check{\eta}) \cdot [\mathrm{IC}_{\mathrm{Gr}^{\check{\mu}+\check{\lambda}}_{G,x}}].$$

Restricting to $\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}$ we obtain that the right hand side of our equation is given by:

$$\dim V^{\check{\lambda}+\check{\eta}}(\check{\mu}+\check{\eta}) \cdot [\mathrm{IC}_{\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}}].$$

By having $U(\check{\mathfrak{n}}^-)$ act on a highest weight vector of $V^{\check{\lambda}+\check{\eta}}$, we observe that for $\check{\eta}$ large enough, $V^{\check{\lambda}+\check{\eta}}(\check{\mu}+\check{\eta})$ is isomorphic to the $(\check{\mu}-\check{\lambda})$-weight component $U(\check{\mathfrak{n}}^-)(\check{\mu}-\check{\lambda})$ of $U(\check{\mathfrak{n}}^-)$.

The similar identification for the left hand side follows from the choice of $\check{\eta}$ (so that $\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}$ identifies with $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$) and the main result of [BFGM02].

Therefore, to prove (2) it suffices to prove that:

$$\chi\Big(\Gamma_{dR,c}\Big(\mathfrak{Z}^{\check{\lambda}}_x, \beta^{\check{\lambda},*}_x(\iota^*(\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}}))\Big)\Big) = 0.$$

Even better: by the geometric Casselman-Shalika formula [FGV01], this cohomology itself vanishes, so its Euler characteristic does too.

□

4.5. Now we give the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. We proceed by induction on $(\rho, \check{\lambda})$.

By factorization and induction, we see that $\pi^{\check{\lambda}}_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})$ is concentrated on the *main diagonal* $X \subseteq \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$. Its ($* =!$-)restriction to X is the !-pushforward along $\mathfrak{Z}^{\check{\lambda}} \to X$ of $\beta^{\check{\lambda},*}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})$. Moreover, since $\mathfrak{Z}^{\check{\lambda}} \to X$ is a "fibration" (i.e., locally a product so that our sheaf is an external product with a constant sheaf on X) the cohomologies of $\pi^{\check{\lambda}}_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})$ on X are lisse and the fiber at $x \in X$ is:

$$\Gamma_{dR,c}\Big(\mathfrak{Z}^{\check{\lambda}}_x, \beta^{\check{\lambda},*}_x(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})\Big).$$

By Corollary 4.4.2 (2) the Euler characteristics of the fibers (on the main diagonal) vanish. Therefore, it is enough to show that $\pi^{\check{\lambda}}_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})$ is a perverse sheaf, i.e., lies in $D(\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}})^{\heartsuit}$.

Because $\pi^{\check{\lambda}}$ is affine and $\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}}$ is a perverse sheaf, we have:

$$\pi^{\check{\lambda}}_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}}) \in D(\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}})^{\geqslant 0}.$$

On the other hand, recall that by Theorem 3.16.1 for every decomposition $\check{\lambda} = \check{\eta} + \check{\mu}$ we have:

$$\mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})[-\dim \mathcal{Z}^{\check{\mu}}] \in D(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}})$$

is a constructible sheaf. Moreover, the fibers of the composition:

$$\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}} \xrightarrow{\mathrm{act}^{\check{\eta},\check{\mu}}_{\overset{o}{\mathcal{Z}}}} \mathcal{Z}^{\check{\lambda}} \xrightarrow{\pi^{\check{\lambda}}} \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$$

have dimension $(\rho, \check{\mu})$. Therefore, we deduce that:

$$\pi_! \, \mathrm{act}^{\check{\eta},\check{\mu}}_{\overset{o}{\mathcal{Z}},!} \, \mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})$$

is concentrated in constructible cohomological degrees:

$$\leqslant 2(\rho, \check{\mu}) - \dim \mathcal{Z}^{\check{\mu}} = 0.$$

Moreover, for $x \in X \subseteq \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ the "top" cohomology of this fiber is 0 by Corollary 4.4.2 (1), and therefore the corresponding fiber is concentrated in constructible cohomological degrees $\leqslant -1$. Because:

$$\pi_! \, \mathrm{act}^{\check{\eta},\check{\mu}}_{\overset{o}{\mathcal{Z}},!} \, \mathrm{act}^{\check{\eta},\check{\mu},*}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{*}{\otimes} \overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}})$$

is concentrated on X and lisse along X this implies that it is in perverse degrees $\leqslant 0$ as desired.

But now the vanishing of Euler characteristics noted above immediately implies the result.

$\square$

5. Identification of the Chevalley complex I

5.1. The goal for this section is to identify the Chevalley complex in the cohomology of Zastava space with coefficients in the Whittaker sheaf. This computation will be the main input in §9.

We first give finite-dimensional versions of the computation, and then in §8 we will easily deduce a Ran space version.

5.2. We will use the language of *graded factorization algebras*.

The definition should encode the following: a $\mathbb{Z}^{\geqslant 0}$-graded factorization algebra is a system $\mathcal{F}_n \in D(\operatorname{Sym}^n X)$ such that we have, for every pair m, n we have isomorphisms:

$$\Big(\mathcal{F}_m \boxtimes \mathcal{F}_n\Big)|_{[\operatorname{Sym}^m X \times \operatorname{Sym}^n X]_{disj}} \xrightarrow{\simeq} \Big(\mathcal{F}_{m+n}\Big)|_{[\operatorname{Sym}^m X \times \operatorname{Sym}^n X]_{disj}}.$$

Note that the addition map $\operatorname{Sym}^m X \times \operatorname{Sym}^n X \to \operatorname{Sym}^{m+n} X$ is étale when restricted to the disjoint locus, and therefore the restriction notation above is unambiguous.

Formally, the scheme $\operatorname{Sym} X = \coprod_n \operatorname{Sym}^n X$ is naturally a commutative algebra under correspondences, where the multiplication is induced by the maps:

$$\begin{tikzcd}
 & {[\operatorname{Sym}^n X \times \operatorname{Sym}^m X]_{disj}} \arrow[dl] \arrow[dr] & \\
\operatorname{Sym}^n X \times \operatorname{Sym}^m X & & \operatorname{Sym}^{m+n} X.
\end{tikzcd}$$

Therefore, as in §13 we can apply the formalism of §12 to obtain the desired theory.

Remark 5.2.1. We will only be working with graded factorization algebras in the heart of the t-structure, and therefore the language may be worked out "by hand" as in [BD04], i.e., without needing to appeal to §12.

Similarly, we have the notion of $\check{\Lambda}^{pos}$-graded factorization algebra: it is a collection of D-modules on the schemes $\operatorname{Div}_{\mathrm{eff}}^{\check{\lambda}}$ with similar identifications as above.

5.3. Recall that [BG08] has introduced a certain $\check{\Lambda}^{pos}$-graded commutative factorization algebra, i.e., a commutative factorization D-module on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$. This algebra incarnates the Chevalley complex of $\check{\mathfrak{n}}$. In *loc. cit.*, this algebra is denoted by $\Omega(\check{\mathfrak{n}}_X)$: we use the notation $\Omega_{\check{\mathfrak{n}}}$ instead. We denote the component of $\Omega_{\check{\mathfrak{n}}}$ on $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ by $\Omega_{\check{\mathfrak{n}}}^{\check{\lambda}}$.[13] Recall from *loc. cit.* that each $\Omega_{\check{\mathfrak{n}}}^{\check{\lambda}}$ lies in $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}})^{\heartsuit}$.[14]

Remark 5.3.1. To remind the reader of the relation between $\Omega_{\check{\mathfrak{n}}}$ and the cohomological Chevalley complex $C^\bullet(\check{\mathfrak{n}})$ of $\check{\mathfrak{n}}$, we recall that the !-fiber of $\Omega_{\check{\mathfrak{n}}}$ at a $\check{\Lambda}^{pos}$-colored divisor $\sum_{i=1}^n \check{\lambda}_i \cdot x_i$ (here $\check{\lambda}_i \in \check{\Lambda}^{pos}$ and the $x_i \in X$ are distinct closed points) is canonically identified with:

$$\bigotimes_{i=1}^{n} C^\bullet(\check{\mathfrak{n}})^{-\check{\lambda}_i}$$

where $C^\bullet(\check{\mathfrak{n}})^{-\check{\lambda}_i}$ denotes the $-\check{\lambda}_i$-graded piece of the complex.

Remark 5.3.2. Recall that [BD04] associates to any commutative algebra a canonical (commutative) factorization algebra over X, in the sense of *loc. cit.* The algebra $\Omega_{\check{\mathfrak{n}}}$ arises by a (derived version of a) similar procedure, but by considering $C^\bullet(\check{\mathfrak{n}})$ as a $\check{\Lambda}^{pos}$-graded commutative algebra (through the opposite grading to the natural one).

Remark 5.3.3. As is apparent already, it would be more natural to be using $\check{\Lambda}^{neg} := -\check{\Lambda}^{pos}$ here.

Remark 5.3.4. We emphasize the "miracle" mentioned above and crucially exploited in [BG08] (and below): although $C^\bullet(\check{\mathfrak{n}})$ is a commutative (DG) algebra that is certainly non-classical, its D-module avatar does lie in the heart of the t-structure. Of course, this is no contradiction, since the !- fibers of a D-module in the heart are only required to live in degrees $\geqslant 0$.

[13] In [BG08], the authors use a different sign convention, preferring to denote this component by $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}$.

[14] We explicitly note that in this section we exclusively use the usual perverse t-structure.

5.4. Observe that $\jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})$ naturally factorizes on $\mathcal{Z}$. Therefore, $\mathfrak{s}^!\jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})$ is naturally a factorization D-module in $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$.

The following key identification is essentially proved in [BG08], but we include a proof with detailed references to *loc. cit.* for completeness.

Theorem 5.4.1. *There is a canonical identification:*

$$H^0(\mathfrak{s}^!\jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})) \xrightarrow{\simeq} \Omega_{\check{\mathfrak{n}}}$$

of $\check{\Lambda}^{pos}$-graded factorization algebras.

Remark 5.4.2. To orient the reader on cohomological shifts, we note that for $\check{\lambda} \in \check{\Lambda}^{pos}$ fixed, $\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}}$ is concentrated in degree 0 and therefore the above H^0 is the minimal cohomology group of the complex $\mathfrak{s}^!\jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}})$.

Proof of Theorem 5.4.1. Let $j : \mathrm{Div}_{\mathrm{eff},simple}^{\check{\Lambda}^{pos}}$ denote the open consisting of "simple" divisors, i.e., its geometric points are divisors of the form $\sum_{i=1}^{n} \check{\alpha}_i \cdot x_i$ for $\check{\alpha}_i$ a positive simple coroot and the points $\{x_i\}$ pairwise distinct. For each $\check{\lambda} \in \check{\Lambda}^{pos}$, we let $j^{\check{\lambda}} : \mathrm{Div}_{\mathrm{eff},simple}^{\check{\lambda}} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ denote the corresponding open embedding. Note that j and each embedding $j^{\check{\lambda}}$ is affine.

Observe that $\mathrm{Div}_{\mathrm{eff},simple}$ has a factorization structure induced by that of $\mathrm{Div}_{\mathrm{eff}}$. The restriction of $\Omega_{\check{\mathfrak{n}}}$ to $\mathrm{Div}_{\mathrm{eff},simple}$ identifies canonically with the exterior product over $i \in \mathcal{I}_G$ of the corresponding "sign" (rank 1) local systems under the identification:

$$\mathrm{Div}_{\mathrm{eff},simple}^{\check{\lambda}} \simeq \prod_{i \in \mathcal{I}_G} \mathrm{Sym}_{simple}^{n_i} X$$

where $\check{\lambda} = \sum_{i \in \mathcal{I}_G} n_i \check{\alpha}_i$ and on the right the subscript simple means "simple effective divisor" in the same sense as above. Moreover, these identifications are compatible with the factorization structure in the natural sense.

Let $\overset{o}{\mathcal{Z}}_{simple}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}_{simple}$ denote the corresponding opens in $\overset{o}{\mathcal{Z}}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ obtained by fiber product. Let $\mathfrak{s}_{simple}$ and $\mathfrak{s}^{\check{\lambda}}_{simple}$ denote the corresponding restrictions of $\mathfrak{s}$ and $\mathfrak{s}^{\check{\lambda}}$.

Then $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}_{simple} \xrightarrow{\simeq} \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff},simple} \times \mathbb{G}_m^{(\rho,\check{\lambda})}$ as a $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff},simple}$-scheme by (3.10.1), and these identifications are compatible with factorization.

Therefore, we deduce an isomorphism:

$$H^0(\mathfrak{s}^!_{simple} \jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}_{simple}})) \xrightarrow{\simeq} j^!(\Omega_{\check{\mathfrak{n}}})$$

of factorization D-modules on $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},simple}$ (note that the sign local system appears on the left by the Koszul rule of signs).

Therefore, we obtain a diagram:

$$\begin{tikzcd} H^0(\mathfrak{s}^! \jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})) \arrow[d] & \Omega_{\check{\mathfrak{n}}} \arrow[d] \\ j_* H^0(\mathfrak{s}^!_{simple} \jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}_{simple}})) \arrow[r, "\simeq"] & j_* j^!(\Omega_{\check{\mathfrak{n}}}). \end{tikzcd} \tag{5.4.1}$$

Note that the bottom horizontal arrow is a map of factorization algebras on $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$.

By [BG08] Lemma 4.8 and Proposition 4.9 the vertical maps in (5.4.1) are monomorphisms in $D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})^{\heartsuit}$ and by the analysis in *loc. cit.* §4.10, there is a (necessarily unique) isomorphism $H^0(\mathfrak{s}^! \jmath_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})) \xrightarrow{\simeq} \Omega_{\check{\mathfrak{n}}}$ completing the square (5.4.1). This isomorphism is therefore necessarily an isomorphism of factorizable D-modules.

□

5.5. Observe that the D-module $\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}}$ canonically factorizes on $\overset{o}{\mathcal{Z}}$ and therefore $\jmath_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}})$ factorizes in $D(\mathcal{Z})$.

By Theorem 5.4.1 we have canonical maps:

$$\mathfrak{s}^{\check{\lambda}}_{*,dR} H^0\Big(\mathfrak{s}^{\check{\lambda},!} \jmath^{\check{\lambda}}_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}})\Big) = \mathfrak{s}_{*,dR}(\Omega^{\check{\lambda}}_{\check{\mathfrak{n}}}) \to \jmath^{\check{\lambda}}_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}}).$$

compatible with factorization as we vary $\check{\lambda}$. Note these maps are between objects of $D(\mathcal{Z}^{\check{\lambda}})^{\heartsuit}$ and are monomorphisms in this category.

Applying $\overset{*}{\psi}_{\mathcal{Z}} \overset{*}{\otimes} -$ and using factorization and lissity of $\overset{*}{\psi}_{\mathcal{Z}}$ and the canonical identifications $\mathfrak{s}^{*,dR,\check{\lambda}}(\overset{*}{\psi}_{\mathcal{Z}^{\check{\lambda}}}) \xrightarrow{\simeq} k_{\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}}$ we obtain maps:

$$\eta^{\check{\lambda}} : \mathfrak{s}_{*,dR}(\Omega^{\check{\lambda}}_{\check{\mathfrak{n}}}) \to \jmath^{\check{\lambda}}_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}} \overset{*}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}}).$$

Note that these are maps between objects that are up to a shift in the heart of the t-structure and as such are monomorphisms. Because everything above is compatible with factorization as we vary $\check{\lambda}$, the maps $\eta^{\check{\lambda}}$ are as well.

We let $\eta : \mathfrak{s}_{*,dR}(\Omega_{\check{\mathfrak{n}}}) \to \jmath_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}} \overset{*}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$ denote the induced map of factorizable D-modules on $\mathcal{Z}$.

Theorem 5.5.1. *The map:*

$$\Omega_{\check{\mathfrak{n}}} = \pi_! \mathfrak{s}_!(\Omega_{\check{\mathfrak{n}}}) = \pi_! \mathfrak{s}_{*,dR}(\Omega_{\check{\mathfrak{n}}}) \xrightarrow{\pi_!(\eta)} \pi_! \jmath_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}} \overset{*}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}) = \overset{o}{\pi}_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}} \overset{*}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$$

is an equivalence of factorizable D-modules on $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$.

Remark 5.5.2. In particular, the theorem asserts that all non-zero cohomology D-modules of $\overset{o}{\pi}_!(\overset{*}{\psi}_{\overset{o}{\mathcal{Z}}} \overset{*}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$ vanish.

Proof of Theorem 5.5.1. It suffices to show for fixed $\check{\lambda} \in \check{\Lambda}^{pos}$ that $\pi^{\check{\lambda}}_!(\eta^{\check{\lambda}})$ is an equivalence.

Recall from [BG08] Corollary 4.5 that we have an equality:

$$[\jmath^{\check{\lambda}}_!(\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}})] = \sum_{\substack{\check{\mu},\check{\eta} \in \check{\Lambda}^{pos} \\ \check{\mu}+\check{\eta}=\check{\lambda}}} [\mathrm{act}^{\check{\eta},\check{\mu}}_{\mathcal{Z},*,dR}(\Omega^{\check{\eta}}_{\check{\mathfrak{n}}} \boxtimes \mathrm{IC}_{\mathcal{Z}^{\check{\mu}}})] \in K_0(D^b_{rh}(\mathcal{Z}^{\check{\lambda}})).$$

in the Grothendieck group of regular holonomic D-modules. Therefore, because $\overset{*}{\psi}_{\mathcal{Z}}$ is lisse, we obtain a similar equality:

$$[j_!^{\check\lambda}(\overset{*}{\psi}_{\overset{\circ}{\mathcal{Z}}^{\check\lambda}} \overset{*}{\otimes} \mathrm{IC}_{\overset{\circ}{\mathcal{Z}}^{\check\lambda}})] = \sum_{\substack{\check\mu,\check\eta\in\check\Lambda^{pos} \\ \check\mu+\check\eta=\check\lambda}} [\mathrm{act}^{\check\eta,\check\mu}_{\mathcal{Z},*,dR}(\Omega^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes (\overset{*}{\psi}_{\overset{\circ}{\mathcal{Z}}^{\check\mu}} \overset{*}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}}))]$$

by the projection formula.

For every $\check\mu + \check\eta = \check\lambda$, note that each map $\mathrm{act}^{\check\eta,\check\mu}_{\mathcal{Z}}$ is proper and therefore we have:

$$\pi_!^{\check\lambda}\,\mathrm{act}^{\check\eta,\check\mu}_{\mathcal{Z},*,dR}\left(\Omega^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes (\overset{*}{\psi}_{\overset{\circ}{\mathcal{Z}}^{\check\mu}} \overset{*}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}})\right) = \mathrm{add}^{\check\eta,\check\mu}_{*,dR}\left(\Omega^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes \pi_!^{\check\mu}(\overset{*}{\psi}_{\overset{\circ}{\mathcal{Z}}^{\check\mu}} \overset{*}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}})\right).$$

By Theorem 4.3.1, this term therefore vanishes for $\check\mu \neq 0$.

Therefore, because $\eta^{\check\lambda}$ is a monomorphism in the shifted heart of the t-structure on $D(\mathcal{Z}^{\check\lambda})$, we see that $\pi_!^{\check\lambda}(\eta^{\check\lambda})$ is an equivalence as desired.

□

5.6. We will use a Verdier dual version of the above computations.

We let $\Upsilon_{\check{\mathfrak{n}}}$ denote the $\check\Lambda^{pos}$-graded D-module obtained by termwise taking Verdier duals to the terms $\Omega^{\check\lambda}_{\check{\mathfrak{n}}}$, i.e.:

$$\Upsilon^{\check\lambda}_{\check{\mathfrak{n}}} := \mathbb{D}_{Verdier}(\Omega^{\check\lambda}_{\check{\mathfrak{n}}}).$$

Again, each component of $\Upsilon_{\check{\mathfrak{n}}}$ lies in the heart of the t-structure. Note that $\Upsilon_{\check{\mathfrak{n}}}$ tautologically factorizes, though it is no longer commutative as a factorization algebra.

Remark 5.6.1. Note that $\Upsilon_{\check{\mathfrak{n}}}$ is termwise holonomic, so we may make sense of its $*$-fibers. Moreover, these are canonically identified with the corresponding graded component of the homological Chevalley complex $C_\bullet(\check{\mathfrak{n}})$ for $\check{\mathfrak{n}}$.

Remark 5.6.2. In the setting of Remark 5.3.2, we may say that $\Upsilon_{\check{\mathfrak{n}}}$ is obtained by taking the (derived) $\check\Lambda^{pos}$-graded Lie-$*$ algebra $\check{\mathfrak{n}}_X := \check{\mathfrak{n}}\otimes k_X$, taking the chiral enveloping algebra, then passing to the corresponding factorization algebra. Here k_X is the constant sheaf on X, which of course is in cohomological degree 1.

5.7. We have the following immediate consequence of Theorem 5.5.1, given by passing to Verdier duals.

Corollary 5.7.1. *There is a canonical identification:*

$$\Upsilon_{\check{\mathfrak{n}}} \xrightarrow{\simeq} \overset{o}{\pi}_{*,dR}(\overset{!}{\psi}_{\overset{o}{Z}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$$

of $\check{\Lambda}^{pos}$-graded factorization algebras.

6. Dramatis personae

6.1. The goal for this section is to introduce the semi-infinite flag variety in the context of factorizable geometry, and its associated Whittaker D-modules.

A summary of what is achieved is given in §6.34, and may be motivating to read before the remainder of the section.

6.2. We fix a smooth affine curve X.

We will use the language and notation of factorization categories from Part 2. In particular, we will be constructing chiral categories using the material of §14.

However, we will make the following change for ease of notation: using the 1-affineness of X_{dR} established in [Gai12b] we avoid the language of sheaves of categories used earlier and work with their global sections instead.

We also will require the theory of D-modules on indschemes developed in §16, and will freely appeal to the notions developed in *loc. cit.*

6.3. Let I be a finite set. Let Y be some fixed affine scheme.

We recall in §6.4-6.6 the definition of the jet space $\mathfrak{J}ets_{X^I}(Y)$ and the meromorphic jet space $\mathfrak{J}ets^{mer}_{X^I}(Y)$.

6.4. **Jet spaces.** Let $n \in \mathbb{Z}^{\geq 0}$ be an integer.

For S an affine test scheme, we define the nth jet space $\mathfrak{J}ets_{X^I}(Y)^{(n)}$ to have S-points:

$$\mathfrak{J}ets_{X^I}(Y)^{(n)}(S) = \Big\{ x = (x_i)_{i \in I} : S \to X^I \text{ and } \curlyvee : \Gamma_x^{(n)} \to Y \Big\} \tag{6.4.1}$$

where $\Gamma_x \subseteq X \times S$ is the scheme-theoretic union of the graphs Γ_{x_i} of the maps x_i, and $\Gamma_x^{(n)}$ is the nth infinitesimal neighborhood of Γ_x in $X \times S$. Note that $\mathfrak{J}ets_{X^I}(Y)^{(n)}$ is represented by a scheme of finite type over X^I.

As n varies, the spaces $\mathfrak{J}ets_{X^I}(Y)^{(n)}$ form an inverse system under affine structure maps. We let $\mathfrak{J}ets_{X^I}(Y)$ denote the projective limit.

The following is well-known: we include a proof for completeness.

Lemma 6.4.1. *Suppose Y a smooth scheme. Then for every pair $m, n \in \mathbb{Z}^{\geqslant 0}$, the scheme $\mathfrak{J}ets_{X^I}(Y)^{(n)}$ is smooth, and the structure maps:*

$$\mathfrak{J}ets_{X^I}(Y)^{(n+m)} \to \mathfrak{J}ets_{X^I}(Y)^{(n)}$$

are smooth, affine and surjective on geometric points.

Proof. We have already noted that the map is affine. The surjectivity follows by formal smoothness of Y.

Let S be an X^I-scheme that is affine, and let it be equipped with the structure map $x : S \to X^I$.

A map $S \to \mathfrak{J}ets_{X^I}(Y)^{(n)}$ is equivalent to a map $\curlyvee : \Gamma_x^{(n)} \to Y$, so the cotangent complex $\Omega^1_{\mathfrak{J}ets_{X^I}(Y)^{(n)}/X^I}$ restricts to S as $\pi_* \curlyvee^*(\Omega^1_Y)$, where $\pi = \pi_n$ is the composition $\Gamma_x^{(n)} \hookrightarrow X \times S \to S$.

Because Y is smooth, Ω^1_Y is a vector bundle concentrated in a single cohomological degree. Therefore, the same is true for $\curlyvee^*(\Omega^1_Y)$. Because π is finite flat, $\pi_* \curlyvee^*(\Omega^1_Y)$ is also a vector bundle concentrated in exactly one degree. Therefore, we deduce smoothness of $\mathfrak{J}ets_{X^I}(Y)^{(n)}$ from the fact that the cotangent complex is a vector bundle.

It remains to show smoothness of the structure maps. We perform the relative tangent space computation. For $\curlyvee : \Gamma_x^{(n+m)} \to Y$, the relevant map is:

$$\pi_{n+m,*}(\gamma^*(T_Y)) \to \pi_{n,*}(\gamma^*(T_Y)|_{\Gamma_x^{(n)}})$$

where T_Y is the tangent complex (i.e., tangent sheaf) of Y. Since the maps π_i are affine, it suffices to show the surjectivity on $\Gamma_x^{(n+m)}$, before applying $\pi_{n+m,*}$. But this is obvious: we are dealing with a restriction map for vector bundles on an affine scheme.

$\square$

6.5. **Discs.** Let S be an affine test scheme and let $x = (x_i)_{i\in I} : S \to X^I$ be a map.

We define the *formal disc* $\widehat{\mathcal{D}}_x$ at x to be the formal completion of $X \times S$ along Γ_x. Note that $\widehat{\mathcal{D}}_x$ is an ind-affine indscheme.

We define the *adic disc* $\mathcal{D}_x \in \mathsf{AffSch}$ to be the value of the partially defined left adjoint of the functor $\mathsf{AffSch} \hookrightarrow \mathsf{PreStk}$ evaluated on $\widehat{\mathcal{D}}_x$. Note that ind-affineness of $\widehat{\mathcal{D}}_x$ implies that this functor is defined here: it is the spectrum of the limit of the corresponding commutative rings.

Observe that formation of $\widehat{\mathcal{D}}_x$ is étale local on X in the natural sense.

Note that $\mathfrak{J}ets_{X^I}(Y)$ is equivalently described as the moduli of maps $x : S \to X^I$ plus a map $\widehat{\mathcal{D}}_x \to Y$ or $\mathcal{D}_x \to Y$.

We define the *punctured disc* $\overset{o}{\mathcal{D}}_x \in \mathsf{Sch}$ at x as:

$$\overset{o}{\mathcal{D}}_x := \mathcal{D}_x \backslash \Gamma_x.$$

These constructions organize into the diagram:

$$\begin{tikzcd} \Gamma_x \arrow[r] & \widehat{\mathcal{D}}_x \arrow[r] & \mathcal{D}_x \arrow[d] & \overset{o}{\mathcal{D}}_x \arrow[l] \\ & & X \times S. & \end{tikzcd}$$

6.6. **Loop spaces.** Finally, we define:

$$\mathfrak{J}ets^{mer}_{X^I}(Y)(S) = \Big\{x : S \to X^I \text{ and } \gamma : \overset{o}{\mathcal{D}}_x \to Y.\Big\} \tag{6.6.1}$$

As in [KV04] Proposition 3.5.2, $\mathfrak{J}ets^{mer}_{X^I}(Y)$ is represented by an indscheme (of ind-infinite type), and formation of $\mathfrak{J}ets^{mer}_{X^I}(Y)$ is étale local on X.

Remark 6.6.1. If Z is an affine X-scheme, then we have notions of "relative jets" and "relative meromorphic jets" that generalizes the constructions above when $Z = X \times Y$.

This is actually the level of generality we will be using in practice, but we find it convenient to write the material that follows in the product situation. See §6.10 and 6.15 for more discussion of this point.

Note that representability questions in the relative case reduce to the product case treated in [KV04]: factor the map Z to X through its graph, and then the relative (resp. meromorphic) jets embed as a closed subscheme (resp. sub-indscheme) of the corresponding "absolute" jets.

6.7. **Factorization of the disc.** Let $\mathsf{Set}_{<\infty}$ denote the category of (possibly empty) finite sets under (possibly non-surjective) maps.

Let $f : I \to J$ be a map in $\mathsf{Set}_{<\infty}$, let S be an affine scheme and let $x = (x_j)_{j\in J} : S \to X^J$ be a map. Let $x' = (x'_i) = (x_{f(i)}) : S \to X^I$ be the map induced by f.

Note that $\Gamma^{red}_{x'}$ is a closed subscheme of Γ^{red}_x, giving a canonical map $\mathcal{D}_{x'} \to \mathcal{D}_x$. Therefore, we obtain an op-correspondence:

$$\begin{tikzcd}[column sep=small] & & \mathcal{D}_x \backslash \Gamma_{x'} & & \\ \overset{o}{\mathcal{D}}_{x'} \arrow[r, equal] & \mathcal{D}_{x'} \backslash \Gamma_{x'} \arrow[ur] & & \mathcal{D}_x \backslash \Gamma_x \arrow[ul] \arrow[r, equal] & \overset{o}{\mathcal{D}}_x. \end{tikzcd} \tag{6.7.1}$$

Remark 6.7.1. If f is surjective then the reduced schemes underlying Γ_x and $\Gamma_{x'}$ coincide. Therefore, in this case the right map in (6.7.1) is an isomorphism.

6.8. **Chiral categories.** Varying $I \in \mathsf{fSet}$, we obtain that the rules $I \mapsto \mathfrak{J}ets_{X^I}(Y)$ and $I \mapsto \mathfrak{J}ets^{mer}_{X^I}(Y)$ factorize.

Applying Proposition 16.50.1, we obtain chiral categories (à la §14) on X_{dR}:

$$\Big(I \mapsto D^!(\mathfrak{J}ets_{X^I}(Y))\Big) \text{ and} \Big(I \mapsto D^!(\mathfrak{J}ets^{mer}_{X^I}(Y))\Big).$$

Passing to the limit over I, we obtain the categories $D^!(\mathfrak{J}ets_{\mathrm{Ran}_X}(Y))$ and $D^!(\mathfrak{J}ets^{mer}_{\mathrm{Ran}_X}(Y))$.

We use the notation $D^!(\mathfrak{J}ets(Y)), D^!(\mathfrak{J}ets^{mer}(Y)) \in \mathsf{Cat}^{ch}(X_{dR})$ to denote the corresponding chiral categories.

6.9. **Unital structures.** Suppose Y is an affine scheme of finite type.

Let $f : I \to J$ be a map in $\mathsf{Set}_{<\infty}$. Using the notation of §6.7, let $\mathcal{H}_{Y,f}$ denote the moduli of maps $x : S \to X^J$ plus a map $(\mathcal{D}_x \backslash \Gamma_{x'}) \to Y$, defined formally as in (6.6.1).

Applying (6.7.1), we obtain a correspondence:

$$\begin{tikzcd} & \mathcal{H}_{Y,f} \arrow[dl, "\alpha_{Y,f}"'] \arrow[dr, "\beta_{Y,f}"] & \\ \mathfrak{J}ets^{mer}_{X^I}(Y) & & \mathfrak{J}ets^{mer}_{X^J}(Y). \end{tikzcd} \tag{6.9.1}$$

For f the identity, this correspondence is the identity correspondence. For $f : I \to J$ and $g : J \to K$, we obtain a canonical diagram:

$$\begin{tikzcd} & & \mathcal{H}_{Y,g\circ f} \arrow[dl] \arrow[dr] & & \\ & \mathcal{H}_{Y,f} \arrow[dl] \arrow[dr] & & \mathcal{H}_{Y,g} \arrow[dl] \arrow[dr] & \\ \mathfrak{J}ets^{mer}_{X^I}(Y) & & \mathfrak{J}ets^{mer}_{X^J}(Y) & & \mathfrak{J}ets^{mer}_{X^K}(Y) \end{tikzcd}$$

where the middle diamond is Cartesian.

In other words, we obtain a functor $\mathsf{Set}_{<\infty} \to \mathsf{IndSch}_{corr}$ sending I to $\mathfrak{J}ets^{mer}_{X^I}(Y)$. This functor is compatible with factorization in the natural sense.

Moreover, for f as above, one sees that the map:

$$\beta_{Y,f} : \mathcal{H}_{Y,f} \to \mathfrak{J}ets^{mer}_{X^J}(Y)$$

is finitely presented. Therefore, by §16.44, we obtain that:

$$I \mapsto D^!\Big(\mathfrak{J}ets^{mer}_{X^I}(Y)\Big)$$

defines a unital chiral category on X_{dR}:

$$D^!(\mathfrak{J}ets^{mer}_{X^I}(Y)) \in \mathsf{Cat}^{ch}_{un}(X_{dR})$$

refining our earlier non-unital chiral category.

Remark 6.9.1. For a morphism $f : I \to J \in \mathsf{Set}_{<\infty}$, the corresponding map $D^!(\mathfrak{J}ets^{mer}_{X^I}(Y)) \to D^!(\mathfrak{J}ets^{mer}_{X^J}(Y))$ is the computed by the functor $\beta_{Y,f,*,!-dR}\alpha^!_{Y,f}$. We recall that the functor $\beta_{Y,f,*,!-dR}$ of !-dR $*$-pushforward is defined for any finitely presented morphism and is the functor of §16.44.

Remark 6.9.2. The unit object in $D^!(\mathfrak{J}ets^{mer}_{\mathrm{Ran}_X}(Y))$ is obtained by !-dR $*$-pushforward of $\omega_{\mathfrak{J}ets_{\mathrm{Ran}_X}(Y)}$. Here, the symbol $\omega_{\mathfrak{J}ets_{\mathrm{Ran}_X}(Y)}$ refers to the compatible system of objects $(I \mapsto \omega_{\mathfrak{J}ets_{X^I}(Y)})$ and the term "!-dR $*$-pushforward" refers to the appropriate compatible system of such functors.

Remark 6.9.3. For a morphism $Y_1 \to Y_2$ of schemes of finite type, we obtain canonical maps $\mathfrak{J}ets^{mer}_{X^I}(Y_1) \to \mathfrak{J}ets^{mer}_{X^I}(Y_2)$. These maps are obviously compatible with the correspondences above and therefore define a canonical strictly unital morphism:

$$D^!(\mathfrak{J}ets^{mer}(Y_2)) \to D^!(\mathfrak{J}ets^{mer}(Y_1))$$

computed as !-pullback over each X^I.

Notation 6.9.4. For I and J two finite sets, we will sometimes use the notation $\mathcal{H}_{Y,I,J}$ in place of $\mathcal{H}_{Y,f}$ with f the tautological embedding $I \hookrightarrow I \coprod J$.

6.10. **Forms of algebraic groups.** We will be working with group schemes $\mathcal{G}$ over X that are *forms* of affine algebraic groups. See §6.15 to see the examples we will use.

We will say that two group schemes over X are *forms* of each other if they are isomorphic as group schemes étale[15] locally on X.

Therefore, being a form of an affine algebraic group means that the group scheme $\mathcal{G}$ is a smooth, affine group scheme that is a form of $\mathcal{G}^0 \times X$ for $\mathcal{G}^0$ an affine algebraic group. In this case, we abbreviate the situation in saying that $\mathcal{G}$ is a form of $\mathcal{G}^0$.

For the remainder of this section, we fix $\mathcal{G}$ an affine group scheme over X of the type above.

Example 6.10.1. Every reductive group scheme over X is a form of the associated split reductive group.

6.11. In applying the Beauville-Laszlo principle [BL95],[16] it is convenient to have the following well-known technical result. We include a proof for completeness.

Lemma 6.11.1. *Let $x : S \to X^I$ be a map from an affine scheme S. Let $\mathcal{G}$ be a form of an algebraic group over X. Then the restriction map:*

$$\{\mathcal{G}\text{–}bundles\ on\ \mathcal{D}_x\} \to \{\mathcal{G}\text{–}bundles\ on\ \widehat{\mathcal{D}}_x\}$$

is an equivalence of groupoids.

[15] A warning: There is a risk that taking étale forms means that e.g. the associated affine Grassmannian will be an ind-algebraic space, not an indscheme, which is somewhat problematic since §16 is written for indschemes. However, we note that 1) the forms we will take are *Zariski* locally trivial (c.f. §6.15), removing the problem for us in practice, and 2) the material in *loc. cit.* extends to the setting of algebraic spaces using [Ryd09] and an appropriate generalization of the relevant material of [GR14]. For these reasons, we will ignore the issue in what follows and deal with D-modules on our indschemes without further mention.

[16] Which is necessarily about $\mathcal{D}$ — not $\widehat{\mathcal{D}}$ — since it involves the punctured disc.

Proof. First, we claim that $\mathcal{O}_{\mathcal{G}}$, considered as a representation of $\mathcal{G}$ over X, is a union of subrepresentations that are finite rank vector bundles on X. Indeed, it is always true that comodules for an A-coalgebra B are a union of A-finitely generated submodules, and because X is a smooth curve, submodules of $\mathcal{O}_{\mathcal{G}}$ (which is flat) are necessarily flat.

Pulling $\mathcal{G}$ back to $\mathcal{D}_x$, we see that there are again "enough" vector bundle representations. Therefore, using the Tannakian formalism, we reduce to treating the case $\mathcal{G} = GL_{r,X}$.

Let $S = \mathrm{Spec}(A)$, and let A_n denote the commutative algebra of functions on the (affine) scheme $\Gamma_x^{(n)}$ (so $A_0 = A$). Let $B = \lim_n A_n$, so $\mathrm{Spec}(B) = \mathcal{D}_x$. Let $I_n \subseteq B$ denote the kernel of the (surjective) map $B \to A_n$.

Let $\mathcal{E}$ be a finitely generated projective B-module of rank r. Because $\mathcal{E}$ is a direct summand of a finite rank free B-module, $\mathcal{E} \xrightarrow{\simeq} \lim_n \mathcal{E}/I_n$. This proves fully-faithfulness.

It remains to show essential surjectivity. Here we need to show that the limit $\mathcal{E} := \lim_n \mathcal{E}_n$ of a compatible system $\{\mathcal{E}_n\}$ of rank r projective A_n-modules is a finitely generated projective B-module.

We can write $\mathcal{E}_0 \oplus \mathcal{E}_0' \xrightarrow{\simeq} A_0^{\oplus(r+s)}$ for $\mathcal{E}_0'$ a rank s vector bundle on $\mathrm{Spec}(A)$.

Therefore, by formal smoothness of $GL_{r+s}/GL_r \times GL_s$, we can lift the compatible system $\{\mathcal{E}_n\}$ to a compatible system $(\mathcal{E}_n, \mathcal{E}_n', \mathcal{E}_n \oplus \mathcal{E}_n' \xrightarrow{\simeq} A_n^{\oplus(r+s)})$ such that the $n = 0$ case is given by our earlier choice. But this obviously realizes $\mathcal{E}$ itself as a direct summand of a finite free module.

$\square$

In particular, we obtain the following corollary from formal smoothness of the map $X \to X/\mathcal{G}$.

Corollary 6.11.2. *In the notation above, a $\mathcal{G}$-bundle on $\mathcal{D}_x$ is trivial if and only if its restriction to S is.*

6.12. **The affine Grassmannian.** We will specialize the above material to the case of (relative) jets into $\mathcal{G}$, considered as in Remark 6.6.1.

Fix a finite set I.

In this case, $\mathfrak{J}ets_{X^I}(\mathcal{G})$ is a group scheme over X^I Moreover, since each $\mathfrak{J}ets_{X^I}(\mathcal{G})^{(n)}$ is a smooth group scheme over X^I, $\mathfrak{J}ets_{X^I}(\mathcal{G})$ satisfies the hypotheses of Example 16.29.3 as a group scheme over X^I.

We also have the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\mathcal{G},X^I}$ with the usual $\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$-equivariant (relative to the left action on the source) map $\pi_{\mathcal{G},X^I} : \mathfrak{J}ets^{mer}_{X^I}(\mathcal{G}) \to \mathrm{Gr}_{\mathcal{G},X^I}$.

We recall that $\mathrm{Gr}_{\mathcal{G},X^I}$ parametrizes points $(x_i)_{i\in I}$ of X, a $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ on X, and a trivialization τ of $\mathcal{P}_{\mathcal{G}}$ defined on $X\backslash\{x_i\}_{i\in I}$. This is understood in families in the usual way.

We have the following well-known result (proved by reduction[17] to $G = GL_n$):

Lemma 6.12.1. *The space* $\mathrm{Gr}_{\mathcal{G},X^I}$ *is an ind-algebraic space of ind-finite type. If* $\mathcal{G}$ *is reductive, then* $\mathrm{Gr}_{\mathcal{G},X^I}$ *is ind-proper over* X^I. *If* $\mathcal{G}$ *is Zariski-locally constant,*[18] *then then* $\mathrm{Gr}_{\mathcal{G},X^I}$ *is an indscheme of ind-finite type.*

We deduce:

Proposition 6.12.2. *The map* $\pi_{\mathcal{G},X^I} : \mathfrak{J}ets^{mer}_{X^I}(\mathcal{G}) \to \mathrm{Gr}_{\mathcal{G},X^I}$ *realizes* $\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$ *as an étale-locally trivial* $\mathfrak{J}ets_{X^I}(\mathcal{G})$*-torsor over* $\mathrm{Gr}_{\mathcal{G},X^I}$.[19]

Proof. We follow [BD] Theorem 4.5.1, where this is proved over a point.

After Zariski localization, we can assume that X admits an étale map to $\mathbb{A}^1$, and after étale localization, that $\mathcal{G}$ is constant (in particular, pulled back from $\mathbb{A}^1$), and

[17] This reduction step is justified as in the proof of Lemma 6.11.1.

[18] I.e., Zariski-locally of the form $\mathcal{G}^0 \times X$ for $\mathcal{G}^0$ an affine algebraic group.

[19] In fact, Zariski-locally trivial if $\mathcal{G}$ is a Zariski form.

therefore we reduce to the case $X = \mathbb{A}^1$. We abuse notation in also denoting by $\mathcal{G}$ the corresponding affine algebraic group.

We embed $X = \mathbb{A}^1$ into its compactification $\mathbb{P}^1$ with ∞ denoting the point complementary to $\mathbb{A}^1$.

In this case we will show that $\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G}) \to \mathrm{Gr}_{\mathcal{G},X^I}$ admits a section Zariski-locally on the target. Because $\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$ acts transitively on geometric points of $\mathrm{Gr}_{\mathcal{G},X^I}$, it suffices to show that there is a Zariski neighborhood of the unit $X^I \subseteq \mathrm{Gr}_{\mathcal{G},X^I}$ that admits a section.

Form the fiber product:

$$\mathcal{U} := \mathrm{Gr}_{\mathcal{G},X^I} \underset{\mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)}{\times} \mathbb{B}\mathcal{G}$$

where $\mathbb{B}\mathcal{G} \to \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)$ is the map defined by the trivial bundle. Note that $\mathbb{B}\mathcal{G} \to \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)$ is an open embedding (specifically because we deal with $\mathbb{P}^1$) and therefore the map $\mathcal{U} \to \mathrm{Gr}_{\mathcal{G},X^I}$ is an open embedding. Of course, the map $X^I \to \mathrm{Gr}_{\mathcal{G},X^I}$ factors through $\mathcal{U}$.

The composition:

$$\mathbb{B}\mathcal{G} \hookrightarrow \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1) \xrightarrow{\mathrm{ev}_\infty} \mathbb{B}\mathcal{G}$$

is the identity. Therefore, one obtains that $\mathcal{U}$ is the moduli of $(x_i)_{i\in I}$ in $X = \mathbb{A}^1$ and a map $\mathbb{P}^1\backslash\{(x_i)_{i\in I}\} \to \mathcal{G}$ sending ∞ to $1 \in \mathcal{G}$. We obtain a map $\mathcal{U} \to \mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$ given by taking Laurent expansions, giving the desired section.

$\square$

Convention 6.12.3. For the ease of exposition, we systematically ignore the differences between schemes and algebraic spaces for the remainder of the section (since the forms we will use are Zariski-locally trivial).

The following now results from Example 16.49.4 and Lemma 6.4.1, since $\mathrm{Gr}_{\mathcal{G},X^I}$ is an indscheme of ind-finite type.

Corollary 6.12.4. *$\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$ is a placid indscheme.*

We obtain the following from Construction 16.53.6 of §16.53.

Corollary 6.12.5. *The indscheme $\mathfrak{J}ets^{mer}_{X^I}(\mathcal{G})$ carries a canonical dimension theory $\tau^{\mathcal{G}}$ such that for any finite type subscheme $T \subseteq \mathrm{Gr}_{\mathcal{G},X^I}$ we have:*

$$\tau^{\mathcal{G}}\big(\pi^{-1}_{\mathcal{G},X^I}(T)\big) = \pi^*_{\mathcal{G},X^I}(\dim_T).$$

6.13. Note that $I \mapsto \mathrm{Gr}_{\mathcal{G},X^I}$ defines a *unital factorization indscheme*, i.e., for every $f : I \to J$ we have correspondences:

$$\begin{tikzcd} & \mathrm{Gr}_{\mathcal{G},X^I} \underset{X^I}{\times} X^J \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_{\mathcal{G},X^I} & & \mathrm{Gr}_{\mathcal{G},X^J} \end{tikzcd}$$

where the left map is obvious and the right map is given by sending:

$$\Big((x_j)_{j\in J}, \mathcal{P}_{\mathcal{G}}, \tau \text{ a trivialization of } \mathcal{P}_{\mathcal{G}}|_{X\setminus\{x_{f(i)}\}_{i\in I}}\Big) \in \mathrm{Gr}_{\mathcal{G},X^J}$$

to the point:

$$\Big((x_j)_{j\in J}, \mathcal{P}_{\mathcal{G}}, \tau|_{X\setminus\{x_j\}_{j\in J}}\Big).$$

Here we note that $X\setminus\{x_j\}_{j\in J} \subseteq X\setminus\{x_{f(i)}\}_{i\in I}$, so that this restriction makes sense.

Therefore, $I \mapsto D(\mathrm{Gr}_{\mathcal{G},X^I})$ defines a unital chiral category $D(\mathrm{Gr}_{\mathcal{G}}) \in \mathsf{Cat}^{ch}_{un}(X_{dR})$.

Remark 6.13.1. The natural maps $\pi_{\mathcal{G},X^I} : \mathfrak{J}ets^{mer}_{X^I}(\mathcal{G}) \to \mathrm{Gr}_{\mathcal{G},X^I}$ are compatible with the correspondences (6.9.1) for $\mathfrak{J}ets^{mer}(\mathcal{G})$. Moreover, for every $f : I \to J$, the square:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{G},f} \arrow[d] \arrow[dr] & \\
\mathrm{Gr}_{\mathcal{G},X^I} \underset{X^I}{\times} X^J \arrow[dr] & \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}) \arrow[d] \\
& \mathrm{Gr}_{\mathcal{G},X^J}
\end{tikzcd}$$

is Cartesian. Therefore, the functors $\pi^!_{\mathcal{G},X^I}$ define a strictly unital factorization functor:

$$\pi^!_{\mathcal{G}} : D(\mathrm{Gr}_G) \to D^!(\mathfrak{J}ets^{mer}(\mathcal{G})). \tag{6.13.1}$$

Remark 6.13.2. Formation of the unital factorization indscheme $I \mapsto \mathrm{Gr}_{\mathcal{G},X^I}$ is obviously functorial in $\mathcal{G}$: given a morphism $\mathcal{G}_1 \to \mathcal{G}_2$ we obtain morphisms $\mathrm{Gr}_{\mathcal{G}_1,X^I} \to \mathrm{Gr}_{\mathcal{G}_2,X^I}$ compatible with the unital factorization structures. Moreover, for every $I \to J$, the square:

$$\begin{tikzcd}
& \mathrm{Gr}_{\mathcal{G}_1,X^I} \underset{X^I}{\times} X^J \arrow[dl] \arrow[d] \\
\mathrm{Gr}_{\mathcal{G}_1,X^I} \arrow[d] & \mathrm{Gr}_{\mathcal{G}_1,X^I} \underset{X^I}{\times} X^J \arrow[dl] \\
\mathrm{Gr}_{\mathcal{G}_2,X^I} &
\end{tikzcd}$$

is (obviously) Cartesian.

Therefore, we obtain a strictly unital chiral functor:

$$D(\mathrm{Gr}_{\mathcal{G}_1}) \to D(\mathrm{Gr}_{\mathcal{G}_2})$$

given by de Rham pushforwards (which is well-behaved because all the indschemes present are ind-finite type).

6.14. **Pure inner forms.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two smooth group schemes over X. Recall that they are said to be *pure inner forms* of each other if there is a specified *bitorsor* for these groups: a $\mathcal{G}_1$-torsor $\mathcal{P}$ on X with a commuting $\mathcal{G}_2$-action realizing $\mathcal{P}$ as a $\mathcal{G}_2$-torsor as well.

In this case, we have a canonical isomorphism of stacks:

$$X/\mathcal{G}_1 \xrightarrow{\simeq} X/\mathcal{G}_2.$$

For example, the map $X/\mathcal{G}_1 \to X/\mathcal{G}_2$ is defined by the $\mathcal{G}_2$-torsor $\mathcal{P}/\mathcal{G}_1$ on $X/\mathcal{G}_1$ (note that we can speak about $\mathcal{G}_2$-torsors on $X/\mathcal{G}_1$ because we have a canonical map $X/\mathcal{G}_1 \to X$).

In particular, if X is proper, we can identify the algebraic stacks:

$$\mathrm{Bun}_{\mathcal{G}_1} \xrightarrow{\simeq} \mathrm{Bun}_{\mathcal{G}_2}. \tag{6.14.1}$$

If $\mathcal{P}$ is a bitorsor for $\mathcal{G}_1$ and $\mathcal{G}_2$, observe $\mathcal{G}_2$ is the group scheme of $\mathcal{G}_1$-automorphisms of $\mathcal{P}$: this follows by considering the local case where $\mathcal{P}$ is trivialized as a $\mathcal{G}_1$-torsor. Therefore, given any group scheme $\mathcal{G}_1$ with a torsor $\mathcal{P}$ we canonically obtain a pure inner form $\mathcal{G}_2$ of $\mathcal{G}_1$ as the group scheme of automorphisms. Moreover, we see that pure inner forms of $\mathcal{G} = \mathcal{G}_1$ are classified by $\mathcal{G}_1$-torsors.

To summarize, for any $\mathcal{G}$ with torsor $\mathcal{P}$, we obtain a form $\mathcal{G}' := \mathrm{Aut}_{\mathcal{G}}(\mathcal{P})$.

6.15. Recall the torsors $\mathcal{P}_T^{can}$, $\mathcal{P}_B^{can}$, $\mathcal{P}_{B^-}^{can}$ and $\mathcal{P}_G^{can}$ from §3.13.

Let G^{can}, B^{can}, and $B^{-,can}$ denote the corresponding pure inner forms of G, B and B^- respectively. Note that commutativity of T means that T^{can} is a constant family.

Let $N^{-,can}$ denote the form of N^- obtained by twisting $\mathcal{P}_{B^-}^{can}$ by the adjoint action of B^- on N^-. Note that $N^{-,can}$ is *not* an inner form of N^-. We treat N^{can} similarly.

Example 6.15.1. Suppose that $G = GL_2$. Then G^{can} is the group scheme whose sections are matrices:

$$\begin{pmatrix} f & \varphi \\ \omega & g \end{pmatrix}$$

with $f, g \in \mathcal{O}_X$, $\omega \in \Omega_X$, and $\varphi \in \Omega_X^{-1}$, and with determinant $fg - \varphi \otimes \omega \in \mathcal{O}_X$ everywhere non-zero.

Convention 6.15.2. To avoid including twists in the notation everywhere, we will write e.g. $\mathfrak{J}ets_{X^I}^{mer}(G)$ for the relative jets into G^{can} (in the sense of Remark 6.6.1). The same goes for $\mathfrak{J}ets$, $\mathfrak{J}ets^{mer}$ and Gr, etc. of G and our other groups.

The truth is that these twists do not play a role at all until we discuss Whittaker invariants, and we could work just as well with any other twists of our groups until then (including the constant one). However, for reasons of notation, we choose to make the official policy to include these twists at every step.

Remark 6.15.3. By (6.14.1), this twist gives rise to the same automorphic forms as the split form of G.

Notation 6.15.4. We will use the notation $\mathfrak{p}_{X^I}^{loc}$ and $\mathfrak{q}_{X^I}^{loc}$ for the maps:

$$\begin{tikzcd} & \mathrm{Gr}_{B,X^I} \arrow[dl, "\mathfrak{p}_{X^I}^{loc}"'] \arrow[dr, "\mathfrak{q}_{X^I}^{loc}"] & \\ \mathrm{Gr}_{G,X^I} & & \mathrm{Gr}_{T,X^I} . \end{tikzcd}$$

(Here the notation *loc* indicates that these are "local" counterparts to the maps $\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ and $\mathfrak{q} : \mathrm{Bun}_B \to \mathrm{Bun}_T$ from [BG02]).

By the above, $\mathfrak{p}_{*,dR}^{loc}$ and $\mathfrak{q}_{*,dR}^{loc}$ have canonical structures of (strictly) unital chiral functors.

6.16. **Group actions on categories.** It will be convenient to have the basic aspects of the theory of group action on categories available to us.

Remark 6.16.1. Because we need to work in a relative framework, it is not sufficient for us to appeal to [Ber].

Let S be a base scheme of finite type and let $\mathcal{H} \to S$ be a group indscheme over S that is placid as a mere indscheme.

By Proposition 16.50.1, the category $D^!(\mathcal{H})$ obtains the structure of coalgebra in the symmetric monoidal category $D(S)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) \simeq \mathsf{ShvCat}_{/S_{dR}}$.

Definition 6.16.2. A category *(!–)acted on* by $\mathcal{H}$ (over S) is a left comodule for $D^!(\mathcal{H})$ in $\mathsf{ShvCat}_{/S_{dR}}$. We denote the corresponding category by $\mathcal{H}\text{–}\mathsf{mod}$.

Example 6.16.3. If T is an indscheme over S with an action of $\mathcal{H}$, then by Proposition 16.50.1, $\mathcal{H}$ acts on $D^!(T)$.

Remark 6.16.4. The "Hopf algebra" structure on $\mathcal{H}$ implies that $\mathcal{H}\text{–}\mathsf{mod}$ admits a symmetric monoidal structure with symmetric monoidal forgetful functor $\mathcal{H}\text{–}\mathsf{mod} \to \mathsf{ShvCat}_{/S_{dR}}$. For $\mathsf{C}, \mathsf{D} \in \mathcal{H}\text{–}\mathsf{mod}$, the coaction map on $\mathsf{C} \underset{D(S)}{\otimes} \mathsf{D}$ is induced in the obvious way from the coaction for C and D separately, and the !-restriction functor $D^!(\mathcal{H} \times_S \mathcal{H}) \to D^!(\mathcal{H})$ induced by the diagonal $\mathcal{H} \to \mathcal{H} \times_S \mathcal{H}$.

Remark 6.16.5. The forgetful functor $\mathcal{H}\text{–}\mathsf{mod} \to \mathsf{ShvCat}_{/S_{dR}}$ admits a right adjoint $\mathsf{C} \mapsto \mathsf{C} \otimes_{D(S)} D^!(\mathcal{H})$.

Moreover, we claim that $\mathcal{H}\text{–}\mathsf{mod} \to \mathsf{ShvCat}_{/S_{dR}}$ commutes with limits. Note that $D^!(\mathcal{H})$ is dualizable as an object of $\mathsf{ShvCat}_{/S_{dR}}$ by placidity and by Proposition 19.12.4 (3). Therefore, tensoring in $\mathsf{ShvCat}_{/S_{dR}}$ with $D^!(\mathcal{H})$ commutes with limits, so the result is proved as [Lur12] Corollary 4.2.3.5.

In particular, we see that every $\mathsf{C} \in \mathcal{H}\text{–}\mathsf{mod}$ admits a bar resolution:

$$\mathsf{C} = \lim_{\boldsymbol{\Delta}} \Big(\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \rightrightarrows \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \ldots \Big)$$

Given C acted on by $\mathcal{H}$, we define the category $\mathsf{C}^{\mathcal{H}}$ of invariants C as the limit of the bar construction:

$$\mathsf{C}^{\mathcal{H}} := \lim_{[n]\in\mathbf{\Delta}} \Big(\mathsf{C} \rightrightarrows D^!(\mathcal{H}) \underset{D(S)}{\otimes} \mathsf{C} \Rrightarrow \dots \Big)$$

There is a tautological functor:

$$\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}.$$

Example 6.16.6. The category $D^!(\mathcal{H})$ acts on itself, and we have $D(S) \xrightarrow{\simeq} D^!(\mathcal{H})^{\mathcal{H}}$ by splitting the relevant cosimplicial object. Here the corresponding functor $D(S) \xrightarrow{\simeq} D^!(\mathcal{H})^{\mathcal{H}} \xrightarrow{\mathrm{Oblv}} D^!(\mathcal{H})$ is !-pullback.

Remark 6.16.7. Suppose that $\mathcal{H} = \cup_i \mathcal{H}_i$ is an ind-group scheme. Then for every C acted on by $\mathcal{H}$, we have:

$$\mathsf{C}^{\mathcal{H}} \xrightarrow{\simeq} \lim_i \mathsf{C}^{\mathcal{H}_i}.$$

Indeed, this follows by commuting limits with limits.

We recall that $D^!(\mathcal{H})$ is dualizable as a $D(S)$-module category with dual $D^*(\mathcal{H})$ because $\mathcal{H}$ is assumed placid. Under this duality, the coalgebra structure on $D^!(\mathcal{H})$ transfers to the canonical algebra structure on $D^*(\mathcal{H}) \in \mathsf{ShvCat}_{/S_{dR}}$ induced by the multiplication on $\mathcal{H}$.[20]

We therefore obtain:

Proposition 6.16.8. *Under the above hypotheses on $\mathcal{H}$, categories acted on by $\mathcal{H}$ are canonically equivalent to left $D^*(\mathcal{H})$-modules in $\mathsf{ShvCat}_{/S_{dR}}$.*

[20]Here we are repeatedly using the canonical identification from [GR14] of $(f^!)^\vee$, the functor dual to $f^!$, with $f_{*,dR}$ for a morphism f of finite type schemes.

For C acted on by $\mathcal{H}$, we refer to the corresponding $D^*(\mathcal{H})$-action as *convolution*.

For the remainder of this discussion, we suppose that $\mathcal{H}$ is a group scheme over S, and moreover that $\mathcal{H}$ satisfies the hypotheses of Example 16.29.3, i.e., $\mathcal{H}$ is a filtered limit of smooth affine S-group schemes under smooth surjective homomorphisms.

By Proposition 16.38.1, the pullback $D(S) \to D^!(\mathcal{H})$ then admits a right adjoint in $\mathsf{ShvCat}_{/S_{dR}}$ of renormalized de Rham pushforward functor of §16.36.

We refer to [Lur12] Theorem 6.2.4.2 and [Gai11] §4.4.7 for an introduction to the Beck-Chevalley formalism used below.

Proposition 6.16.9. *Under the above hypotheses on $\mathcal{H}$, the cosimplicial object defining $\mathsf{C}^{\mathcal{H}}$ satisfies the Beck-Chevalley conditions.*

Corollary 6.16.10. *The functor* $\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}$ *admits a right adjoint* $\mathrm{Av}_{\mathcal{H},\mathsf{C},*} = \mathrm{Av}_{\mathcal{H},*} = \mathrm{Av}_*$ *in $D(S)$. In particular, formation of* Av_* *commutes with base-change of the (finite type) scheme S.*

Moreover, for a morphism $\mathsf{C} \to \mathsf{D}$ of categories acted on by $\mathcal{H}$, the diagram:

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, "\mathrm{Av}_*"] & \mathsf{D} \arrow[d, "\mathrm{Av}_*"] \\ \mathsf{C}^{\mathcal{H}} \arrow[r] & \mathsf{D}^{\mathcal{H}} \end{tikzcd}$$

commutes (i.e., the relevant natural transformation is a natural isomorphism). More precisely, Av_* *is given by convolution with $\omega_{\mathcal{H}}^{ren}$, this object being defined by the dimension theory on $\mathcal{H}$ obtained by pullback from the standard dimension theory on S.*

We we will use the following in the proof of Proposition 6.16.9.

Lemma 6.16.11. *For C acted on by $\mathcal{H}$, let*

$$\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \to \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})$$

be the endofunctor induced by the coaction map:

$$\mathsf{C} \to \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})$$

and considering the right hand side as a $(D^!(\mathcal{H}), \overset{!}{\otimes})$*-module.*

Then this endofunctor is an equivalence.

Proof. Recall that $D^!(\mathcal{H})$ is dualizable as a $D(S)$-module category. Therefore, by Remark 6.16.5 we reduce to the case where $\mathsf{C} = \mathsf{D} \otimes_{D(S)} D^!(\mathcal{H})$ for $\mathsf{D} \in \mathsf{ShvCat}_{/S_{dR}}$. Here the result is obvious.

□

Proof of Proposition 6.16.9. For every integer n, the functor:

$$\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{n \text{ times}} \to \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(n+1) \text{ times}}$$

coming from tensoring on the right with the pullback $D(S) \to D^!(\mathcal{H})$ admits a right adjoint, as noted before. Moreover, we claim that for every morphism $[n] \to [m] \in \mathbf{\Delta}$, we need to show that the following diagram commutes (i.e., the base-change map should be an equivalence):

$$\begin{tikzcd}
\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(n+1) \text{ times}} \arrow[r] \arrow[d] & \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{n \text{ times}} \arrow[d] \\
\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(m+1) \text{ times}} \arrow[r] & \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{m \text{ times}}
\end{tikzcd}$$

where horizontal arrows are these left adjoints and vertical arrows are the structure maps, $[n+1] \to [m+1]$ being induced from $[n] \to [m]$ by adjoining a new infimum.

Rather than get bogged down in notation, we prove this instead for "representative" morphisms $[n] \to [m]$, the general argument being the same.

Namely, suppose that $n = 0$ and $m = 1$. If $0 \mapsto 1$, the commutativity is tautological. Therefore, suppose that $0 \mapsto 0$. Then the corresponding map $\mathsf{C} \to \mathsf{C} \otimes_{D(S)} D^!(\mathcal{H})$ is the coaction map, and we should prove that the diagram:

$$\begin{tikzcd}
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \arrow[rr] \arrow[d, "\mathrm{coact} \otimes \mathrm{id}"] & & \mathsf{C} \arrow[d, "\mathrm{coact}"] \\
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) \arrow[r] & \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
\end{tikzcd}$$

commutes, where the horizontal arrows are given by taking renormalized de Rham cohomology in the last variable.

Intertwining the lower two terms using Lemma 6.16.11, we see that this diagram is isomorphic to:

$$\begin{tikzcd}
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \arrow[rr] \arrow[d] & & \mathsf{C} \arrow[d] \\
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) \arrow[r] & \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
\end{tikzcd} \tag{6.16.1}$$

where now the two vertical arrows are induced by tensoring appropriately with the pullback $D(S) \to D^!(\mathcal{H})$.

To see that the diagram (6.16.1) commutes, it suffices to show that in the diagram:

$$\begin{tikzcd}
\mathcal{H} \underset{S}{\times} \mathcal{H} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathcal{H} \arrow[d, "\pi"] \\
\mathcal{H} \arrow[r, "\pi"] & S
\end{tikzcd}$$

the natural transformation $p_2^! \pi_{*,ren} \to p_{1,*,ren}\pi^!$ arising from adjunction is an equivalence. To this end, we extend the diagram to:

$$\begin{tikzcd}
\mathcal{H} \underset{S}{\times} \mathcal{H} \arrow[r] \arrow[d, "p_2"] & \mathcal{H} \times \mathcal{H} \arrow[r, "p_1"] \arrow[d, "\mathrm{id}_{\mathcal{H}} \times \pi"] & \mathcal{H} \arrow[d, "\pi"] \\
\mathcal{H} \arrow[r, "\Gamma_\pi"] & \mathcal{H} \times S \arrow[r, "p_2"] & S
\end{tikzcd}$$

where Γ_π indicates the graph of the map π. Now base-change is obvious for the right square, and for the left square it follows from Proposition 16.38.1.

□

6.17. **The unipotent case.** Let S be a finite type base scheme again.

Definition 6.17.1. A *unipotent S-group scheme* is a smooth S-group scheme that has a central filtration by smooth S-group schemes with subquotients forms (in the sense of 6.10) of $\mathbb{G}_a \times S$.

A *prounipotent group S-scheme* is a group S-scheme that is a projective limit of unipotent S-group schemes under smooth surjective group homomorphisms.

A *unipotent group indscheme* over S is a group indscheme over S that is a union of closed subgroup schemes each of which is prounipotent.

Example 6.17.2. Any form $\mathcal{H}$ of a unipotent group $\mathcal{H}^0$ over $\mathrm{Spec}(k)$ is unipotent: indeed, this follows from comparing the lower central series of $\mathcal{H}$ with that of $\mathcal{H}^0$. The group scheme $\mathfrak{J}ets_{X^I}(N)$ is prounipotent. For any form $\mathcal{G}$ of an algebraic group, Ker $\big(\mathfrak{J}ets_{X^I}(\mathcal{G}) \to \mathcal{G}\big)$ is prounipotent. The group indscheme $\mathfrak{J}ets^{mer}_{X^I}(N)$ is a unipotent group indscheme over X^I.

Remark 6.17.3. Obviously unipotent group indschemes are placid.

Let $\mathcal{H}$ be a unipotent group indscheme over S for the remainder of this section.

The key feature for our purposes is the following:

Proposition 6.17.4. *For every* C *acted on by* $\mathcal{H}$*, the functor:*

$$\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}$$

is fully-faithful in $\mathsf{ShvCat}_{/S_{dR}}$.

Proof. By Remark 6.16.7 and Corollary 19.4.5, we reduce to proving this in the case when $\mathcal{H}$ is a prounipotent group scheme over S.

In this case, note that $D(S) \to D^!(\mathcal{H})$ is fully-faithful and admits a fully-faithful right adjoint in $\mathsf{ShvCat}_{/S_{dR}}$. Indeed, under the identification $D^! \simeq D^*$, $f^!$ identifies with $f^{*,dR}$ by Proposition 16.38.1, so the result follows from the contractibility of affine space.

Therefore, for any $\mathsf{D} \in \mathsf{ShvCat}_{/S_{dR}}$, the induced functor:

$$\mathsf{D} \to \mathsf{D} \underset{D(S)}{\otimes} D^!(\mathcal{H})$$

is fully-faithful.

By Lemma 6.16.11, we see that each morphism in the semicosimplicial diagram (underlying the cosimplicial diagram) defining $\mathsf{C}^{\mathcal{H}}$ is fully-faithful. By contractibility of the category of the semisimplex category (i.e., finite totally ordered sets under injections preserving the orders), we deduce the result from Corollary 19.4.5.

∎

6.18. **Semi-infinite Borel.** Let $\mathfrak{J}ets^{mer}_{X^I}(B)^0$ denote the "connected component of the identity,"[21] i.e., the group indscheme over X^I:

$$\mathfrak{J}ets^{mer}_{X^I}(B)^0 := \mathfrak{J}ets^{mer}_{X^I}(B) \underset{\mathfrak{J}ets^{mer}_{X^I}(T)}{\times} \mathfrak{J}ets_{X^I}(T).$$

Remark 6.18.1. Note that $\mathfrak{J}ets^{mer}_{X^I}(B)^0$ is an ind-group scheme: indeed, choose a coordinate t on X and then $\mathfrak{J}ets^{mer}_{X^I}(B)^0$ is the union of the subgroups $\mathrm{Ad}_{-\check{\lambda}(t)}(\mathfrak{J}ets_{X^I}(B))$ for

[21]We remark that this is poor terminology scheme-theoretically: for example, $T(O)$ is not the connected component of the identity of $T(K)$ due to the existence of nilpotents.

$\check{\lambda}$ a dominant coweight, and one readily checks that these subgroups do not depend on the choice of coordinate.

Remark 6.18.2. Varying the finite set I, one sees that $\mathfrak{J}ets^{mer}_{X^I}(B)^0$ is another factorization group scheme. It has a unital structure under correspondences induced by that of $\mathfrak{J}ets^{mer}_{X^I}(B)$.

6.19. **Semi-infinite flag variety.** In this section, we consider $\mathfrak{J}ets^{mer}_{X^I}(G)$ acting on itself through the right action.

We define $D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I})$ as the $\mathfrak{J}ets^{mer}_{X^I}(B)^0$-coinvariants category of $D^!(\mathfrak{J}ets^{mer}_{X^I}(G))$.

We have a tautological functor:

$$\mathfrak{p}^{\frac{\infty}{2}}_{I,*,ren} : D^!(\mathfrak{J}ets^{mer}_{X^I}(G)) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I}).$$

These categories are compatible with restrictions between X^I as $I \in \mathsf{fSet}$ varies by Proposition 16.50.1 and by the base-change results of §16.44. Therefore, we obtain the category $D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{\mathrm{Ran}_X})$, which arises as the global sections on an underlying sheaf of categories $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$ on $\mathrm{Ran}_{X_{dR}}$, equipped with the tautological functor:

$$\mathfrak{p}^{\frac{\infty}{2}}_{\mathrm{Ran}_X,*,ren} : D^!(\mathfrak{J}ets^{mer}_{\mathrm{Ran}_X}(G)) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{\mathrm{Ran}_X}).$$

There is an evident structure of chiral category on $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$ (which we will upgrade to *unital* chiral category in what follows), equipped with the functor $\mathfrak{p}^{\frac{\infty}{2}}_{*,ren} : D^!(\mathfrak{J}ets^{mer}(G)) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$.

Remark 6.19.1. While the semi-infinite flag variety $\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I}$ does not exist as an indscheme, the notation follows the standard convention in the literature to pretend that it does. Then $\mathfrak{p}^{\frac{\infty}{2}}_I$ would be map $\mathfrak{J}ets^{mer}_{X^I}(G) \to \mathfrak{Fl}^{\frac{\infty}{2}}_{X^I}$.

Remark 6.19.2. As discussed in §1.20, we could have chosen to work with invariants instead here. The present choice is more natural for the purposes of §9.

6.20. **Intermediate Grassmannian.** We will need the following intermediate space between the semi-infinite flag variety $\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I}$ and Gr_{G,X^I}.

For each finite set I, let Gr_{G,B,X^I} be the *intermediate Grassmannian* parametrizing a point $x = (x_i)_{i \in I} \in X^I$, a G^{can}-bundle $\mathcal{P}$ on X with a trivialization on $X \backslash x = X \backslash \{x_i\}_{i \in I}$ and a reduction to B^{can} on $\widehat{\mathcal{D}}_x$ (this is understood in families in the usual manner).

Remark 6.20.1. For a closed point $x \in X$ with a trivialization of $\Omega_X^{1/2}|_{\mathcal{D}_x}$ (to eliminate the twist of §6.15), the fiber of $\mathrm{Gr}_{G,B,X}$ over a closed point $x \in X$ is the indscheme (of ind-infinite type) $G(K_x)/B(O_x)$.

We have obvious maps $\mathrm{Gr}_{G,B,X^I} \to \mathrm{Gr}_{G,X^I}$, and by Proposition 6.12.2, Gr_{G,B,X^I} is a placid indscheme. Clearly $I \mapsto \mathrm{Gr}_{G,B,X^I}$ factorizes.

Moreover, the unital structure (in the sense of correspondences) on $(I \mapsto \mathfrak{J}ets^{mer}_{X^I}(G))$ defines a unital structure on $(I \mapsto \mathrm{Gr}_{G,B,X^I})$. For example, the unit map over X^I is given by the correspondence:

$$\begin{tikzcd} & \mathfrak{J}ets_{X^I}(G)/\mathfrak{J}ets_{X^I}(B) \arrow[dl] \arrow[dr] & \\ X^I & & \mathrm{Gr}_{G,B,X^I} . \end{tikzcd} \tag{6.20.1}$$

Therefore, the assignment:

$$I \mapsto D^!(\mathrm{Gr}_{G,B,X^I})$$

defines a unital factorization category $D^!(\mathrm{Gr}_{G,B}) \in \mathsf{Cat}^{ch}_{un}(X_{dR})$ on X_{dR}.

6.21. We can more explicitly express the category $D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I})$ by realizing it as a localization of $D^!(\mathrm{Gr}_{G,B,X^I})$ as follows.

We have a canonical functor:

$$\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren} : D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I})$$

obtained by writing $D^!(\mathrm{Gr}_{G,B,X^I})$ as the $\mathfrak{J}ets_{X^I}(B)$-coinvariants of $D^!(\mathfrak{J}ets_{X^I}^{mer}(G))$ via Proposition 16.48.1.

This is a functor of $D(X^I)$-module categories (i.e., sheaves of categories on X^I_{dR}), and we will show in §6.22 that it is a localization functor as such.

6.22. As in Remark 6.18.1, we can write $\mathfrak{J}ets_{X^I}^{mer}(B)^0$ as a filtered union of subgroup schemes $\mathcal{K}_\alpha$ beginning with $\mathfrak{J}ets_{X^I}(B)$ and such that the subquotients are locally finite-dimensional affine spaces over X^I.

It follows tautologically that:

$$D^!(\mathfrak{Fl}_{X^I}^{\frac{\infty}{2}}) \simeq \operatorname*{colim}_\alpha D^!(\mathfrak{J}ets_{X^I}^{mer}(G))_{\mathcal{K}_\alpha}$$

with the coinvariant category on the right defined as the colimit of the appropriate bar construction.

By Proposition 16.48.1, we have a canonical identification:

$$D^!(\mathfrak{J}ets_{X^I}^{mer}(G))_{\mathfrak{J}ets_{X^I}(B)} \simeq D^!(\mathrm{Gr}_{G,B,X^I})$$

with the equivalence induced by the functor of renormalized de Rham pushforward along $\mathfrak{J}ets_{X^I}^{mer}(G) \to \mathrm{Gr}_{G,B,X^I}$.

We claim that for each of our distinguished subgroups $\mathcal{K}_\alpha$, the functor:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \simeq D^!(\mathfrak{J}ets_{X^I}^{mer}(G))_{\mathfrak{J}ets_{X^I}(B)} \to D^!(\mathfrak{J}ets_{X^I}^{mer}(G))_{\mathcal{K}_\alpha} \tag{6.22.1}$$

admits a fully-faithful left adjoint.

Indeed, there is a canonical indscheme (of ind-infinite type):

$$\mathfrak{J}ets_{X^I}^{mer}(G)/\mathcal{K}_\alpha$$

so that $\mathfrak{J}ets_{X^I}^{mer}(G) \to \mathfrak{J}ets_{X^I}^{mer}(G)/\mathcal{K}_\alpha$ is a $\mathcal{K}_\alpha$-torsor (for $\mathcal{K}_\alpha = \mathfrak{J}ets_{X^I}(B)$, we obtain Gr_{G,B,X^I}).

By Proposition 16.48.1, we have:

$$D^!(\mathfrak{J}ets^{mer}_{X^I}(G))_{\mathcal{K}_\alpha} \simeq D^!(\mathfrak{J}ets^{mer}_{X^I}(G)/\mathcal{K}_\alpha)$$

so that the functor (6.22.1) corresponds to the renormalized pushforward:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathfrak{J}ets^{mer}_{X^I}(G)/\mathcal{K}_\alpha).$$

Then the existence of the left adjoint follows from Proposition 16.59.1: it is computed as the upper-! functor under this dictionary. Moreover, the fact that the fibers of our map are affine spaces implies the fully-faithfulness of this left adjoint.

Passing to the colimit over the groups $\mathcal{K}_\alpha$ and applying Proposition 19.7.3, we obtain that the functor $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren}$ is a localization functor as desired.

Remark 6.22.1. Note that $D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I})$ is *not* a localization of $D^!(\mathfrak{J}ets^{mer}_{X^I}(G))$: the problem is that $B(O)$ admits the non-trivial reductive quotient T.

6.23. **Unitality of the semi-infinite flag variety.** For every finite set I, let $\mathcal{K}_I$ denote the kernel of the functor $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren}$.

For I and J two finite sets, let:

$$\begin{tikzcd} & \mathcal{H}_{G,B,I,J} := \mathcal{H}_{G,B,f} \arrow[dl, "\alpha_{G,B}"'] \arrow[dr, "\beta_{G,B}"] & \\ X^I \times \mathrm{Gr}_{G,B,X^J} & & \mathrm{Gr}_{G,B,X^{I\coprod J}} \end{tikzcd} \tag{6.23.1}$$

denote the associated unit correspondence, where $f : I \hookrightarrow I \coprod J$ is the tautological inclusion.

Lemma 6.23.1. *The unit functor $\beta_{G,B,*,!-dR}\alpha^!_{G,B}$ maps $D(X^I) \otimes \mathcal{K}_J$ to $\mathcal{K}_{I \coprod J}$.*

Proof. Suppose that $\mathcal{F} \in D(X^I) \otimes \mathcal{K}_J$. We need to show that:

$$\mathfrak{p}^{\frac{\infty}{2},int}_{X^I \coprod J,*,ren} \beta_{G,B,*,!-dR} \alpha^!_{G,B}(\mathcal{F}) = 0.$$

Step 1. First, let us show that the left hand side is zero when restricted to $[X^I \coprod X^J]_{disj}$, the locus where the corresponding point in $\mathrm{Ran}_X \times \mathrm{Ran}_X$ lies in $[\mathrm{Ran}_X \times \mathrm{Ran}_X]_{disj}$.

Each of our functors is intertwined by this restriction to this open: indeed, this is obvious for $\beta_{G,B,*,!-dR}$ and $\alpha^!_{G,B}$, and for $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I \coprod J,*,ren}$ this follows by combining the analysis of §6.22 with Proposition 16.59.1.

Then the claim follows because our correspondence restricts to the obvious correspondence:

$$\begin{tikzcd}
& {[\mathfrak{J}ets_{X^I}(G)/\mathfrak{J}ets_{X^I}(B) \times \mathrm{Gr}_{G,B,X^J}]} \arrow[dl] \arrow[dr] & \\
{[X^I \times \mathrm{Gr}_{G,B,X^J}]_{disj}} & & {[\mathrm{Gr}_{G,B,X^I} \times \mathrm{Gr}_{G,B,X^J}]_{disj}}
\end{tikzcd}$$

Here the notation $[-]_{disj}$ everywhere indicates that we restrict to $[X^I \times X^J]_{disj}$. Moreover, the map $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I \coprod J,*,ren}$ restricts to this locus as $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren} \otimes \mathfrak{p}^{\frac{\infty}{2},int}_{X^J,*,ren}$. From here, the claim is obvious.

Step 2. To complete the above analysis, we need the following digression.

Suppose that we are given $I = I_1 \coprod I_2$ and a map $\varepsilon : I_2 \to J$.

We associate to this datum a locally closed subscheme $Z \hookrightarrow X^I \times X^J$, defined as the locus of points $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big)$ such that, for every $i \in I_1, j \in J$, we have $x_i \neq x_j$, and for every $i \in I_2$ we have $x_i = x_{\varepsilon(i)}$. (The scheme-theoretic meaning of $x_i \neq x_j$ for S-points is that the map $(x_i, x_j) : S \to X \times X$ factors through the complement to the diagonal).

For example, if $I_1 = I$, $I_2 = \varnothing$, then $Z = [X^I \times X^J]_{disj}$. In general, Z is isomorphic to $[X^{I_1} \times X^J]_{disj}$, and the map $Z \to X^I \times X^J$ factors as:

$$Z = [X^{I_1} \times X^J]_{disj} \hookrightarrow [X^{I_1} \times (X^{I_2 \coprod J})]_{disj} \hookrightarrow X^I \times X^J \tag{6.23.2}$$

where the first map is the diagonal embedding defined by the surjection $I_2 \coprod J \overset{\varepsilon \times \mathrm{id}_J}{\twoheadrightarrow} J$.

Note that as the data $(I = I_1 \coprod I_2, \varepsilon : I_2 \to J)$ vary, the associated locally closed subschemes cover $X^I \times X^J$. Indeed, given a geometric point $\big((x_i)_{i \in I}, (x_j)_{j \in J}\big) \in X^I \times X^J$, let I_1 be the set of i such that $x_i \neq x_j$ for all $j \in J$, let I_2 be its complement, and define $\varepsilon : I_2 \to J$ by choosing for each $i \in I_2$ some $j \in J$ such that $x_i = x_j$.

We remark that this construction does not form a partition: there is some redundancy.

Step 3. Let $I = I_1 \coprod I_2$, $\varepsilon : I_2 \to J$ and Z be as above.

Using factorization and the composition (6.23.2), we see that the restriction of (6.23.1) to Z is isomorphic to:

$$\begin{tikzcd}[column sep=small] & \mathfrak{J}ets_{X^{I_1}}(G)/\mathfrak{J}ets_{X^{I_1}}(B) \times X^{I_2} \times \mathrm{Gr}_{G,B,X^J} \arrow[dl] \arrow[dr] & \\ X^I \times \mathrm{Gr}_{G,B,X^J} & & \mathrm{Gr}_{G,B,X^{I_1}} \times X^{I_2} \times \mathrm{Gr}_{G,B,X^J} . \end{tikzcd}$$

The same argument as in Step 1 implies that our functors are intertwined by !-restriction to Z in the obvious way. Therefore, we see that $\mathfrak{p}^{\frac{\infty}{2},int}_{X^{I \coprod J},*,ren} \beta_{G,B,*,!-dR} \alpha^!_{G,B}(\mathcal{F})$ has vanishing !-restriction to the locus:

$$\mathrm{Gr}_{G,B,X^{I \coprod J}} \underset{X^{I \coprod J}}{\times} Z.$$

But this suffices, since varying our choice of $I = I_1 \coprod I_2$ and $\varepsilon : I_2 \to J$ we obtain a cover of $X^I \times X^J$ by locally closed subschemes.

□

Therefore, varying I and J, we see that $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$ has a canonical structure of unital sheaf of categories. We will denote the corresponding object of $\mathsf{ShvCat}_{/\mathrm{Ran}^{un}_{X_{dR}}}$ by the same notation.

Lemma 6.23.2. *Let $f : I \to J$ be a surjection of finite sets. Then the functor:*

$$\mathcal{K}_I \underset{D(X^I)}{\otimes} D(X^J) \to \mathcal{K}_J$$

induced by !-restriction is an equivalence.

Proof. Let $\mathcal{K}_{X^I,\alpha} \subseteq \mathfrak{Jets}^{mer}_{X^I}(B)^0$ be a subgroup scheme as in §6.22 (there denoted $\mathcal{K}_\alpha$, where there was only one finite set at play). Let $\mathcal{K}_{X^J,\alpha}$ denote the restriction of $\mathcal{K}_{X^I,\alpha}$ along the closed embedding:

$$\mathrm{Gr}_{G,B,X^J} = \mathrm{Gr}_{G,B,X^I} \underset{X^I}{\times} X^J \hookrightarrow \mathrm{Gr}_{G,B,X^I}\,. \tag{6.23.3}$$

Note that $\mathcal{K}_{X^J,\alpha} \subseteq \mathfrak{Jets}^{mer}_{X^J}(B)^0$ is a subgroup scheme of the same type as considered in §6.22.

Define $\mathcal{K}_{I,\alpha}$ and $\mathcal{K}_{J,\alpha}$ respectively as the kernels of the renormalized pushforward functors:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathfrak{Jets}^{mer}_{X^I}(G)/\mathcal{K}_{X^I,\alpha})$$

$$\text{resp. } D^!(\mathrm{Gr}_{G,B,X^J}) \to D^!(\mathfrak{Jets}^{mer}_{X^J}(G)/\mathcal{K}_{X^J,\alpha}).$$

Because these pushforward functors admit fully-faithful left adjoints, the corresponding functors:

$$\mathcal{K}_{X^I,\alpha} \hookrightarrow D^!(\mathrm{Gr}_{G,B,X^I})$$

$$\mathcal{K}_{X^J,\alpha} \hookrightarrow D^!(\mathrm{Gr}_{G,B,X^J})$$

do as well. Moreover, they are $D(X^I)$-equivariant. Applying this to I, we see that the functor:

$$\mathcal{K}_{I,\alpha} \underset{D(X^I)}{\otimes} D(X^J) \to D^!(\mathrm{Gr}_{G,B,X^I}) \underset{D(X^I)}{\otimes} D(X^J)$$

is fully-faithful as well. By Proposition 16.50.1, the functor:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \underset{D(X^I)}{\otimes} D(X^J) \to D^!(\mathrm{Gr}_{G,B,X^J})$$

is an equivalence, so we see that:

$$\mathcal{K}_{I,\alpha} \underset{D(X^I)}{\otimes} D(X^J) \to \mathcal{K}_{J,\alpha} \tag{6.23.4}$$

is fully-faithful.

Now observe that (6.23.3) is a finitely presented closed embedding (having been obtained by base-change from $X^J \hookrightarrow X^I$), and therefore the !-restriction functor admits a fully-faithful left adjoint of !-dR $*$-pushforward. This left adjoint is a morphism of $D(X^I)$-module categories by Remark 16.15.5. Moreover, by Proposition 16.39.1, we see that this !-dR $*$-pushforward functor coincides with renormalized pushforward up to cohomological shift, and therefore it maps $\mathcal{K}_{J,\alpha}$ to $\mathcal{K}_{I,\alpha}$.

Therefore, we see that (6.23.4) is essentially surjective and therefore an equivalence.

The proof of Proposition 19.7.3 shows that the colimit $\mathrm{colim}_\alpha \mathcal{K}_{I,\alpha}$ considered as a subcategory of $D^!(\mathrm{Gr}_{G,B,X^I})$ coincides with $\mathcal{K}_I$; comparing with the same expression for $\mathcal{K}_J$, we obtain the result.

□

Therefore, we see that the conditions of §13.5 are satisfied, so that $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$ obtains a canonical structure of unital chiral category. As such, it is equipped with the canonical strictly unital functor:

$$\mathfrak{p}^{\frac{\infty}{2},int}_{*,ren} : D^!(\mathrm{Gr}_{G,B}) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}) \in \mathsf{Cat}^{ch}_{un}(X_{dR}).$$

6.24. Fix a finite set I. Let $i_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{G,B,X^I}$ denote the canonical map induced by the embedding $B \hookrightarrow G$. As in Remark 6.9.3, these maps give a canonical strictly unital chiral functor:

$$i^! : D^!(\mathrm{Gr}_{G,B}) \to D(\mathrm{Gr}_B).$$

Proposition 6.24.1. *There is a unique unital chiral functor:*

$$i^{\frac{\infty}{2},!} : D^!(\mathfrak{Fl}^{\frac{\infty}{2}}) \to D(\mathrm{Gr}_T) \in \mathsf{Cat}^{ch}(X_{dR}). \tag{6.24.1}$$

with an isomorphism:

$$i^{\frac{\infty}{2},!} \circ \mathfrak{p}^{\frac{\infty}{2},int}_{*,ren} \simeq \mathfrak{q}^{loc}_{*,dR} \circ i^! : D^!(\mathrm{Gr}_{G,B}) \to D(\mathrm{Gr}_T) \in \mathsf{Cat}^{ch}_{un}(X_{dR}).$$

The unital functor $i^{\frac{\infty}{2},!}$ is strictly unital.

Proof. By construction of the factorization structure on $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$, it suffices to show that for every finite set I, the kernel of the functor

$$\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren} : D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}_{X^I})$$

is annihilated by the functor $\mathfrak{q}^{loc}_{X^I,*,dR} \circ i^!_{X^I}$. Here $i_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{G,B,X^I}$ is the obvious map.

Let $\mathcal{K}_\alpha$ be a subgroup scheme of $\mathfrak{J}ets^{mer}_{X^I}(B)^0$ as in §6.22. It suffices to show that $i^!_{X^I}$ maps the kernel of the functor (6.22.1) into the kernel of the pushforward functor $\mathfrak{q}^{loc}_{X^I,*,dR}$ for the map $\mathfrak{q}^{loc}_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{T,X^I}$.

As in *loc. cit.*, (6.22.1) may be realized as the renormalized pushforward along the placid morphism:

$$\mathrm{Gr}_{G,B,X^I} \to \mathfrak{J}ets^{mer}_{X^I}(G)/\mathcal{K}_\alpha.$$

Therefore, the result follows by the base-change property of Proposition 16.59.1, as applied to the (Cartesian) square in the diagram:

$$\begin{tikzcd}
\mathrm{Gr}_{B,X^I} \arrow[r] \arrow[d, "i_{X^I}"] & \mathfrak{J}ets^{mer}_{X^I}(B)/\mathcal{K}_\alpha \arrow[r] \arrow[d] & \mathrm{Gr}_{T,X^I} \\
\mathrm{Gr}_{G,B,X^I} \arrow[r] & \mathfrak{J}ets^{mer}_{X^I}(G)/\mathcal{K}_\alpha. &
\end{tikzcd}$$

□

Remark 6.24.2. As in Remark 6.19.1, the notation $i^{\frac{\infty}{2}}$ refers to the would-be embedding:

$$\mathrm{Gr}_T = \mathfrak{J}ets^{mer}(B)/\mathfrak{J}ets^{mer}(B)^0 \hookrightarrow \mathfrak{Fl}^{\frac{\infty}{2}}.$$

6.25. **Whittaker conditions.** The remainder of this section is devoted to imposing the Whittaker condition on $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$, and especially to establishing its structure as a unital chiral category.

6.26. **Whittaker character.** Observe that we have a canonical homomorphism:

$$\mathfrak{J}ets^{mer}_{X^I}(N^-) \to \mathfrak{J}ets^{mer}_{X^I}(N^-/[N^-,N^-]) = \mathfrak{J}ets^{mer}_{X^I}(\oplus_{i\in\mathcal{I}_G}\Omega^1_X) \xrightarrow{\mathrm{Res}} \prod_{i\in\mathcal{I}_G}\mathbb{G}_a \overset{\mathrm{sum}}{\to} \mathbb{G}_a$$

and we let $\overset{!}{\psi}_{X^I} \in D^!(\mathfrak{J}ets^{mer}_{X^I}(N^-))$ denote the induced character D-module on $\mathfrak{J}ets^{mer}_{X^I}(N^-)$ given by !-pulling back the character D-module $\overset{!}{\psi} \in D(\mathbb{G}_a)$. Note that $\overset{!}{\psi}_{X^I}$ canonically descends to an object:

$$\widetilde{\psi}_{X^I} \in D(\mathrm{Gr}_{N^-,X^I}).$$

Let $D(X^I)^\psi$ denote the category $D(X^I)$ considered as a category acted on by $\mathfrak{J}ets^{mer}_{X^I}(N^-))$ via the character D-module ψ^{loc}. Let $D(X^I)^{-\psi}$ denote the same, but with the character D-module $\overset{!}{\psi}_{X^I}$ replaced by its pullback under the inversion map on $\mathfrak{J}ets^{mer}_{X^I}(N^-)$.

6.27. For any category C acted on by $\mathfrak{J}ets^{mer}_{X^I}(N^-)$, we let $\mathsf{Whit}_{X^I}(\mathsf{C}) = \mathsf{Whit}(\mathsf{C})$ denote the (!–) *Whittaker category*:

$$(\mathsf{C} \underset{D(X^I)}{\otimes} D(X^I)^{-\psi})^{\mathfrak{J}ets^{mer}_{X^I}(N^-)}.$$

By unipotence, the functor:

$$\mathsf{Whit}(\mathsf{C}) \to \mathsf{C}$$

is locally fully-faithful.

Example 6.27.1. We have $\widetilde{\psi}_{X^I} \in \mathsf{Whit}(\mathrm{Gr}_{N^-,X^I})$. In fact, the functor $D(X^I) \to \mathsf{Whit}(\mathrm{Gr}_{N^-,X^I})$ given by tensoring with $\widetilde{\psi}_{X^I}$ is an equivalence.

Remark 6.27.2. The category constructed above is sometimes called the !-Whittaker category. It plays the role of Whittaker invariants. There is a dual construction of Whittaker coinvariants sometimes called the *-Whittaker category.

For further discussion of these points, see [Gai10b] and [Ber].

6.28. For each finite set I, define $\mathsf{Whit}^{abs}_{X^I}$ the *absolute Whittaker category over* X^I as $\mathsf{Whit}_{X^I}\big(D^!(\mathfrak{J}ets^{mer}_{X^I}(G))\big)$.

Varying I, we obtain a chiral category:

$$I \mapsto \mathsf{Whit}^{abs}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(\mathfrak{J}ets^{mer}_{X^I}(G))\big)$$

Similarly, we obtain the chiral categories:

$$I \mapsto \mathsf{Whit}^{sph}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(\mathrm{Gr}_{G,X^I})\big)$$

$$I \mapsto \mathsf{Whit}^{int}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(\mathrm{Gr}_{G,B,X^I})\big).$$

6.29. **Unital structures on Whittaker categories.** We now describe the construction of unital chiral category structures on Whittaker categories.

Our key technical tool for this is the following lemma.

Lemma 6.29.1. *Let Z be one of the factorization spaces $\mathfrak{J}ets^{mer}(G)$, Gr_G, or $\mathrm{Gr}_{G,B}$. Then for each pair I, J of finite sets, we have:*

(1) The unit functor:

$$D(X^I) \otimes D^!(Z_{X^J}) \to D^!(Z_{X^{I \coprod J}})$$

admits a $D(X^I) \otimes D(X^J)$-linear right adjoint.

(2) This right adjoint:

$$D^!(Z_{X^{I \coprod J}}) \to D(X^I) \otimes D^!(Z_{X^J})$$

preserves the Whittaker subcategories.

(3) The induced functor:

$$\mathsf{Whit}(D^!(Z_{X^{I \coprod J}})) \to D(X^I) \otimes \mathsf{Whit}(D^!(Z_{X^J}))$$

admits a $D(X^I) \otimes D(X^J)$-linear left adjoint.

We will prove (1) and (2) in §6.30-6.31. The proof of (3) requires the introduction of some new ideas that are orthogonal to our current purposes, so we will delay this part of the argument to §7.

Corollary 6.29.2. *The chiral category Whit^{abs} admits a unique structure of unital chiral category such that $\mathsf{Whit}^{abs} \to D^!(\mathfrak{J}ets^{mer}(G))$ upgrades to a unital chiral functor.*

For I and J two finite sets, the corresponding unit functor:

$$D(X^I) \otimes \mathsf{Whit}^{abs}_{X^J} \to \mathsf{Whit}^{abs}_{X^I \coprod J}$$

is the left adjoint of Lemma 6.29.1 (3)*.*

The same results hold with $\mathfrak{J}ets^{mer}(G)$ replaced by Gr_G *(resp.* $\mathrm{Gr}_{G,B}$*) and* Whit^{abs} *replaced by* Whit^{sph} *(resp.* Whit^{int}*).*

Remark 6.29.3. We emphasize that in Corollary 6.29.2, e.g. the inclusion functor $\mathsf{Whit}^{abs} \to D^!(\mathfrak{J}ets^{mer}(G))$ is lax unital, not strictly unital.

Proof that Lemma 6.29.1 implies Corollary 6.29.2. Lemma 6.29.1 exactly implies that the hypotheses of Proposition 13.4.2 are satisfied, and therefore *loc. cit.* implies the result.

□

6.30. Let $\mathcal{G}$ be as in §6.10 and fix finite sets I and J.

We claim that the corresponding unit map:

$$D(X^I) \otimes D^!(\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})) \to D^!(\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G}))$$

admits a continuous right adjoint, and we claim that this functor is a morphism of $D(X^I \times X^J)$-module categories.

Indeed, form the correspondence, using Notation 6.9.4:

$$\begin{tikzcd} & \mathcal{H}_{\mathcal{G},I,J} \arrow[dl, "\alpha=\alpha_{\mathcal{G}}"'] \arrow[dr, "\beta_{\mathcal{G}}=\beta"] & \\ X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}) & & \mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G}) \end{tikzcd} \tag{6.30.1}$$

with $f : I \hookrightarrow I \coprod J$ the tautological embedding. Then the unit map is computed as $\beta_{*,!-dR} \circ \alpha^!$.

Note that $\mathcal{H}_{\mathcal{G},I,J}$ is placid because $\mathcal{H}_{\mathcal{G},I,J} \to \mathfrak{J}ets^{mer}_{X^{I \coprod J}}(\mathcal{G})$ is a finitely presented closed embedding. We record for future use the observation that $\mathcal{H}_{\mathcal{G},I,J}$ therefore inherits a dimension theory from §16.54.

We immediately see from §16.46 that $\beta_{*,!-dR}$ has right adjoint $\beta^!$.

Lemma 6.30.1. *The map:*

$$\alpha : \mathcal{H}_{\mathcal{G},I,J} \to X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$$

is a placid morphism.[22]

Proof. We will prove this by an explicit construction.

Let $n, m \geqslant -1$ be two fixed integers. Define the indscheme $\mathcal{H}^{n,m}_{\mathcal{G},I,J}$ parametrizing:

$$\left\{ \begin{array}{c} x_I = (x_i)_{i \in I} \in X^I,\, x_J = (x_j)_{j \in J} \in X^J,\, \mathcal{P}_{\mathcal{G}} \text{ a } \mathcal{G}\text{-bundle on } X, \\ \tau \text{ a trivialization of } \mathcal{P}_{\mathcal{G}}|_{X \setminus \{x_j\}_{j \in J}}, \\ \sigma \text{ a trivialization of } \mathcal{P}_{\mathcal{G}} \text{ on } \Gamma^{(n)}_{x_I} \cup \Gamma^{(m)}_{x_J}. \end{array} \right\}$$

Here, we use the natural convention that $\Gamma^{(-1)}_x = \varnothing$ for any $x : S \to X^K$. We emphasize that the symbol $\cup$ here indicates sum of effective divisors.

As in Lemma 6.4.1, as n and m vary, we obtain a projective system under maps that are affine smooth covers. Since for $n = m = -1$, we obtain $X^I \times \mathrm{Gr}_{\mathcal{G},X^J}$, we see that the $\mathcal{H}^{n,m}_{\mathcal{G},I,J}$ actually are indschemes.

By Lemma 6.11.1, we have:

$$\lim_{n,m} \mathcal{H}^{n,m}_{\mathcal{G},I,J} = \mathcal{H}_{\mathcal{G},I,J}$$

$$\lim_{m} \mathcal{H}^{-1,m}_{\mathcal{G},I,J} = X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}).$$

[22]This subsection requires the most subtle use of the notion of placid morphism, so we recall (as in §16.9) that the notion of *placid morphism* is introduced in §16.37 and §16.58, and is something like a pro-smooth morphism. The key point is Proposition 16.59.1, which roughly says that placid morphisms behave like smooth morphisms in this setting, and the implicit dimension shifts in the infinite-dimensional D-module theory make $\alpha^!$ behave like $\alpha^{*,dR}$.

Therefore, taking for $\mathcal{I}$ the filtered category $\mathbb{Z}^{\geqslant -1} \times \mathbb{Z}^{\geqslant -1}$ (with $\mathbb{Z}^{\geqslant -1}$ considered as a category by its ordering), we see that the map α can be written as obtained from the compatible affine smooth covering maps:

$$\lim_{n,m} \mathcal{H}^{n,m}_{\mathcal{G},I,J} \to \lim_{m} \mathcal{H}^{-1,m}_{\mathcal{G},I,J}$$

giving the result.

□

One easily shows that the dimension theories on $\mathcal{H}_{\mathcal{G},I,J}$ coming from α and β respectively coincide. Therefore, by Proposition 16.59.1, $\alpha^!$ admits the right adjoint $\alpha_{*,ren}$.

We record the following feature of $\alpha_{*,ren}$ for future use.

Lemma 6.30.2. *Suppose that $\mathcal{G}$ is a form of a unipotent algebraic group. Then the functor $\alpha^!$ is fully-faithful, i.e., the counit for the adjunction $(\alpha^!, \alpha_{*,ren})$ is an equivalence.*

Proof. We use the same notation as in Lemma 6.30.1.

Unipotence implies that the pullback functors for each of the maps:

$$\mathcal{H}^{n,m}_{\mathcal{G},I,J} \to \mathcal{H}^{n',m'}_{\mathcal{G},I,J}$$

are fully-faithful, since the fibers are fibrations with affine space fibers.

The argument easily follows from here — we form the commutative square:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{G}} \arrow[r, "\alpha"] \arrow[d] & X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}) \arrow[d] \\
\mathcal{H}^{n,m}_{\mathcal{G},I,J} \arrow[r] & \mathcal{H}^{-1,m}_{\mathcal{G},I,J}.
\end{tikzcd}$$

and note that, by definition, it suffices to check that the counit is an equivalence after pushing forward to $\mathcal{H}^{-1,m}_{\mathcal{G},I,J}$ for every m. Moreover, we can check this after applying the counit to objects pulled back from $\mathcal{H}^{-1,m}_{\mathcal{G},I,J}$ (by smoothness of these structure maps). From here the claim is obvious.

$\square$

Variant 6.30.3. We use the notation of (6.23.1) for the unit correspondence for Gr_{G,B,X^J}. Note that in general we have:

$$\mathcal{H}_{G,B,I,J} = \mathcal{H}_{G,I,J}/\mathfrak{J}ets_{X^I \coprod J}(B).$$

As above, the unit functor $\beta_{G,B,*,!-dR} \circ \alpha^!_{G,B}$ admits the right adjoint $\alpha_{G,B,*,ren} \circ \beta^!_{G,B}$.

We also note that the corresponding statement for Gr_G is true and vacuous.

6.31. In the setting of §6.30 with $\mathcal{G}$ our twisted form of G, we claim that the functor $\alpha_{G,*,ren}\beta^!_G$ preserves the corresponding Whittaker equivariant subcategories on each side.

In the diagram:

$$\begin{tikzcd}
 & \mathcal{H}_{N^-,I,J} := \mathcal{H}_{N^-,f} \arrow[dl, "\alpha_{N^-}"'] \arrow[dr, "\beta_{N^-}"] & \\
X^I \times \mathfrak{J}ets^{mer}_{X^J}(N^-) & & \mathfrak{J}ets^{mer}_{X^I \coprod J}(N^-)
\end{tikzcd}$$

the two corresponding character D-modules on $\mathcal{H}_{N^-,I,J}$ obtained by pullback from α or β obviously coincide.

Therefore, we can make sense of the Whittaker category of $D^!(\mathcal{H}_{G,I,J})$. Moreover, $\beta^!_G$ obviously preserve Whittaker categories. Therefore, it suffices to show that $\alpha_{*,ren}$ preserves these Whittaker equivariant categories.

We begin by showing that $\alpha_{G,*,ren}$ maps the $\mathfrak{J}ets_{X^I \coprod J}(N^-)$-equivariant category of $D^!(\mathcal{H}_{N^-,I,J})$ to the $\mathfrak{J}ets_{X^J}(N^-)$-equivariant (i.e., $X^I \times \mathfrak{J}ets_{X^J}(N^-)$-equivariant) category of $D^!(X^I \times \mathfrak{J}ets^{mer}_{X^J}(G))$.

We have the diagram:

$$\begin{tikzcd}
\mathfrak{J}ets_{X^I \coprod J}(N^-) \underset{X^I \times X^J}{\times} \mathcal{H}_{G,I,J} \arrow[r, "\mathrm{act}"] \arrow[d, "\alpha'_G"] & \mathcal{H}_{G,I,J} \arrow[d, "\alpha_G"] \\
X^I \times \mathfrak{J}ets_{X^J}(N^-) \underset{X^I \times X^J}{\times} X^I \times \mathfrak{J}ets^{mer}_{X^J}(G) \arrow[r, "\mathrm{act}"] & X^I \times \mathfrak{J}ets^{mer}_{X^J}(G).
\end{tikzcd} \tag{6.31.1}$$

Noting that the horizontal maps are placid, we claim:

Lemma 6.31.1. *The base-change map:*

$$\mathrm{act}^! \, \alpha_{G,*,ren} \to \alpha'_{G,*,ren} \, \mathrm{act}^!$$

is an equivalence.

Proof. The diagram (6.31.1) is isomorphic in the usual way to:

$$\begin{tikzcd}
\mathfrak{J}ets_{X^I \coprod J}(N^-) \underset{X^I \times X^J}{\times} \mathcal{H}_{G,I,J} \arrow[r, "p_2"] \arrow[d, "\alpha'_G"] & \mathcal{H}_{G,I,J} \arrow[d, "\alpha_G"] \\
X^I \times \mathfrak{J}ets_{X^J}(N^-) \underset{X^I \times X^J}{\times} X^I \times \mathfrak{J}ets^{mer}_{X^J}(G) \arrow[r, "p_2"] & X^I \times \mathfrak{J}ets^{mer}_{X^J}(G).
\end{tikzcd}$$

Therefore, it suffices to see that the base-change map is an isomorphism for this diagram.

We enlarge this diagram to:

$$\begin{tikzcd}
\mathfrak{J}ets_{X^I \coprod J}(N^-) \underset{X^I \times X^J}{\times} \mathcal{H}_{G,I,J} \arrow[r, "\Delta"] \arrow[d, "\alpha'"] & \mathfrak{J}ets_{X^I \coprod J}(N^-) \times \mathcal{H}_{G,I,J} \arrow[r, "p_2"] \arrow[d, "\alpha_{N^-} \times \alpha_G"] & \mathcal{H}_{G} \arrow[d] \\
X^I \times \mathfrak{J}ets_{X^J}(N^-) \underset{X^I \times X^J}{\times} X^I \times \mathfrak{J}ets^{mer}_{X^J}(G) \arrow[r, "\Delta"] & X^I \times \mathfrak{J}ets_{X^J}(N^-) \times X^I \times \mathfrak{J}ets^{mer}_{X^J}(G) \arrow[r, "p_2"] & X^I \times \mathfrak{J}et
\end{tikzcd}$$

where we have abused notation in several ways, not least of all that α_{N^-} denotes the restriction of α_{N^-} to $\mathfrak{J}ets_{X^I \coprod J}(N^-)$. It suffices to show the base-change property for each of these squares separately.

For the left square above, note that this square is Cartesian, and that the maps Δ are finitely presented because $X^I \times X^J$ is finite type. Therefore, Proposition 16.59.1 implies the base-change property.

For the right square, the result follows immediately from Lemma 6.30.2.

$\square$

From the lemma and Lemma 6.30.2, it is obvious that $\alpha_{G,*,ren}$ maps the $\mathfrak{J}ets_{X^I \coprod J}(N^-)$-equivariant category of $D^!(\mathcal{H}_{N^-,I,J})$ to the $\mathfrak{J}ets_{X^J}(N^-)$-equivariant (i.e., $X^I \times \mathfrak{J}ets_{X^J}(N^-)$-equivariant) category of $D^!(X^I \times \mathfrak{J}ets_{X^J}^{mer}(G))$.

The same argument as above applies verbatim to larger congruence subgroups with (or just as well, without) the twist by the Whittaker character (which restricts to $\mathfrak{J}ets(N^-)$ as the trivial character). Exhausting $\mathfrak{J}ets_{X^I \coprod J}^{mer}(N^-)$ by these compact open subgroups, we obtain the result.

Variant 6.31.2. As in Variant 6.30.3, the right adjoints to the unit functors for $\mathrm{Gr}_{G,B}$ and Gr_G also preserve the Whittaker subcategories.

6.32. As was mentioned in §6.29, we now postpone the proof of the third condition from *loc. cit.* to §7, assuming it (and therefore Corollary 6.29.2) for the remainder of this section.

6.33. Let I be a finite set. Define $\mathsf{Whit}_{X^I}^{\frac{\infty}{2}} \in \mathsf{ShvCat}_{/X^I_{dR}}$ as the $\mathfrak{J}ets_{X^I}^{mer}(B)^0$-coinvariants of $\mathsf{Whit}_{X^I}^{abs}$. Varying I, we obtain a chiral category $\mathsf{Whit}^{\frac{\infty}{2}} \in \mathsf{Cat}^{ch}(X_{dR})$.[23]

The lemmas of §6.23 apply verbatim, and therefore $\mathsf{Whit}^{\frac{\infty}{2}}$ inherits a unital chiral category structure. The tautological functor:

[23]It is natural to ask if formation of these coinvariants commute with the formation of the Whittaker invariants. Over a point, this is true by §17, and for $G = GL_n$ it follows from work in progress by Beraldo, extending his results [Ber] to the factorization setting.

$$\mathfrak{p}_{*,ren}^{\frac{\infty}{2},int} : \mathsf{Whit}^{int} \to \mathsf{Whit}^{\frac{\infty}{2}}$$

is again strictly unital.

Moreover, we have an obvious lax unital chiral functor:

$$\mathsf{Whit}^{\frac{\infty}{2}} \to D^!(\mathfrak{Fl}^{\frac{\infty}{2}}). \tag{6.33.1}$$

6.34. The results of this section may be summarized as follows:

We have a diagram:

$$\mathrm{Gr}_G \longleftarrow \mathfrak{J}ets^{mer}(G) \longrightarrow \mathrm{Gr}_{G,B}^{int} \longrightarrow \mathfrak{Fl}^{\frac{\infty}{2}}$$

where subscripts have been removed and the right map is a fiction in the style of Remark 6.19.1. This induces a diagram:

$$\begin{tikzcd} \mathsf{Whit}^{sph} \arrow[r] \arrow[d] & \mathsf{Whit}^{abs} \arrow[r] \arrow[d] & \mathsf{Whit}^{int} \arrow[r] \arrow[d] & \mathsf{Whit}^{\frac{\infty}{2}} \arrow[d] \\ D(\mathrm{Gr}_G) \arrow[r] & D^!(\mathfrak{J}ets^{mer}(G)) \arrow[r] & D^!(\mathrm{Gr}_{G,B}^{int}) \arrow[r] & D^!(\mathfrak{Fl}^{\frac{\infty}{2}}) \end{tikzcd}$$

of unital chiral categories. Here all functors are (lax) unital chiral functors defined appropriately as !-pullback or renormalized pushforward, and the the two horizontal lines consist of strictly unital chiral functors.

7. Fusion with the Whittaker sheaf (a technical point)

7.1. This purpose of this section is to the complete the proof of Lemma 6.29.1 by proving (3) of *loc. cit.* The proof of the proposition is given by combining a *fusion* construction with some well-known facts about Drinfeld's compactification of Gr_{N^-}.

7.2. Before proceeding, we begin with a somewhat informal description of the method in the case when I and J are singleton sets, and say for definiteness that $Z = \mathfrak{J}ets^{mer}(G)$. We will use e.g. the notation:

$$G(K) \times G(K) \rightsquigarrow G(K)$$

for the space $\mathfrak{J}ets^{mer}_{X^2}(G)$, where this should be read as describing a factorization space that is $G(K_x) \times G(K_y)$ away from the diagonal specializing to $G(K_x)$ over the diagonal.

Suppose that $\mathcal{F} \in \mathsf{Whit}^{abs}_X := \mathsf{Whit}(\mathfrak{J}ets^{mer}_X(G))$. We are supposed to show e.g. that we can !-average the induced object:

$$\delta_{\mathfrak{J}ets_X(G)} \boxtimes \mathcal{F} \rightsquigarrow \mathcal{F}$$

with respect to the Whittaker character (here $\delta_{\mathfrak{J}ets_X(G)}$ is the δ D-module on meromorphic jets supported on regular jets).[24]

We construct a space:

$$\mathrm{Gr}_{N^-} \times G(K) \rightsquigarrow G(K)$$

encoding the action of $N^-(K)$ on $G(K)$. Moreover, we show that given $\mathcal{F} \in \mathsf{Whit}(\mathfrak{J}ets^{mer}_X(G))$, we can form an object:

$$\overset{!}{\psi}_X \boxtimes \mathcal{F} \rightsquigarrow \mathcal{F} \tag{7.2.1}$$

encoding the Whittaker equivariance of $\mathcal{F}$. These constructions we refer to as *fusion*.

We moreover have a space:

$$\mathrm{Gr}_G \times G(K) \rightsquigarrow G(K)$$

[24]We note that the required task appears completely obvious in the given notation, due to the holonomicity of $\delta_{\mathfrak{J}ets_X(G)}$. However, this ignores the important "interaction" occurring over the diagonal, preventing such a naive argument from going through.

encoding the action of $G(K)$ on itself. Moreover, the $*$-extension of (7.2.1) to this locus coincides with the !-extension. Indeed, it suffices to see this over the closure of $\big((\mathrm{Gr}_{N^-} \times G(K) \rightsquigarrow G(K)\big)$, and here it follows from the usual considerations of the Whittaker character of $N^-(K)$.

We then show that the pullback to $\big(G(K) \times G(K) \rightsquigarrow G(K)\big)$ of this D-module computes the desired left adjoint.

7.3. We begin by studying the semi-infinite orbits of Gr_G in the factorization setting. Fix a finite set I and $\check{\lambda} = (\check{\lambda}_i)$ a collection of coweights for G defined for each $i \in I$.

Observe that there is a canonical section:

$$X^I \to \mathrm{Gr}_{T,X^I}$$

associated to $\check{\lambda}$. Indeed, it suffices to define a relative Cartier divisor valued in $\check{\Lambda}$ on the relative curve $X \times X^I \to X^I$, and we take $\sum_i \check{\lambda}_i \cdot [x_i]$, where $x_{i_0} : X^I \to X \times X^I$ is the section defined by:

$$(x_i)_{i \in I} \mapsto \Big(x_{i_0}, (x_i)_{i \in I}\Big)$$

and $[x_i]$ is the associated effective Cartier divisor.

Note that every geometric point of Gr_{T,X^I} is in the image of one of these sections for appropriate choice of $\check{\lambda}$.

7.4. We define $\mathrm{Gr}_{B,X^I}^{\check{\lambda}}$ as the fiber product:

$$\mathrm{Gr}_{B,X^I}^{\check{\lambda}} := \mathrm{Gr}_{B,X^I} \underset{\mathrm{Gr}_{T,X^I}}{\times} X^I$$

where the map $X^I \to \mathrm{Gr}_{T,X^I}$ is the section defined by $\check{\lambda}$.

Example 7.4.1. Suppose that $I = \{1, 2\}$. Then the fiber of Gr_{B,X^2} over $(x, y) \in X^2$ is $\mathrm{Gr}_{B,x}^{\check{\lambda}_1} \times \mathrm{Gr}_{B,y}^{\check{\lambda}_2}$ for $x \neq y$, and is $\mathrm{Gr}_{B,x}^{\check{\lambda}_1+\check{\lambda}_2}$ for $x = y$.

7.5. We give a variant of $\mathrm{Gr}_B^{\check{\lambda}}$ with $\overline{\mathrm{Gr}}_B$ replacing Gr_B.

First, note that we can define $\overline{\mathrm{Gr}}_{B,X^I}$ to parametrize points $x = (x_i)_{i\in I}$ in X^I, a G-bundle on X with a Drinfeld reduction to B, and a trivialization of this data away from $\{x_i\}_{i\in I}$, incorporating twists by $\mathcal{P}_T^{can}$ in the obvious way.

Remark 7.5.1. One easily finds that $\mathrm{Gr}_{B,X^I} \to \overline{\mathrm{Gr}}_{B,X^I}$ is a Zariski open embedding (in particular, schematic).

It is easy to see that the morphism:

$$\overline{\mathrm{Gr}}_{B,X^I} \to \mathrm{Gr}_{G,X^I} \underset{X^I}{\times} \mathrm{Gr}_{T,X^I}$$

is an ind-closed embedding, and in particular, that $\overline{\mathrm{Gr}}_{B,X^I}$ is an ind-proper indscheme.

We then define $\overline{\mathrm{Gr}}_{B,X^I}^{\check{\lambda}}$ using the map $\overline{\mathrm{Gr}}_{B,X^I} \to \mathrm{Gr}_{T,X^I}$, as with $\mathrm{Gr}_{B,X^I}^{\check{\lambda}}$. Note that $\overline{\mathrm{Gr}}_{B,X^I}^{\check{\lambda}} \to \mathrm{Gr}_{G,X^I}$ is an ind-closed embedding.

In the special case $\check{\lambda} = 0$ (i.e., each $\check{\lambda}_i = 0$), we use the notation $\overline{\mathrm{Gr}}_{N,X^I}$ for $\overline{\mathrm{Gr}}_{B,X^I}^{0}$.

7.6. We have similarly spaces $\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}}$, $\overline{\mathrm{Gr}}_{B^-,X^I}^{\check{\lambda}}$, and $\overline{\mathrm{Gr}}_{N^-,X^I}$ defined again as fiber products with the section $X^I \to \mathrm{Gr}_{T,X^I}$ defined by $\check{\lambda}$, via the natural map e.g. $\mathrm{Gr}_{B^-,X^I} \to \mathrm{Gr}_{T,X^I}$.

Observe that $\mathfrak{J}ets_{X^I}^{mer}(N^-)$ acts on $\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}}$ and $\overline{\mathrm{Gr}}_{B^-,X^I}^{\check{\lambda}}$ for each $\check{\lambda}$.

By the usual conductor considerations, one finds:

$$\mathsf{Whit}(D(\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}})) = 0$$

when $-\check{\lambda}$ is not a dominant coweight.

Let $\jmath_{N^-,X^I}$ denote the open embedding $\mathrm{Gr}_{N^-,X^I} \hookrightarrow \overline{\mathrm{Gr}}_{N^-,X^I}$. As in Example 6.27.1, we have:

$$\jmath_{N^-,X^I,*,dR}(\widetilde{\psi}_{X^I}) \in \mathsf{Whit}(D(\overline{\mathrm{Gr}}_{N^-,X^I}))$$

and the above remarks imply that the induced functor:

$$D(X^I) \to \mathsf{Whit}(D(\overline{\mathrm{Gr}}_{N^-,X^I})) \tag{7.6.1}$$

given by tensoring with this object is an equivalence.

Variant 7.6.1. The above considerations also apply to describe the Whittaker coinvariants of $D(\overline{\mathrm{Gr}}_{N^-,X^I})$. Here one finds that the functor:

$$D(\overline{\mathrm{Gr}}_{N^-,X^I}) \to D(X^I)$$

given by !-restriction to Gr_{N^-,X^I} followed by twisting by the character $\widetilde{\psi}_{X^I}$ and then applying de Rham pushforward to X^I is an equivalence after applying Whittaker coinvariants. Indeed, this again follows by analysis of strata.

7.7. **From actions to fusion.** Fix $\mathcal{G}$ over X a form of an affine algebraic group and I and J two finite sets. Suppose that Z is an indscheme over X^J with an action of $\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$.

Under certain hypotheses, we will construct a new indscheme $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$ that lives over $X^{I\coprod J}$, and that over the disjoint locus of the base is isomorphic to the restriction of $\mathrm{Gr}_{\mathcal{G},X^I} \times Z$. The construction is inspired by [Gai01].

Recall the space $\mathcal{H}_{\mathcal{G},I,J}$ from §6.9 (see Notation 6.9.4 in particular). We have a morphisms:

$$\begin{tikzcd} & \mathcal{H}_{\mathcal{G},I,J} \arrow[dl] \arrow[dr] & \\ \mathfrak{J}ets^{mer}_{X^{I\coprod J}}(\mathcal{G}) & & X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}) \end{tikzcd} \tag{7.7.1}$$

between placid group indschemes over $X^{I\coprod J}$. In particular, $\mathcal{H}_{\mathcal{G},I,J}$ acts on $X^I \times Z$, using the action of $\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$ on Z and the right leg of (7.7.1). We consider $\mathcal{H}_{\mathcal{G},I,J}$ acting on the right on $\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})$ via the left leg of (7.7.1). We obtain the diagonal action of $\mathcal{H}_{\mathcal{G},I,J}$ on:

$$\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G}) \underset{X^I \coprod J}{\times} X^I \times Z. \tag{7.7.2}$$

Definition 7.7.1. We say that the action of $\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$ on Z is *fusive* if the quotient of (7.7.2) by the action of $\mathcal{H}_{\mathcal{G},I,J}$ exists as an indscheme for each I.

When the action is fusive, we let $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$ denote the corresponding quotient; see Remark 7.7.5 for a description of what the resulting space looks like.

Note that there is a canonical action of $\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})$ on $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$ arising from the action of $\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})$ on (7.7.2) through its action of the left on the first factor of *loc. cit.*

Example 7.7.2. Suppose that $Z = \mathrm{Gr}_{\mathcal{G},X^J}$, equipped with the usual action. This action is fusive: one easily finds that the desired quotient is $\mathrm{Gr}_{\mathcal{G},X^I \coprod J}$, where the structure map:

$$\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G}) \underset{X^I \coprod J}{\times} (X^I \times \mathrm{Gr}_{\mathcal{G},X^J}) \to \mathrm{Gr}_{\mathcal{G},X^I \coprod J}$$

is defined by the action of $\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})$ on $\mathrm{Gr}_{\mathcal{G},X^I \coprod J}$ and the unit map $X^I \coprod \mathrm{Gr}_{\mathcal{G},X^J} \to \mathrm{Gr}_{\mathcal{G},X^I \coprod J}$.

Counterexample 7.7.3. The trivial action of $\mathcal{G}$ (i.e., its action as a group scheme over X on X itself) is *not* fusive.

Example 7.7.4. Suppose that $Z = \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$, equipped with the left action. This action is again fusive: in this case, the desired quotient $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G}))$ is the moduli of points $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^{I \coprod J}$, a $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ on X trivialized away from the points $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big)$, and with an additional trivialization on the formal neighborhood of the points $(x_j)_{j\in J}$. One shows that this moduli is a placid indscheme in the usual way, using the increasing infinitesimal neighborhoods of the points x_j.

We have an obvious map $X^I \times \mathfrak{Jets}^{mer}_{X^J}(\mathcal{G}) \to \mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G}))$, realizing the latter as the locus where the $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ is instead trivialized on the complement to the points $(x_j)_{j \in J}$. There is also an obvious action of $\mathfrak{Jets}^{mer}_{X^I \coprod J}(\mathcal{G})$ on $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G}))$, essentially coming from the action of jets on the affine Grassmannian. Therefore, as in Example 7.7.2, we obtain the structure map:

$$\mathfrak{Jets}^{mer}_{X^I \coprod J}(\mathcal{G}) \underset{X^I \coprod J}{\times} (X^I \times \mathfrak{Jets}^{mer}_{X^J}(\mathcal{G})) \to \mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G}))$$

by combining these two observations.

Remark 7.7.5. It is instructive to analyze the space $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$ in the combinatorially simplest case, in which $I = J = *$. In this case, away from the diagonal of X^2, we have $\mathcal{H}_{\mathcal{G},I,J} \simeq \mathfrak{Jets}_X(\mathcal{G}) \times \mathfrak{Jets}^{mer}_X(\mathcal{G})$, while over the diagonal it is isomorphic to $\mathfrak{Jets}^{mer}_X(\mathcal{G})$. Therefore, we have:

$$\mathfrak{Fus}^{\mathcal{G}}_{*,*}(Z)|_{X^2 \setminus \Delta} \simeq \mathfrak{Jets}^{mer}_X(\mathcal{G}) \times \mathfrak{Jets}^{mer}_X(\mathcal{G}) \overset{\mathfrak{Jets}_X(\mathcal{G}) \times \mathfrak{Jets}^{mer}_X(\mathcal{G})}{\underset{X^2}{\times}} Z|_{X^2 \setminus \Delta} \overset{p_1 \times \mathrm{act}}{\simeq} \mathrm{Gr}_{G,X} \times Z|_{X^2 \setminus \Delta}$$

$$\mathfrak{Fus}^{\mathcal{G}}_{*,*}(Z)|_{\Delta} \simeq \mathfrak{Jets}^{mer}_X(\mathcal{G}) \overset{\mathfrak{Jets}^{mer}_X(\mathcal{G})}{\underset{X}{\times}} Z \overset{\mathrm{act}}{\simeq} Z.$$

Here the superscript of a group over a Cartesian product indicates that we take the quotient by the appropriate diagonal action.

7.8. **Fusion of sheaves.** Suppose in the setting of §7.7 that $\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G})$ acts fusively on $Z \to X^J$. Suppose moreover that $\mathcal{F}$ is a $\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G})$-equivariant D-module on Z, i.e., $\mathcal{F}$ is an object of the equivariant category:

$$D^!(Z)^{\mathfrak{Jets}^{mer}_{X^J}(\mathcal{G})}.$$

We obtain a new D-module:

$$\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F}) \in D^!(\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathfrak{Jets}^{mer}_{X^I \coprod J}(\mathcal{G})} \tag{7.8.1}$$

by the following construction:

Note that:

$$\omega_{X^I} \boxtimes \mathcal{F} \in D^!(X^I \times Z) \tag{7.8.2}$$

is $X^I \times \mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$-equivariant (i.e., equipped with an equivariant structure), and therefore equivariant for $\mathcal{H}_{\mathcal{G},I,J}$ acting through the right leg of (7.7.1). Pulling back (7.8.2) along the map:

$$\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G}) \underset{X^I \coprod J}{\times} \left(X^I \times Z\right) \to X^I \times Z$$

we obtain a D-module equivariant for the diagonal action of $\mathcal{H}_{\mathcal{G},I,J}$ considered in §7.7, and for the left action of $\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})$ on the first factor of this space.

Descending to $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$ via the first of these equivariance observations, and appealing to the second, we obtain (7.8.1) as desired.

Example 7.8.1. In the setting of Remark 7.7.5, the D-module $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F})$ is isomorphic to $\omega_{\mathrm{Gr}_{\mathcal{G},X}} \boxtimes \mathcal{F}$ away from the diagonal, and isomorphic to $\mathcal{F}$ over the diagonal.

Variant 7.8.2. Given $\widetilde{\mathcal{F}} \in D(X^I) \otimes D^!(Z)^{\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})}$, we claim that we can generalize the above construction to produce:

$$\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\widetilde{\mathcal{F}}) \in D^!(\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})}.$$

in such a way in the case $\widetilde{\mathcal{F}} = \omega_{X^I} \boxtimes \mathcal{F}$, we recover our earlier construction of $\mathfrak{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F})$.

Indeed, we simply replace $\omega_{X^I} \boxtimes \mathcal{F}$ in (7.8.2) by $\widetilde{\mathcal{F}}$.

Observe that this new construction is $D(X^I) \otimes D(X^J)$-linear.

Remark 7.8.3. We can reformulate this construction in the following way. The map:

$$X^I \times Z \to \mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)$$

induces a restriction functor:

$$D^!(\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathfrak{J}ets^{mer}_{X^I \coprod J}(\mathcal{G})} \to D^!(X^I \times Z)^{\mathcal{H}_{\mathcal{G},I,J}}$$

that is an equivalence (c.f. Proposition 16.48.1) with inverse $\mathfrak{Fus}$.

Remark 7.8.4. The above construction can be performed more generally on any sheaf of categories on X^J_{dR} acted on by $\mathfrak{J}ets^{mer}_{X^J}(\mathcal{G})$.

7.9. **Compactification.** Suppose now that $\mathcal{G}$ is our preferred form of our reductive group G and that $Z \to X^J$ is acted on fusively by G.

We have a canonical map:

$$\mathfrak{Fus}^{N^-}_{I,J}(Z) \hookrightarrow \mathfrak{Fus}^{G}_{I,J}(Z).$$

We will presently use Drinfeld's method to construct $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)$, a "compactification" of this map.

Example 7.9.1. We begin by explicitly treating the case of $Z = \mathrm{Gr}_{G,X^J}$ from Example 7.7.2.

In this case, we define $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$ as the moduli of $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^{I \coprod J}$, a G-bundle $\mathcal{P}$ on X with a polar Drinfeld reduction to N^- (in the $\mathcal{P}^{can}_T$-twisted sense), the poles being at the points x_j, and a trivialization of this datum on $X \backslash \{x_i, x_j\}_{i\in I, j\in J}$. Here a *polar Drinfeld reduction* of the specified type means that we give a Drinfeld reduction defined on the complement to the union of the graphs of the points x_j.

Remark 7.9.2. As in Remark 7.7.5, it is instructive to see what happens when $I = J = *$. In this case, one easily finds:

$$\mathfrak{Fus}^{N^-}_{*,*}(\mathrm{Gr}_{G,X})|_{X^2\backslash\Delta} \simeq \mathrm{Gr}_{N^-,X} \times \mathrm{Gr}_{G,X}\,|_{X^2\backslash\Delta}$$

$$\mathfrak{Fus}^{N^-}_{*,*}(\mathrm{Gr}_{G,X})|_{\Delta} \simeq \mathrm{Gr}_{G,X}$$

It is easy to see that the tautological map $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) \to \mathrm{Gr}_{G,X^{I\coprod J}}$ is an ind-closed embedding, and the natural map:

$$\mathfrak{Fus}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) \to \overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$$

is an ind-open embedding.

Remark 7.9.3. Recall from [FGV01] that for X a proper curve, the moduli space of a point of $x = (x_j) \in X^J$ and G-bundle on X with a polar Drinfeld reduction to N^- defined away from the points x_j is an ind-algebraic stack $\overline{\mathrm{Bun}}^{pol}_{N^-,X^J}$ locally of finite type (proof: bound the order of the poles allowed). Then $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$ may be computed as the fiber product:

$$\begin{tikzcd}
\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) \arrow[r] \arrow[d] & \mathrm{Gr}_{G,X^{I\coprod J}} \arrow[d] \\
X^I \times \mathrm{Bun}^{pol}_{N^-,X^J} \arrow[r] & \mathrm{Bun}^{pol}_{N^-,X^{I\coprod J}}.
\end{tikzcd}$$

Before giving $\overline{\mathfrak{Fus}}^{N^-}$ in the general case, we need to observe the existence of a certain group action.

Construction 7.9.4. Recall from §6.12 that $\pi_{G,X^{I\coprod J}}$ denotes the structure map $\mathfrak{J}ets^{mer}_{X^{I\coprod J}}(G) \to \mathrm{Gr}_{G,X^{I\coprod J}}$. We will construct an action of $\mathcal{H}_{G,I,J}$ on $\pi^{-1}_{G,X^{I\coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}))$ (the action is on the right, so to speak).

Indeed, we have:

$$\pi^{-1}_{G,X^{I\coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) = \left\{ \begin{array}{l} x = \big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^{I\coprod J}, \\ \text{a } G\text{-bundle } \mathcal{P}_G \text{ on } X \text{ with a} \\ \quad \mathcal{P}_T^{can}\text{-twisted Drinfeld reduction to } N^- \text{ on } X\backslash\{x_j\}, \\ \text{a trivialization of this datum on } X\backslash\{x_i, x_j\}_{i\in I, j\in J}, \\ \text{and a trivialization of } \mathcal{P}_G \text{ on } \widehat{\mathcal{D}}_x. \end{array} \right\}$$

and Beauville-Laszlo allows us to rewrite this as:

$$\left\{ \begin{array}{l} x = \big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^{I\coprod J}, \\ \text{a } \mathcal{P}_T^{can}\text{-twisted map } \delta : \mathcal{D}_x\backslash\Big(\cup_{j\in J}\Gamma_{x_j}\Big) \to \overline{G/N}, \\ \text{and a lift of } \delta|_{\overset{o}{\mathcal{D}}_x} \text{ to a map } \overset{o}{\mathcal{D}}_x \to G. \end{array} \right\}.$$

The action of:

$$\mathcal{H}_{G,I,J} = \{x = \big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^{I\coprod J}, \mathcal{D}_x\backslash\Big(\cup_{j\in J}\Gamma_{x_j}\Big) \to G\}$$

on this space is now clear: it arises from the G-equivariant map $G \to \overline{G/N}/T$.

Construction 7.9.5. We are now equipped to define $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)$.

We take it to be the quotient of:

$$\pi^{-1}_{G,X^{I\coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I\coprod J}}{\times} X^I \times Z. \tag{7.9.1}$$

by the diagonal action of $\mathcal{H}_{G,I,J}$. Note that $\mathfrak{J}ets^{mer}_{X^{I\coprod J}}(N^-)$ acts $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)$ through its left action on $\pi^{-1}_{G,X^{I\coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}))$.

Remark 7.9.6. The quotient of:

$$\pi^{-1}_{G,X^{I\coprod J}}(\mathfrak{Fus}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I\coprod J}}{\times} X^I \times Z$$

by $\mathcal{H}_{G,I,J}$ is obviously isomorphic to the quotient of:

$$\mathfrak{Jets}^{mer}_{X^I \coprod J}(N^-) \underset{X^{I \coprod J}}{\times} X^I \times Z$$

by $\mathcal{H}_{N^-,I,J}$.

Lemma 7.9.7. *The restriction functor:*

$$\mathsf{Whit}_{X^{I \coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)) \to \mathsf{Whit}_{X^{I \coprod J}}(\mathfrak{Fus}^{N^-}_{I,J}(Z)) \tag{7.9.2}$$

is an equivalence.

Proof. Note that the map:

$$\pi^{-1}_{G,X^{I \coprod J}}(\mathfrak{Fus}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I \coprod J}}{\times} X^I \times Z \hookrightarrow \pi^{-1}_{G,X^{I \coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I \coprod J}}{\times} X^I \times Z$$

is an open embedding of ind-finite type.

Therefore, the functor (7.9.2) admits a right adjoint in $\mathsf{ShvCat}_{/X^{I \coprod J}_{dR}}$ given by $(*, dR)$-extension. It suffices to check that the unit of the adjunction is an equivalence, and we can check this after restriction using a covering of $X^I \times X^J$ as in the proof of Lemma 6.23.1. Now the result follows because (7.6.1) is an equivalence.

∎

7.10. Suppose that Z is an indscheme over X^J acted on fusively by $\mathfrak{Jets}^{mer}_{X^J}(G)$, and let $\widetilde{\mathcal{F}}$ be an object of $D(X^I) \otimes \mathsf{Whit}(D^!(Z))$. Twisting and untwisting by the character ψ and applying Variant 7.8.2, we form $\widetilde{\mathfrak{Fus}}^{N^-}_{I,J}(\widetilde{\mathcal{F}}) \in \mathsf{Whit}_{X^{I \coprod J}}(D^!(\mathfrak{Fus}^{\mathcal{G}}_{I,J}(Z)))$. By Lemma 7.9.7, this object canonically lifts to an object:

$$\overline{\mathfrak{Fus}}^{N^-}_{I,J}(\widetilde{\mathcal{F}}) \in \mathsf{Whit}_{X^{I \coprod J}}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)).$$

Moreover, the assignment $\widetilde{\mathcal{F}} \mapsto \overline{\mathfrak{Fus}}^{N^-}_{I,J}(\widetilde{\mathcal{F}})$ is obviously $D(X^I) \otimes D(X^J)$-linear.

We claim that the functor:

$$\mathsf{Whit}_{X^I\coprod J}(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)) \to D(X^I)\otimes \mathsf{Whit}_{X^J}(Z) \tag{7.10.1}$$

induced by restriction along the map:

$$X^I \times Z \to \overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)$$

is an equivalence, with inverse provided by $\overline{\mathfrak{Fus}}^{N^-}_{I,J}$. Indeed, this follows by combining Remark 7.8.3 with Lemma 7.9.7, and the observation that the functor:

$$D^!(X^I \times Z)^{\mathcal{H}_{N^-,I,J},\psi} \to D^!(X^I)\otimes \mathsf{Whit}(D^!(Z))$$

is an equivalence, where the superscript ψ indicates that we take invariants twisted with respect to the character of $\mathfrak{J}ets^{mer}_{X^I\coprod J}(N^-)$. We note that the last observation is trivial: the functor is fully-faithful since both are subcategories of $D^!(X^I \times Z)$, and is then an equivalence since $\mathcal{H}_{N^-}$ acts on $X^I \times Z$ through $X^I \times \mathfrak{J}ets^{mer}_{X^J}(N^-)$.

7.11. We now obtain that the !-restriction functor:

$$\mathsf{Whit}(D^!(\mathfrak{Fus}^G_{I,J}(Z))) \to D(X^I)\otimes \mathsf{Whit}_{X^J}(D^!(Z))$$

admits a left adjoint. Indeed, from the equivalence (7.10.1), we need to show that the functor:

$$\mathsf{Whit}(D^!(\mathfrak{Fus}^G_{I,J}(Z))) \to \mathsf{Whit}(D^!(\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z)))$$

admits a left adjoint. But the map $\overline{\mathfrak{Fus}}^{N^-}_{I,J}(Z) \hookrightarrow \mathfrak{Fus}^G_{I,J}(Z)$ is a finitely presented closed embedding, so the functor of !-dR *-pushforward provides the desired left adjoint.

7.12. We now establish the third point of Lemma 6.29.1. First, we specialize to the case $Z = \mathfrak{J}ets^{mer}_{X^I}(G)$.

Recall that e.g. $\mathsf{Whit}^{abs}_{X^I}$ denotes the category of Whittaker D-modules on $\mathfrak{J}ets^{mer}_{X^I}(G)$.

We have the Cartesian diagram:

$$\begin{tikzcd}
\mathcal{H}_{G,I,J} \arrow[r, "\beta_G"] \arrow[d, "\alpha_G"] & \mathfrak{J}ets^{mer}_{X^{I\coprod J}}(G) \arrow[d] \\
X^I \times \mathfrak{J}ets^{mer}_{X^J}(G) \arrow[r] & \mathfrak{Fus}^G_{I,J}(\mathfrak{J}ets^{mer}_{X^J}(G)).
\end{tikzcd} \tag{7.12.1}$$

We are supposed to show that the functor:

$$\alpha_{G,*,ren}\beta^!_G : \mathsf{Whit}^{abs}_{X^{I\coprod J}} \to D(X^I) \otimes \mathsf{Whit}^{abs}_{X^J}$$

admits a left adjoint.

As in Lemma 6.30.1, the right and left vertical maps in (7.12.1) are placid. Therefore, by Proposition 16.59.1 we may compute $\alpha_{G,*,ren}\beta^!_G$ by base-change. Then the existence of the left adjoint follows from placidity of the right vertical map, Proposition 16.59.1, and §7.11.

The other cases for Z work similarly, since in each case the corresponding indscheme over $X^{I\coprod J}$ maps placidly to $\mathfrak{Fus}^G_{I,J}(Z_{X^J})$.

8. Identification of the Chevalley complex II

8.1. The goal for this section is to deduce Ran space counterparts to the computations of §5.

8.2. Fix a non-empty finite set I. For each $\check{\lambda} \in \check{\Lambda}^{pos}$ we have a canonical incidence scheme:

$$\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff},X^I} \subseteq \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}} \times X^I$$

consisting of pairs $(D, \{x_i\}_{i\in\mathcal{I}})$ of a $\check{\Lambda}^{pos}$-divisor of degree $\check{\lambda}$ and an I-tuple of points such that the divisor D is supported set-theoretically at the points $\{x_i\}$, i.e., its restriction to $X\backslash\{x_i\}_{i\in\mathcal{I}}$ is the empty divisor. Let $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},X^I}$ be the corresponding union over $\check{\lambda} \in \check{\Lambda}^{pos}$.

Note that we have a canonical closed embedding:

$$\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},X^I} \hookrightarrow \mathrm{Gr}_{\check{T},X^I}.$$

We define:

$$\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}_{X^I} \subseteq \overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \times X^I := \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff},X^I} \underset{\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}} \times X^I}{\times} \overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \times X^I$$
$$\overset{o}{\mathcal{Z}}_{X^I} \subseteq \overset{o}{\mathcal{Z}} \times X^I := \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},X^I} \underset{\mathrm{Div}_{\mathrm{eff}} \times X^I}{\times} \overset{o}{\mathcal{Z}} \times X^I.$$

Note that as I varies in fSet, these schemes form a covariant system. Passing to the colimit over I, we obtain $\overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X}$ and $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},\mathrm{Ran}_X}$, both living over Ran_X.

We denote by $\overset{o}{\pi}_{\mathrm{Ran}_X} : \overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X} \to \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},\mathrm{Ran}_X}$ the structure map, or where there is no confusion, for the corresponding map to $\mathrm{Gr}_{\check{T},\mathrm{Ran}_X}$.

We introduce the notation:

$$\begin{aligned} \rho_{\overset{o}{\mathcal{Z}}} &: \overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X} \to \overset{o}{\mathcal{Z}} \\ \rho_{\mathrm{Div}} &: \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},\mathrm{Ran}_X} \to \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \end{aligned} \tag{8.2.1}$$

for the structure maps.

Remark 8.2.1. By construction, $\overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X}$ and $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},\mathrm{Ran}_X}$ are pseudo-indschemes in the sense of [Gai11]. In particular, we can make sense of D-modules: it is the limit under !-restriction of the categories of D-modules on the corresponding indschemes of ind-finite type, or equivalently, the colimit in DGCat_{cont} of the corresponding categories under de Rham pushforwards.

Remark 8.2.2. Recall that $\overset{o}{\zeta}$ denotes the stack $B^-\backslash G/B$.

The space $\overset{o}{\mathcal{Z}}_{X^I}$ can be realized as the moduli of a point $x = (x_i)_{i\in I} \in X^I$ and a $\mathcal{P}^{can}_T$-twisted map $X \to \overset{o}{\zeta}$ with a trivialization of the induced map $X\backslash\{x_i\}_{i\in I} \to \overset{o}{\zeta}$ (i.e., this map should factor through $\mathbb{B}T \subseteq \overset{o}{\zeta}$).

8.3. Next, in §8.4, we compare two Ran space versions of $\Upsilon_{\check{\mathfrak{n}}}$, the main factorization algebra of our interest (c.f. §1.25).

Here's why it is necessary: we want an intrinsic Ran space characterization of $\Upsilon_{\check{\mathfrak{n}}}$, namely, as the chiral enveloping algebra of the $\check{\Lambda}$-graded Lie-* algebra $\check{\mathfrak{n}} \otimes k_X$ (c.f. §8.4).

However, in §5, we used a version of $\Upsilon_{\check{\mathfrak{n}}}$ that did not involve Ran space: it only involved the finite-dimensional geometry of symmetric powers of the curve. In particular, Corollary 5.7.1 involves this finite-dimensional version.

The comparison between these two constructions (and the details of the first construction) are given below.

8.4. Observe that $\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}$ factorizes compatibly with the factorization structure on Ran_X, so defines a factorization category on X_{dR} with global sections $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$. We will abuse notation in denoting this factorization category by the same notation as its global sections.

Moreover, the addition structure on $\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}$ defines on $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$ the structure of *commutative* factorization category.[25] Therefore, we may speak of Lie-$*$ algebras in this category, as in §15.

As in Remark 5.6.2, the $\check{\Lambda}^{pos}$-grading on $\check{\mathfrak{n}}$ defines a Lie-$*$ structure on $\check{\mathfrak{n}} \otimes k_X$ in the category $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$. Therefore, we may form the chiral enveloping algebra $U^{ch}(\check{\mathfrak{n}} \otimes k_X)$ and define $\Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}$ to be the associated factorization algebra in $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$.

Lemma 8.4.1. *There is a canonical isomorphism $\Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X} \simeq \rho_{\mathrm{Div}}^!(\Upsilon_{\check{\mathfrak{n}}})$ of factorization algebras in $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$.*

Proof. The framework of §13 and §15 works just as well for $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$, and we use the corresponding language.

[25] In fact, it is the commutative factorization category associated with the symmetric monoidal category of $\check{\Lambda}^{pos}$-graded vector spaces by the procedure of §15.

One can consider $\check{\mathfrak{n}} \otimes k_X$ as a Lie-$*$ algebra in $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$. Note that this Lie-$*$ algebra is supported only on the locus of those divisors concentrated at a single point.

Therefore, one easily finds that $\check{\mathfrak{n}} \otimes k_X$ pulls back along $\rho^!_{\mathrm{Div}}$ to give the same-named Lie-$*$ algebra in $D(\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}})$.

Moreover, one readily shows that $\rho^!_{\mathrm{Div}}$ commutes with Koszul duality. Then using the chiral PBW theorem, one shows that it commutes with taking chiral envelopes.

We then immediately obtain the result from Remark 5.6.2.

□

8.5. Let $\overset{!}{\psi}_{\overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X}}$ denote the !-pullback of the sheaf $\overset{!}{\psi}_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}$ via the structure map:

$$\rho_{\overset{o}{\mathcal{Z}}} : \overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X} \to \overset{o}{\mathcal{Z}}.$$

The main result of this section is the following:

Theorem 8.5.1. *There is a canonical equivalence:*

$$\overset{o}{\pi}_{\mathrm{Ran}_X,*,dR}(\overset{!}{\psi}_{\overset{o}{\mathcal{Z}}_{\mathrm{Ran}_X}}) \xrightarrow{\simeq} \Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}. \tag{8.5.1}$$

Proof. Immediate by base-change from Corollary 5.7.1 and Lemma 8.4.1.

□

9. Construction of the functor

9.1. In this section, we perform the main construction of this thesis. This is a routine matter of drawing together material already developed in other parts of this thesis.

9.2. Recall from Proposition 6.24.1 that we have a unital chiral functor $i^{\frac{\infty}{2},!} : D^!(\mathfrak{Fl}^{\frac{\infty}{2}}) \to D(\mathrm{Gr}_T)$. We obtain a (lax) unital chiral functor $\mathsf{Whit}^{\frac{\infty}{2}} \to D(\mathrm{Gr}_T)$ by composition with the structure map (6.33.1) from $\mathsf{Whit}^{\frac{\infty}{2}}$ to $D^!(\mathfrak{Fl}^{\frac{\infty}{2}})$. We also denote this functor by $i^{\frac{\infty}{2},!}$.

Theorem 9.2.1. *The functor* $i^{\frac{\infty}{2},!} : \mathsf{Whit}^{\frac{\infty}{2}} \to D(\mathrm{Gr}_T)$ *sends the unit object to* $\Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}$.

Proof. By §6.34, we have strictly unital chiral functors:

$$\mathsf{Whit}^{sph} \to \mathsf{Whit}^{int} \xrightarrow{\mathfrak{p}^{\frac{\infty}{2},int}_{*,ren}} \mathsf{Whit}^{\frac{\infty}{2}}.$$

As in Proposition 6.24.1, we have an identification:

$$i^{\frac{\infty}{2},!} \circ \mathfrak{p}^{\frac{\infty}{2},int}_{*,ren} \simeq \mathfrak{q}^{loc}_{*,dR} \circ i^! : D^!(\mathrm{Gr}_{G,B}) \to D(\mathrm{Gr}_T) \in \mathsf{Cat}^{ch}_{un}(X_{dR}).$$

Therefore, it suffices to compute where the unit of Whit^{sph} maps to under $\mathfrak{q}^{loc}_{*,dR} \circ i^!$.

By construction, the unit object of Whit^{sph} is the $*$-extension of the Whittaker sheaf on Gr_{N^-}. Therefore, by base-change, the image of the unit is obtained by pulling and pushing this Whittaker sheaf along the diagram:

$$\begin{tikzcd} & \mathrm{Gr}_{N^-} \underset{\mathrm{Gr}_G}{\times} \mathrm{Gr}_B \arrow[dl] \arrow[dr] & & \\ \mathrm{Gr}_{N^-} & & \mathrm{Gr}_B \arrow[r] & \mathrm{Gr}_T . \end{tikzcd}$$

Noting that that fiber product is the open Zastava space $\overset{o}{\mathcal{Z}}$, we obtain the result from Theorem 8.5.1.

$\square$

9.3. Recall from Proposition 14.14.1 that $I \mapsto \Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,X^I}))$ defines a weak chiral category $\Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_T))$. Moreover, this proposition combined with Theorem 9.2.1 implies that we obtain a functor:

$$\mathsf{Whit}^{\frac{\infty}{2}} \to \Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_T))$$

of unital weak chiral categories (the left hand side being a true chiral category). This functor is obviously strictly unital in the obvious sense.

9.4. We conclude with the following concrete conjecture concerning this functor, which appears to be very much in reach.

Conjecture 1. *Define* $\mathsf{Whit}^{\frac{\infty}{2},ren}_{X^I}$ *as the compactly generated category whose compact objects are the full subcategory of* $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$ *generated from compact objects in* $\mathsf{Whit}^{sph}_{X^I}$ *using the functor* $\mathsf{Whit}^{sph}_{X^I} \to \mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$ *(c.f. §6.34) and the action of compact objects in* $D(\mathrm{Gr}_{T,X^I})$ *under its action on* $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$*.*

Define $\Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,X^I}))^{ren}$ *to be compactly generated by modules induced from compact Lie-$*$ modules for the Lie-$*$ algebra* $\check{\mathfrak{n}}^- \otimes k_X$*.*

Then the induced functor:

$$\left(I \mapsto \mathsf{Whit}^{\frac{\infty}{2},ren}_{X^I}\right) \to \left(I \mapsto \Upsilon_{\check{\mathfrak{n}},\mathrm{Ran}_X}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,X^I}))^{ren}\right)$$

is an equivalence of factorization categories.

Remark 9.4.1. This conjecture amounts to proving the main conjecture (c.f. §1.22) in the formal neighborhood of regular local systems inside of all local systems.

Part 2. Chiral categories

10. A guide for the perplexed

10.1. The goal of the following foundational sections is to develop a theory of chiral categories, chiral algebras in them, and chiral modules for these chiral algebras. This material has been heavily influenced by [BD04], [FG12], [Lur12] §5, [Gai08], and private conversations with Dennis Gaitsgory.

10.2. Our goals in developing the theory of chiral categories are modest, and the material itself is technical. These technicalities largely are due to the use of derived categories: the combinatorial aspects of [BD04] need to be replaced by more abstract formulations to be used in higher category theory.

We find it convenient in presenting this material to describe the goals and motivation in isolation from its technical implementation. The present section is devoted exactly to giving an introduction to these ideas, beyond what was already said in §1.

The hope is to provide some general narrative structure for the technical material that follows, and to help equip the reader who so desires to skip most of Part 2 and refer back to it only as necessary. In particular, we draw the reader's attention to §10.12 below, which explicitly spells out what is accomplished in Part 2 with regard to constructing the functor (1.22.1).

Remark 10.2.1. We note from the onset that most of the technicalities occur only in the unital setting, where the meaning of the word *unital* is indicated below.

Remark 10.2.2. Below, we discuss everything at a very heuristic level. In particular, we ignore *higher compatibilities* (such as associativity) throughout.

10.3. **Sheaves of categories.** Let X be a scheme of finite type.

To discuss chiral categories in analogy with chiral (or more appropriately: factorization) algebras, we need a "linear algebra" of categories over X, meant to be one categorical level higher than quasi-coherent sheaves or D-modules on X.

This theory is provided by the theory of sheaves of categories from [Gai12b] (see also §19). Recall that there is a notion of (DG) category $\mathcal{C}$ over X: for $X = \mathrm{Spec}(A)$ an affine (DG) scheme, this amounts to a cocomplete DG category enriched over the symmetric monoidal DG category $A\text{–}\mathsf{mod}$, and for general X the notion is obtain by gluing. Categories over schemes are contravariantly functorial with respect to morphisms of schemes.

Moreover, we have a general notion of category $\mathcal{C}$ over X with a *connection*, also known as a crystal of categories. This amounts to saying that given any two infinitesimally close points of X, we identify the fibers of $\mathcal{C}$ in a functorial way satisfying the (higher) cocycle conditions.

The notion of crystal of categories on X can be summarized more succinctly: we have the prestack X_{dR}, and there is a general notion of sheaf of categories on a prestack. Crystals of categories on X are equivalent to sheaves of categories on X_{dR}, since X_{dR} is the quotient of X by its universal infinitesimal groupoid (c.f. [GR14]).

We want to have quasi-coherent and D-module versions of the theory of chiral algebras and chiral categories, and therefore we replace X with a general prestack $\mathcal{X}$, so that for $\mathcal{X} = X$ we obtain the quasi-coherent version and for $\mathcal{X} = X_{dR}$ we obtain the D-module version.

Note that there is a canonical sheaf of categories $\mathsf{QCoh}_{\mathcal{X}}$ on the prestack $\mathcal{X}$, whose global sections (in the sense of sheaves of categories) is the category $\mathsf{QCoh}(\mathcal{X})$ of quasi-coherent sheaves on $\mathcal{X}$. This sheaf of categories plays the role that $\mathcal{O}_{\mathcal{X}}$ plays one categorical level down.

Convention 10.3.1. We use the language of quasi-coherent sheaves in what follows, noting that the D-module language is a special case by the above.

Terminology 10.3.2. Recall that [BD04] defines notions of both *chiral* and *factorization* algebra on X_{dR}, and proves that the two notions are equivalent by means of a non-trivial functor (e.g., it doesn't commute with the forgetful functor to D-modules).

The notion of chiral algebra is much less flexible than that of factorization algebra: e.g., it can only be defined in the de Rham setting, not in the general quasi-coherent setting. In particular, only the factorization perspective generalizes to categories.

Therefore, we use the terms *chiral category* and *factorization category* interchangeably in the categorical setting because there is no risk for ambiguity. However, for sheaves, we will be much more conservative in the use of the word *chiral*.

10.4. **Ran's space.** Next, we recall the Ran space construction from [BD04].

The idea of Ran space Ran_X is to parametrize non-empty finite subsets of a space X.

Remark 10.4.1. Any construction of Ran_X builds it out of the schemes X^I for I a finite set. This translates to saying that specialization in Ran_X allows points to collide.

It has been treated formally in algebraic geometry in a number of ways, and we follow [FG12] and [Gai11] in treating it as a prestack. The construction is defined for any prestack $\mathcal{X}$, giving rise to a prestack $\mathrm{Ran}_{\mathcal{X}}$.

The key point is that quasi-coherent sheaves $\mathcal{F}$ on $\mathrm{Ran}_{\mathcal{X}}$ are equivalent to systems of quasi-coherent sheaves $\mathcal{F}_{\mathcal{X}^I}$ on each $\mathcal{X}^I$ as I varies under non-empty finite sets, and such that these sheaves are compatible along diagonal restrictions (note that we consider the reordering of coordinates as a diagonal restriction, so these quasi-coherent sheaves are automatically equivariant for the symmetric group). The same holds for sheaves of categories.

Remark 10.4.2. One may heuristically think that a quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{Ran}_{\mathcal{X}}$ is an assignment of a vector space $\mathcal{F}_{x_1,\ldots,x_n}$ for every finite subset $\{x_i\} \subseteq X$, such that these vector spaces behave "continuously" as points move and collide. Similarly, a sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$ is a continuous assignment of cocomplete DG categories $\mathcal{C}_{x_1,\ldots,x_n}$.

10.5. **Unital sheaves on** $\mathrm{Ran}_{\mathcal{X}}$**.** There is also a notion of *unital* quasi-coherent sheaf of $\mathrm{Ran}_{\mathcal{X}}$, implicit in [BD04] §3.4.5, and appearing again in [Gai10a], [Gai11], and [Bar12].

Here we are again given quasi-coherent sheaves $\mathcal{F}_{\mathcal{X}^I}$ for each finite set I, now also allowing the empty set as well. For every morphism $f : I \to J$ of finite sets, giving rise to the map $\Delta_f : \mathcal{X}^J \to \mathcal{X}^I$, we should be given:

$$\Delta_f^*(\mathcal{F}_{X^I}) \to \mathcal{F}_{X^J}$$

in a way compatible with compositions of morphisms of finite sets, and such that, if Δ_f is a diagonal embedding (i.e., f is surjective), this map should be an isomorphism. In particular, for every I we have a canonical *unit* map:

$$\mathcal{F}_{\varnothing} \otimes_k \mathcal{O}_{X^I} \to \mathcal{F}_{X^I}.$$

Similarly, we have a notion of unital sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$.

Obviously, unital quasi-coherent sheaves on $\mathrm{Ran}_{\mathcal{X}}$ are quasi-coherent sheaves on $\mathrm{Ran}_{\mathcal{X}}$ with additional structure.

Remark 10.5.1. Unital quasi-coherent sheaves on $\mathrm{Ran}_{\mathcal{X}}$ do not quite fall under the purview of quasi-coherent sheaves on prestacks. However, in §11, we show that the language of *lax prestacks* — moduli problems valued in categories rather than groupoids — does suffice.

Namely, we define a lax prestack $\mathrm{Ran}_{\mathcal{X}}^{un}$ whose points are morally the (possibly empty) finite subsets of $\mathcal{X}$, considered as a category by taking morphisms that are inclusions of finite subsets, and show that this lax prestack gives a good theory of unital quasi-coherent sheaves.

Remark 10.5.2. In the heuristic of Remark 10.4.2, a unital quasi-coherent sheaf $\mathcal{F}$ on Ran_X is a continuous assignment:

$$\big(\{x_1, \ldots, x_n\} \subseteq X\big) \mapsto \mathcal{F}_{x_1,\ldots,x_n} \in \mathsf{Vect}$$

as before (now allowing $n = 0$), and such that for every inclusion:

$$\{x_1, \ldots, x_n\} \subseteq \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\} \subseteq X \tag{10.5.1}$$

we have a map:

$$\mathcal{F}_{x_1,\ldots,x_n} \to \mathcal{F}_{x_1,\ldots,x_m} \tag{10.5.2}$$

satisfying the natural compatibilities.

Remark 10.5.3 (Lax unital functors). The heuristic notion of unital sheaf of categories is identical to the discussion of Remark 10.5.2. However, a difference emerges in the notion of *morphism* of unital sheaves of categories.

Given unital sheaves of categories $\mathcal{C}$ and $\mathcal{D}$ on $\mathrm{Ran}_{\mathcal{X}}$, we have two notions functor $\mathcal{C} \to \mathcal{D}$, *strict* and *lax*.

For a *strict* functor, we require that we are given functors:

$$F_{x_1,\ldots,x_n} : \mathcal{C}_{x_1,\ldots,x_n} \to \mathcal{D}_{x_1,\ldots,x_n}$$

such that, for every inclusion (10.5.1), the diagram:

$$\begin{tikzcd}
\mathcal{C}_{x_1,\ldots,x_n} \arrow[r, "F_{x_1,\ldots,x_n}"] \arrow[d] & \mathcal{D}_{x_1,\ldots,x_n} \arrow[d] \\
\mathcal{C}_{x_1,\ldots,x_m} \arrow[r, "F_{x_1,\ldots,x_m}"] & \mathcal{D}_{x_1,\ldots,x_n}
\end{tikzcd}$$

commutes, where the vertical arrows come from the unital structure.

For a lax functor, we merely require that the diagram *lax commute*, i.e., we are given a natural transformation:

$$\begin{tikzcd}
\mathcal{C}_{x_1,\ldots,x_n} \arrow[r, "F_{x_1,\ldots,x_n}"] \arrow[d] & \mathcal{D}_{x_1,\ldots,x_n} \arrow[d] \arrow[dl, Rightarrow] \\
\mathcal{C}_{x_1,\ldots,x_m} \arrow[r, "F_{x_1,\ldots,x_m}"] & \mathcal{D}_{x_1,\ldots,x_n}
\end{tikzcd}$$

This difference is a general feature of working with sheaves of categories on lax prestacks that is different from the more restricted theory of sheaves of categories on usual prestacks. It is discussed in detail in §11, where we remove the adjective "lax" from the term "lax functor."

For the importance of working with lax functors of unital sheaves of categories, see the discussion of Remark 10.6.3 below.

10.6. **Factorization algebras.** The heuristic idea of a *factorization algebra* in a factorization category $\mathcal{C}$ is that we are given have $\mathcal{A} \in \mathsf{QCoh}(\mathrm{Ran}_X)$, and for $\{x_1, \ldots, x_n\} \subseteq X$, we are given isomorphisms:

$$\mathcal{A}_{x_1,\ldots,x_n} \simeq \mathcal{A}_{x_1} \otimes \ldots \otimes \mathcal{A}_{x_n} \tag{10.6.1}$$

that are continuous as we vary the points x_i. There is a somewhat subtle requirement as points collide: if we choose $1 \leqslant k < n$, then we require that the induced isomorphisms:

$$\mathcal{A}_{x_1,\ldots,x_n} \simeq \mathcal{A}_{x_1,\ldots,x_k} \otimes \mathcal{A}_{x_{k+1},\ldots,x_n}$$

extend only when we allow points x_i to collide with points x_j only when $1 \leqslant i, j \leqslant k$ or $k < i, j \leqslant n$. In particular, for a pair $\{x, y\}$ of distinct points of X, we do not at all specify the behavior of the isomorphism:

$$\mathcal{A}_{x,y} \simeq \mathcal{A}_x \otimes \mathcal{A}_y \tag{10.6.2}$$

as x and y collide.

Remark 10.6.1. In practice, it is unreasonable (except for $\mathcal{A} = \mathcal{O}_X$) to require that the isomorphisms (10.6.2) to extend when x and y collide. However, we may require a map to exist in one direction: this gives the theory of *commutative* factorization sheaves, that we develop in §15.

Similarly, we have the notion of *unital* factorization sheaf. Here we require that the isomorphisms (10.6.1) be compatible in the natural sense with the unital maps (10.5.2).

Again, the notion of (resp. unital) chiral category can be described similarly. Note that we can speak about factorization algebras inside of a chiral category $\mathcal{C}_x$: this is a continuous assignment of objects $\mathcal{A}_{x_1,\ldots,x_n} \in \mathcal{C}_{x_1,\ldots,x_n}$ with identifications:

$$\mathcal{A}_{x_1,\ldots,x_n} \simeq \mathcal{A}_{x_1} \otimes \ldots \otimes \mathcal{A}_{x_n}$$

in the identified (by chirality) categories:

$$\mathcal{C}_{x_1,\ldots,x_n} \simeq \mathcal{C}_{x_1} \otimes \ldots \otimes \mathcal{C}_{x_n}.$$

Remark 10.6.2 (Unit objects). The unital factorization conditions force $\mathcal{C}_{\varnothing} \simeq \mathsf{Vect}$ canonically. Considering $\varnothing \hookrightarrow \{x\}$, we see that $\mathcal{C}_x$ contains a canonical *unit object* $\mathrm{unit}_{\mathcal{C},x}$ which by definition is the image of $k \in \mathsf{Vect}$ under the induced functor:

$$\mathsf{Vect} = \mathcal{C}_{\varnothing} \to \mathcal{C}_x.$$

Remark 10.6.3 (Unital factorization functors). What a factorization functor should be should be clear in the above heuristics: it is a functor $F : \mathcal{C} \to \mathcal{D}$ of categories over $\mathrm{Ran}_{\mathcal{X}}$, such that, e.g., for every pair of distinct points $x, y \in \mathcal{X}$, the diagram:

$$\begin{tikzcd}
\mathcal{C}_{x,y} \arrow[r, "F_{x,y}"] \arrow[d, "\simeq"] & \mathcal{D}_{x,y} \arrow[d, "\simeq"] \\
\mathcal{C}_x \otimes \mathcal{C}_y \arrow[r, "F_x \otimes F_y"] & \mathcal{D}_x \otimes \mathcal{D}_y.
\end{tikzcd} \tag{10.6.3}$$

As in Remark 10.5.3, there are two notions of unital factorization functor, *lax* and *strict*.

The difference primarily occurs at the level of underlying sheaves of categories, i.e., in the setting of *loc. cit.* That is to say, we still require the diagram (10.6.3)

The key distinction between lax and strict here is that a strictly unital factorization functor preserves unit objects, while for a lax unital factorization functor, we only have a morphism:

$$\mathrm{unit}_{\mathcal{D}} \to F(\mathrm{unit}_{\mathcal{C}}).$$

This is relevant for the purposes of this thesis because, as in §1.26, the factorization functor we are interested in *does not* preserve unit objects; rather, it is merely lax unital (c.f. also to Footnote 5).

10.7. The idea for implementing §10.6 is to exploit the *chiral multiplication* of $\mathrm{Ran}_{\mathcal{X}}$ and $\mathrm{Ran}_{\mathcal{X}}^{un}$, that we describe below.

Recall that if $\mathcal{S} \in \mathsf{PreStk}$ is equipped with a commutative and associative multiplication, we can speak of *multiplicative* quasi-coherent sheaves on $\mathcal{S}$; for m the multiplication operation, these are quasi-coherent sheaves $\mathcal{A} \in \mathsf{QCoh}(\mathcal{S})$ with isomorphisms:

$$m^*(\mathcal{A}) \simeq \mathcal{A} \boxtimes \mathcal{A}$$

satisfying the natural commutativity and associativity requirements.

Note that $\mathrm{Ran}_{\mathcal{X}}$ admits a natural commutative semigroup structure: the multiplication operation is given by union of subsets of $\mathcal{X}$. Similarly, $\mathrm{Ran}_{\mathcal{X}}^{un}$ has a commutative monoid structure given in the same way.

Remark 10.7.1. We only say "semigroup" here because $\mathrm{Ran}_{\mathcal{X}}$ does not contain the empty subset of $\mathcal{X}$, which would correspond to the unit: this should only ever be regarded as a minor issue.

The *chiral multiplication* can be thought of as a *partially-defined* multiplication, where we are only allowed to add two subsets of $\mathrm{Ran}_{\mathcal{X}}$ if they are disjoint.

Then we say that e.g. a factorization sheaf on $\mathrm{Ran}_{\mathcal{X}}$ is a multiplicative sheaf with respect to this partially-defined multiplication.

10.8. **Correspondences.** However, there is still a substantive technical issue: what do we mean by "partially-defined multiplication?"

One convenient approach here is to use the formalism of *correspondences* here, developed in the homotopical setting in [GR14].

Recall that if $\mathcal{C}$ is a category with fiber products, the category $\mathcal{C}_{corr}$ is defined to have the same objects as $\mathcal{C}$, with morphisms $X \to Y$ given by hats:

$$\begin{tikzcd} & H \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in $\mathcal{C}$. Composition of morphisms is defined by fiber products, i.e., we regard diagrams:

$$\begin{tikzcd} & & H_3 \arrow[dl] \arrow[dr] & & \\ & H_1 \arrow[dl] \arrow[dr] & & H_2 \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

with inner square Cartesian as realizing the correspondence $(X \leftarrow H_3 \to Z)$ as the composition of the morphisms $X \to Y$ and $Y \to Z$ in $\mathcal{C}_{corr}$.

If $\mathcal{C}$ is equipped with a symmetric monoidal structure, then $\mathcal{C}_{corr}$ inherits a symmetric monoidal structure in the obvious way.

Remark 10.8.1. We recall the construction from [GR14] in more detail in §20.

10.9. **Chiral multiplication via correspondences.** We can now say that chiral multiplication is a (non-unital) commutative algebra structure on $\mathrm{Ran}_{\mathfrak{X}}$ when regarded as an object of PreStk_{corr}, where the multiplication operation is defined by the correspondence:

$$\begin{tikzcd} & {[\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj}} \arrow[dl] \arrow[dr] & \\ \mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}} & & \mathrm{Ran}_{\mathfrak{X}} \end{tikzcd}$$

where the notation $disj$ indicates that we take the locus of this product where points are pairwise disjoint, and where the right map is the addition map.

In §12, we develop a theory of multiplicative sheaves of categories on lax prestacks with commutative algebra structures defined using correspondences, giving a definition of factorization category. This is specialized to the case of Ran space in §13.

10.10. **Factorization modules.** Next, we discuss the idea of factorization modules.

Let $\mathcal{A}$ be a factorization algebra and let x_0 be a point of X. A factorization module structure at x_0 for a vector space M is essentially a rule that associates to every finite set $\{x_0, x_1, \ldots, x_n\}$ of points of X a vector space $M_{x_0,x_1,\ldots,x_n}$ such that, for every $0 \leqslant k < n$ we have identifications:

$$M_{x_0,\ldots,x_n} \simeq M_{x_0,\ldots,x_k} \otimes \mathcal{A}_{x_{k+1},\ldots,x_n}$$

compatible with refinements in the obvious sense.

This notion generalizes in the usual ways: we can allow the x_0 to move, or to take factorization modules at several points at once, or to take unital factorization modules, or to take factorization module categories for a chiral category, etc.

An important point is Theorem 13.13.2, which says that under certain hypotheses, modules for the unit factorization algebra in a unital chiral category are just objects of the underlying category.

A second important point is the construction of *external fusion* from §13.12, that takes chiral modules at two distinct points (or disjoint subsets of points) and produces a module at their union.

Remark 10.10.1. Heuristically, external fusion should make factorization modules for a factorization algebra into a factorization category. However, since the tensor product of DG categories is unwieldy in many respects, we expect that this is only true after appropriate *renormalization* in the sense of [FG09]. In general, the only structure is that of *lax factorization category*, as is discussed in §14.

10.11. **Factorization without** $\mathrm{Ran}_{\mathcal{X}}$**.** In §14, we present an alternative approach to chiral categories.

This approach is much more combinatorial than the approach using prestacks and correspondences. Proofs of foundational results, while largely possible in this setting, are much less clean. However, this second approach has the advantage that it only uses finite-dimensional geometry (say if $\mathcal{X} = X$ or X_{dR}), without explicit recourse to the Ran space.

Roughly, in this perspective a factorization sheaf $\mathcal{A}$ on $\mathrm{Ran}_{\mathcal{X}}$ is a compatible system $\mathcal{A}_{\mathcal{X}^I}$ of D-modules on each $\mathcal{X}^I$, and with identifications:

$$\mathcal{A}_{\mathcal{X}^I} \boxtimes \mathcal{A}_{\mathcal{X}^J}|_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \simeq \mathcal{A}_{\mathcal{X}^{I \coprod J}}|_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}}.$$

10.12. **User's guide.** There are two basic results in Part 2 that we will need for Part 1.

(1) Proposition-Construction 11.26.1, and its consequence Proposition 13.4.2. These results will be used for constructing unital chiral category structures on various Whittaker categories, and ultimately, on $\mathsf{Whit}^{\frac{\infty}{2}}$.

For simplicity, here is what these propositions say we should do to construct a unital structure on $\mathsf{Whit}^{sph} := \mathsf{Whit}(D(\mathrm{Gr}_G))$ (i.e., Whittaker sheaves on Gr_G).

First, we construct a unital structure on $D(\mathrm{Gr}_G)$. For $\{x_1, \ldots, x_n\} \subseteq \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\} \subseteq X$ as in Remark 10.5.2, the corresponding unit maps (10.5.2) are given by:

$$D(\mathrm{Gr}_{G,x_1}) \otimes \ldots \otimes D(\mathrm{Gr}_{G,x_n}) \simeq D(\mathrm{Gr}_{G,x_1}) \otimes \ldots \otimes D(\mathrm{Gr}_{G,x_n}) \otimes \mathsf{Vect} \otimes \ldots \otimes \mathsf{Vect} \to$$
$$D(\mathrm{Gr}_{G,x_1}) \otimes \ldots \otimes D(\mathrm{Gr}_{G,x_n}) \otimes D(\mathrm{Gr}_{G,x_{n+1}}) \otimes \ldots \otimes D(\mathrm{Gr}_{G,x_m})$$

where for each $n < i \leqslant m$, the map $\mathsf{Vect} \to D(\mathrm{Gr}_{G,x_i})$ sends k to the δ D-module concentrated at the unit point in Gr_{G,x_i}.

For Whittaker sheaves, this construction does not work verbatim because $\mathsf{Vect} \to D(\mathrm{Gr}_{G,x_i})$ does not factor through the subcategory of Whittaker sheaves. Therefore, we further compose it with the functor of !-averaging against the Whittaker character.[26]

The precise conditions that are needed for this format — which are somewhat more subtle than they appear above because we need to allow points to collide — are discussed in Remark 11.26.2.

(2) Next, under certain favorable circumstances, we show in Theorem 13.13.2 that for a unital chiral category $\mathcal{C}$ with unit object $\mathrm{unit}_{\mathcal{C}}$, we have $\mathrm{unit}_{\mathcal{C}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\mathcal{C}) \simeq \mathcal{C}$, where these symbols are made sense of in §13. I.e., the result says that the structure of unital module for the unit object is no extra structure at all — certainly a familiar kind of statement!

We apply this result as follows.

As was discussed in §1.26, we have a !-restriction functor $D(\mathfrak{Fl}^{\frac{\infty}{2}}) \to D(\mathrm{Gr}_T)$ inducing a composite functor:

$$F : \mathsf{Whit}^{\frac{\infty}{2}} \to D(\mathrm{Gr}_T)$$

sending the unit object of $\mathsf{Whit}^{\frac{\infty}{2}}$ to the factorization algebra $\Upsilon_{\check{\mathfrak{n}}} \in D(\mathrm{Gr}_T)$ from §1.25 (c.f. Theorem 1.26.1). This functor is a lax unital functor of unital chiral categories, as in Remark 10.6.3 above.

By functoriality of modules for factorization algebras, this induces a functor:

$$\mathsf{Whit}^{\frac{\infty}{2}} \simeq \mathrm{unit}_{\mathsf{Whit}^{\frac{\infty}{2}}}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{Whit}^{\frac{\infty}{2}}) \to \Upsilon\text{–}\mathsf{mod}^{\mathrm{fact}}(D(\mathrm{Gr}_T))$$

as desired.

[26]Working in families, there's no a priori reason why this !-averaging should be defined, since we deal with non-holonomic D-modules. This is essentially be the subject of §7.

11. Lax prestacks and the unital Ran space

11.1. In this section, we introduce Ran space as a prestack and its unital counterpart as a lax prestack. We discuss sheaves on lax prestacks in detail.

An important point is Proposition-Construction 11.26.1, which we will use to construct certain important unital sheaves of categories on Ran space.

11.2. **Notation for categories of sets.** Let Set denote the (1,1)-category of sets. Let $\mathsf{Set}_{<\infty} \subseteq \mathsf{Set}$ denote the full subcategory of finite sets. Let $\mathsf{fSet}_{\varnothing} \subseteq \mathsf{Set}_{<\infty}$ denote the non-full subcategory with the same objects, but in which we only allow surjective morphisms. Finally, let $\mathsf{fSet} \subseteq \mathsf{fSet}_{\varnothing}$ denote the full subcategory of *non-empty* finite sets.

We consider each of these categories as a non-unital symmetric monoidal category under disjoint unions. Of course, in all cases except fSet, this symmetric monoidal structure is in fact unital with unit the empty set.

Remark 11.2.1. The notation fSet is borrowed from [Gai11].

11.3. Let $\mathcal{G} \in \mathsf{Gpd}$ be fixed. We define the groupoids:

$$\mathrm{Ran}_{\mathcal{G}} := \operatorname*{colim}_{I \in \mathsf{fSet}^{op}} \mathcal{G}^I$$

$$\mathrm{Ran}_{\mathcal{G},\varnothing} := \operatorname*{colim}_{I \in \mathsf{fSet}_{\varnothing}^{op}} \mathcal{G}^I$$

Remark 11.3.1. $\mathrm{Ran}_{\mathcal{G},\varnothing}$ is just $\mathrm{Ran}_{\mathcal{G}}$ with a disjoint basepoint adjoined. We denote this basepoint by $\varnothing$ where convenient and unambiguous.

The (resp. non-unital) symmetric monoidal structure on the functor $I \mapsto \mathcal{G}^I$ from $\mathsf{fSet}_{\varnothing}$ (resp. fSet) determines the structure of (resp. non-unital) commutative monoid on $\mathrm{Ran}_{\mathcal{G},\varnothing}$ (resp. $\mathrm{Ran}_{\mathcal{G}}$), using that product in Cat commute with colimits in each variable.

We denote the corresponding maps:

$$\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}} \to \mathrm{Ran}_{\mathcal{G}}$$

$$\mathrm{Ran}_{\mathcal{G},\varnothing} \times \mathrm{Ran}_{\mathcal{G},\varnothing} \to \mathrm{Ran}_{\mathcal{G},\varnothing}$$

both by add.

Example 11.3.2. Suppose that $\mathcal{G} \in \mathsf{Set} \subseteq \mathsf{Gpd}$. In this case, one can show that $\mathrm{Ran}_{\mathcal{G}}$ is actually a set as well, and that it identifies in the obvious way with the set of non-empty finite subsets of $\mathcal{G}$. Similarly, $\mathrm{Ran}_{\mathcal{G},\varnothing}$ then identifies with the set of possibly empty finite subsets of $\mathcal{G}$.

Remark 11.3.3. Observe that $\mathcal{G} \mapsto \mathrm{Ran}_{\mathcal{G}}$ and $\mathcal{G} \mapsto \mathrm{Ran}_{\mathcal{G},\varnothing}$ commute with sifted colimits in the variable $\mathcal{G}$. Indeed, colimits commute with colimits, and for I finite, $\mathcal{G} \mapsto \mathcal{G}^I$ commutes with sifted colimits by definition of sifted.

Therefore, we can recover the functors $\mathcal{G} \mapsto \mathrm{Ran}_{\mathcal{G}}$ and $\mathcal{G} \mapsto \mathrm{Ran}_{\mathcal{G},\varnothing}$ as the left Kan extensions of their restrictions to $\mathsf{Set}_{<\infty}$.

11.4. **Unital Ran categories.** Let $\mathcal{G}$ be a groupoid. We will give three perspectives on a certain category $\mathrm{Ran}^{un}_{\mathcal{G}}$.

11.5. **Partial-ordering.** In the first construction, suppose first that $\mathcal{G}$ is a set. Recall that in this case $\mathrm{Ran}_{\mathcal{G},\varnothing}$ is the set of finite subsets of $\mathcal{G}$. We consider this set as a partially-ordered set under inclusions.

We then declare $\mathrm{Ran}^{un}_{\mathcal{G}} := \mathsf{Poset}_{\mathrm{Ran}_{\mathcal{G},\varnothing}}$ to be the category associated with this partially-ordered set. It is easy to see that this construction commutes with filtered colimits in the variable $\mathcal{G}$.

Following Remark 11.3.3, we then extend this definition to an arbitrary groupoid $\mathcal{G}$ by declaring that it should commute with sifted colimits.

11.6. **Unital Ran as a lax colimit.** We now give a second construction of $\mathrm{Ran}^{un}_{\mathcal{G}}$.

We will begin by defining a second groupoid $'\mathrm{Ran}^{un}_{\mathcal{G}}$, and then in Corollary 11.6.2 we will show that $'\mathrm{Ran}^{un}_{\mathcal{G}}$ is isomorphic to $\mathrm{Ran}^{un}_{\mathcal{G}}$.

Consider the functor $\mathsf{Set}_{<\infty}^{op} \to \mathsf{Gpd}$ defined by $I \mapsto \mathcal{G}^I$. We denote this functor temporarily by $\Psi_{\mathcal{G}}$.

We then form the Cartesian fibration $\mathsf{coGroth}(\Psi_{\mathcal{G}}) \to \mathsf{Set}_{<\infty}^{op}$, and define $'\mathrm{Ran}_{\mathcal{G}}^{un}$ to be the result of inverting all arrows in $\mathsf{coGroth}(\Psi_{\mathcal{G}})$ that are Cartesian and lie over a surjective morphism in $\mathsf{Set}_{<\infty}$, i.e., a morphism in fSet^{op}.

Note that unions induce a canonical symmetric monoidal structure on $'\mathrm{Ran}_{\mathcal{G}}^{un}$ (c.f. §12.15).

Proposition 11.6.1. *(1) The functor $\mathcal{G} \mapsto {}'\mathrm{Ran}_{\mathcal{G}}^{un}$ commutes with sifted colimits.*

(2) For $\mathcal{G}$ a set, the functor:

$$\mathsf{coGroth}(\Psi_{\mathcal{G}}) \to \mathsf{Poset}_{\mathrm{Ran}_{\mathcal{G},\varnothing}} \tag{11.6.1}$$

sending a datum $(I \in \mathsf{Set}_{<\infty}, x \in \mathcal{G}^I)$ to[27] *$x \in \mathrm{Ran}_{\mathcal{G},\varnothing}$ induces an equivalence:*

$$'\mathrm{Ran}_{\mathcal{G}}^{un} \xrightarrow{\simeq} \mathsf{Poset}_{\mathrm{Ran}_{\mathcal{G},\varnothing}}. \tag{11.6.2}$$

Corollary 11.6.2. *There is a functorial equivalence of $\mathrm{Ran}_{\mathcal{G}}^{un} \simeq {}'\mathrm{Ran}_{\mathcal{G}}^{un}$ of symmetric monoidal categories.*

Proof of Proposition 11.6.1. The first part follows easily from the fact that $\mathcal{G} \mapsto \mathcal{G}^I$ commutes with sifted colimits for I finite.

The map (11.6.1) sends Cartesian arrows over fSet^{op} to isomorphisms, and therefore induces the symmetric monoidal functor (11.6.2).

The prove that this functor is an equivalence (and in particular, that the left hand side is a 1-category), we will explicitly construct an inverse.

For $I = \{x_1, \dots, x_n\}$ a finite subset of $\mathcal{G}$, we attach an object of $\mathsf{coGroth}(\Psi_{\mathcal{G}})$ in the tautological way: a point of $\mathsf{coGroth}(\Psi_{\mathcal{G}})$ is a pair of a finite set and a subset of $\mathcal{G}$ indexed

[27]Here we are using that objects of $\mathsf{Poset}_{\mathrm{Ran}_{\mathcal{G},\varnothing}}$ are points of $\mathrm{Ran}_{\mathcal{G},\varnothing}$.

by that finite set, and we attach the finite set I with the tautological associated subset of $\mathcal{G}$. This operation is evidently functorial, and projecting to $'\mathrm{Ran}_{\mathcal{G}}^{un}$ evidently provides an inverse.

$\square$

11.7. **Unital Ran space via tuples of finite sets.** We now give a final construction that more explicitly describes $\mathrm{Ran}_{\mathcal{G}}^{un}$ as a category by essentially describing its objects and morphisms and composition law. More precisely, we will describe its complete Segal groupoid.

11.8. Recall that $[n]$ denotes the totally ordered set $\{0, 1, \ldots, n\}$ of order $n+1$.

Let $\mathsf{fSet}^{\rightarrow}_{\varnothing,[n]}$ denote the $(1,1)$-category whose objects are data:

$$I_0 \xrightarrow{\Upsilon_1} I_1 \xrightarrow{\Upsilon_2} \ldots \xrightarrow{\Upsilon_n} I_n$$

with each I_i a (possibly empty) finite set and Υ_i an arbitrary map of sets, and where morphisms are given by commutative diagrams:

$$\begin{tikzcd} I_0 \arrow[r, "\Upsilon_1"] \arrow[d] & I_1 \arrow[r, "\Upsilon_2"] \arrow[d] & \ldots \arrow[r, "\Upsilon_n"] & I_n \arrow[d] \\ J_0 \arrow[r, "\delta_1"] & J_1 \arrow[r, "\delta_2"] & \ldots \arrow[r, "\delta_n"] & J_n. \end{tikzcd}$$

The data $[n] \mapsto \mathsf{fSet}^{\rightarrow}_{\varnothing,[n]}$ defines a simplicial category in the obvious way.

Example 11.8.1. For $n = 0$, we recover the category $\mathsf{fSet}_{\varnothing}$ by this construction. This is the reason we include $\varnothing$ in the notation.

Variant 11.8.2. We let $\mathsf{fSet}^{\rightarrow}_{[n]}$ denote the subcategory of $\mathsf{fSet}^{\rightarrow}_{\varnothing}$ in which we only allow *non-empty* finite sets to appear.

11.9. For $\mathcal{G}$ a groupoid, we obtain a functor:

$$\mathsf{fSet}^{\rightarrow,op}_{\varnothing,[n]} \to \mathsf{Gpd}$$

$$I_0 \xrightarrow{\Upsilon_1} I_1 \xrightarrow{\Upsilon_2} \ldots \xrightarrow{\Upsilon_n} I_n \mapsto \mathcal{G}^{I_n}.$$

We define $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[n]}$ as the corresponding colimit:

$$\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[n]} := \operatorname*{colim}_{(I_0 \xrightarrow{\Upsilon_1} I_1 \xrightarrow{\Upsilon_2} \ldots \xrightarrow{\Upsilon_n} I_n) \in \mathsf{fSet}^{\rightarrow,op}_{\varnothing,[n]}} \mathcal{G}^{I_n} \in \mathsf{Gpd}. \tag{11.9.1}$$

Example 11.9.1. For $n = 0$, we recover $\mathrm{Ran}_{\mathcal{G},\varnothing}$ through this construction.

Variant 11.9.2. As in Remark 11.8.2, we also obtain groupoids $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},[n]}$ by forming the colimit (11.9.1) over $\mathsf{fSet}^{\rightarrow,op}_{[n]}$ instead of $\mathsf{fSet}^{\rightarrow,op}_{\varnothing,[n]}$.

Example 11.9.3. For $\mathcal{G}$ a set, one can show as in Example 11.3.2 that $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[n]}$ is the set with elements data $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n \subseteq \mathcal{G}$ with each S_i finite.

$\mathrm{Ran}^{\rightarrow}_{\mathcal{G},[n]}$ is similar, but with each S_i additionally assumed non-empty.

11.10. We observe that the assignment $[n] \mapsto \mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[n]}$ defines a simplicial groupoid.

Indeed, for $p : [m] \to [n]$ a map in $\mathbf{\Delta}$, we are supposed to specify a map:

$$\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[n]} \to \mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing,[m]}. \tag{11.10.1}$$

We construct it explicitly below.

Recall that $[n] \mapsto \mathsf{fSet}^{\rightarrow}_{\varnothing,[n]}$ is functorial for $[n] \in \mathbf{\Delta}^{op}$. For p as above and $I_0 \xrightarrow{\Upsilon_1} I_1 \xrightarrow{\Upsilon_2} \ldots \xrightarrow{\Upsilon_n} I_n \in \mathsf{fSet}^{\rightarrow}_{\varnothing,[n]}$, the induced object of $\mathsf{fSet}^{\rightarrow}_{\varnothing,[m]}$ is:

$$I_{p(0)} \xrightarrow{\Upsilon_{p(1)}} I_{p(1)} \xrightarrow{\Upsilon_p(2)} \ldots \xrightarrow{\Upsilon_p(m)} I_{p(m)} \in \mathsf{fSet}^{\rightarrow}_{\varnothing,[n]} \in \mathsf{fSet}^{\rightarrow}_{\varnothing,[m]}.$$

Observe that we have a corresponding map:

$$\mathcal{G}^{I_n} \to \mathcal{G}^{I_{p(m)}}.$$

Indeed, there is a canonical map $I_{p(m)} \to I_n$, and we restrict along it to obtain $\mathcal{G}^{I_n} = \mathrm{Hom}(I_n, \mathcal{G}) \to \mathrm{Hom}(I_{p(m)}, \mathcal{G}) = \mathcal{G}^{I_{p(m)}}$.

This gives a map:

$$\mathcal{G}^{I_n} \to \mathcal{G}^{I_{p(m)}} \to \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing,[m]}$$

inducing (11.10.1) as desired.

Example 11.10.1. In Example 11.9.3, this is the obvious simplicial structure.

11.11. One easily finds that the simplicial groupoid $[n] \mapsto \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing,[n]}$ is a complete Segal space, and therefore defines a category $''\mathrm{Ran}^{un}_{\mathcal{G}}$.

Proposition 11.11.1. $''\mathrm{Ran}^{un}_{\mathcal{G}}$ *is canonically identified with* $\mathrm{Ran}^{un}_{\mathcal{G}}$*.*

Proof. For $\mathcal{G}$ a set, this follows from Example 11.9.3. But one clearly has that $\mathcal{G} \mapsto \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing,[n]}$ commutes with sifted colimits.

□

Remark 11.11.2. That $[n] \mapsto \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing,[n]}$ is a simplicial commutative monoid gives rise to the symmetric monoidal structure on $''\mathrm{Ran}^{un}_{\mathcal{G}}$. The above comparison with $\mathrm{Ran}^{un}_{\mathcal{G}}$ evidently extends to match up these two symmetric monoidal structures.

11.12. Before moving on, we record for later use some notation for the most important cases of the constructions. The reader may safely skip this section and refer back to it as necessary.

First, we follow [Gai11] is using the notations:

$$\mathrm{Ran}^{\to}_{\mathcal{G}} := \mathrm{Ran}^{\to}_{\mathcal{G},[1]} \, \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing} := \mathrm{Ran}^{\to}_{\mathcal{G},\varnothing,[1]} \, .$$

Our simplicial structure gives rise to the following natural maps:

We have the *left* and *right forgetful maps*:

$$\mathrm{Oblv}^{\leftarrow} : \mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing} \to \mathrm{Ran}_{\mathcal{G},\varnothing}$$

$$\mathrm{Oblv}^{\rightarrow} : \mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing} \to \mathrm{Ran}_{\mathcal{G},\varnothing}$$

normalized so that for $\mathcal{G}$ a set, we have:

$$\mathrm{Oblv}^{\leftarrow}(S \subseteq T \subseteq \mathcal{G}) = S$$

$$\mathrm{Oblv}^{\rightarrow}(S \subseteq T \subseteq \mathcal{G}) = T.$$

We also have the map:

$$\sigma : \mathrm{Ran}_{\mathcal{G},\varnothing} \to \mathrm{Ran}^{\rightarrow}_{\mathcal{G},\varnothing}$$

$$(S \subseteq \mathcal{G}) \mapsto (S \subseteq S \subseteq \mathcal{G})$$

(the formula being literally true for $\mathcal{G}$ a set, and given the obvious meaning otherwise). Note that σ serves as a simultaneous section to both $\mathrm{Oblv}^{\leftarrow}$ and $\mathrm{Oblv}^{\rightarrow}$.

11.13. **The disjoint loci.** It is convenient to record the following constructions before proceeding.

Recall that a *monomorphism* of groupoids is synonymous with "fully-faithful functor." In other words $\mathcal{G}_1 \to \mathcal{G}_2$ is a monomorphism if the morphism $\pi_0(\mathcal{G}_1) \to \pi_0(\mathcal{G}_2)$ is an injective morphism of sets, and the canonical morphism:

$$\mathcal{G}_1 \to \mathcal{G}_2 \underset{\pi_0(\mathcal{G}_2)}{\times} \pi_0(\mathcal{G}_1)$$

is an equivalence. Note that, for $\mathcal{G}_2$ fixed, the assignment $(\mathcal{G}_1 \hookrightarrow \mathcal{G}_2) \mapsto \pi_0(\mathcal{G}_1) \subseteq \pi_0(\mathcal{G}_2)$ defines a bijection between monomorphisms $\mathcal{G}_1 \to \mathcal{G}_2$ and subsets of $\pi_0(\mathcal{G}_2)$.

Returning to $\mathcal{G}$ our fixed, groupoid, define the monomorphism:

$$[\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}]_{disj} \to \mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}$$

by allowing those (homotopy) points in $\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}$ whose class in:

$$\pi_0(\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}) = \mathrm{Ran}_{\pi_0(\mathcal{G})} \times \mathrm{Ran}_{\pi_0(\mathcal{G})} = \{S, T \subseteq \pi_0(\mathcal{G}) \text{ pairs of finite subsets}\}$$

is given by a pair of disjoint subsets of $\pi_0(\mathcal{G})$.

On the other hand, for I, J two non-empty finite sets, we also have the monomorphism:

$$[\mathcal{G}^I \times \mathcal{G}^J]_{disj} \to \mathcal{G}^I \times \mathcal{G}^J \tag{11.13.1}$$

defined in the same way, or equivalently, as:

$$[\mathcal{G}^I \times \mathcal{G}^J]_{disj} := (\mathcal{G}^I \times \mathcal{G}^J) \underset{\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}}{\times} [\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}]_{disj}.$$

We have the canonical morphism:

$$\underset{I,J \in \mathsf{fSet}^{op}}{\mathrm{colim}} [\mathcal{G}^I \times \mathcal{G}^J]_{disj} \to [\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}]_{disj}. \tag{11.13.2}$$

Lemma 11.13.1. *The morphism* (11.13.2) *is an equivalence.*

Proof. Immediate from the universality of colimits in PreStk.

□

Variant 11.13.2. Because the 1-full subcategory of $\mathrm{Ran}^{un}_{\mathcal{G}}$ formed by invertible morphisms identifies with $\mathrm{Ran}_{\mathcal{G}}$, we obtain the corresponding full subcategory $[\mathrm{Ran}^{un}_{\mathcal{G}} \times \mathrm{Ran}^{un}_{\mathcal{G}}]_{disj}$ of $\mathrm{Ran}^{un}_{\mathcal{G}} \times \mathrm{Ran}^{un}_{\mathcal{G}}$.

11.14. **Lax prestacks.** We will digress temporarily to introduce the following convenient formalism.

Definition 11.14.1. A *lax prestack* is an (accessible) functor $\mathsf{AffSch}^{op} \to \mathsf{Cat}$.

We denote the 2-category of lax prestacks by PreStk^{lax}. We have an obvious embedding $\mathsf{PreStk} \hookrightarrow \mathsf{PreStk}^{lax}$ that admits a right adjoint we will denote by $\mathcal{Y} \mapsto \mathcal{Y}^{\mathsf{PreStk}}$. Note that for $\mathcal{Y}$ a lax prestack and S an affine scheme, $\mathcal{Y}^{\mathsf{PreStk}}(S)$ is computed as the maximal subgroupoid of $\mathcal{Y}(S)$.

We say a lax prestack is locally almost of finite type if it is obtained by left Kan extension from AffSch_{laft}.

11.15. For any lax prestack $\mathcal{Y}$, we can make sense of $\mathsf{QCoh}(\mathcal{Y})$ as the category of natural transformations $\mathcal{Y} \to \mathsf{QCoh} : \mathsf{AffSch}^{op} \to \mathsf{Cat}$.

Remark 11.15.1. Because we require that $\mathcal{Y}$ take values in small categories, $\mathsf{QCoh}(\mathcal{Y})$ is locally small.

If $\mathcal{Y}$ is locally almost of finite type, then we similarly have categories $\mathsf{IndCoh}(\mathcal{Y})$ and $D(Y)$. Note that formation of QCoh, IndCoh and D are contravariant in Y, and we denote restriction functors in the usual ways.

Note that if $\mathcal{Y}$ is a usual prestack, i.e., $\mathcal{Y}$ takes values in $\mathsf{Gpd} \subseteq \mathsf{Cat}$, then the above notions coincide with the usual ones.

11.16. Somewhat more explicitly, e.g. a quasi-coherent sheaf $\mathcal{F}$ on a lax prestack $\mathcal{Y}$ is an assignment:

$$\begin{gathered}\Big(f : S \to \mathcal{Y}, S \in \mathsf{AffSch}\Big) \mapsto f^*(\mathcal{F}) \in \mathsf{QCoh}(S) \\ \Big(T \xrightarrow{g} S \xrightarrow{f} \mathcal{Y}, S, T \in \mathsf{AffSch}) \mapsto g^* f^*(\mathcal{F}) \simeq (f \circ g)^*(\mathcal{F}) \\ \Big(\varepsilon : f \to g \in \mathcal{Y}(S)\Big) \mapsto f^*(\mathcal{F}) \to g^*(\mathcal{F}).\end{gathered} \tag{11.16.1}$$

11.17. The notion of sheaf of categories on a lax prestack is somewhat more subtle: some 2-categorical problems play a role.

Here is what we want to model:

As in §10.5.3, for $\mathcal{Y}$ a lax prestack we want to define two categories $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}$ and $\mathsf{ShvCat}_{/\mathcal{Y}}$ of sheaves of categories on $\mathcal{Y}$. The objects are the same, but $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive} \subseteq \mathsf{ShvCat}_{/\mathcal{Y}}$ is merely a 1-full subcategory.

Sheaves of categories on $\mathcal{Y}$ admit a description as in (11.16.1). Then morphisms $\mathsf{C} \to \mathsf{D}$ in $\mathsf{ShvCat}_{/\mathcal{Y}}$ amount to the data:

$$\Big(f : S \to \mathcal{Y}, S \in \mathsf{AffSch}\Big) \mapsto \eta_f : f^*(\mathsf{C}) \to f^*(\mathsf{D})$$

$$\Big(\varepsilon : f \to g \in \mathcal{Y}(S)\Big) \mapsto \begin{tikzcd} f^*(\mathsf{C}) \arrow[r] \arrow[d, "\eta_f"] & g^*(\mathsf{C}) \arrow[d, "\eta_g"] \\ f^*(\mathsf{D}) \arrow[r] \arrow[ur, Rightarrow] & g^*(\mathsf{D}) \end{tikzcd}$$

$$\Big(T \xrightarrow{g} S \xrightarrow{f} \mathcal{Y}, S, T \in \mathsf{AffSch}) \mapsto g^*(\eta_f) \simeq \eta_{f\circ g}.$$

Here the notation on the second line means that we specify a 2-morphism between the compositions:

$$\Big(f^*(\mathsf{C}) \to f^*(\mathsf{D}) \to g^*(\mathsf{D})\Big) \Longrightarrow \Big(f^*(\mathsf{C}) \to g^*(\mathsf{C}) \to g^*(\mathsf{D})\Big).$$

A morphism as above is a morphism in $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}$ if and only if these natural transformations are natural equivalences.

Example 11.17.1. For $\mathsf{C} = \mathsf{D} = \mathsf{QCoh}_{\mathcal{Y}}$, we have the canonical equivalence:

$$\mathsf{Hom}_{\mathsf{ShvCat}_{/\mathcal{Y}}}(\mathsf{QCoh}_{\mathcal{Y}}, \mathsf{QCoh}_{\mathcal{Y}}) = \mathsf{QCoh}(\mathcal{Y}).$$

Indeed, this is the main motivation for constructing $\mathsf{ShvCat}_{/\mathcal{Y}}$ as we have.

By comparison, if $\mathcal{Y}^{inv}$ is the prestack obtained from $\mathcal{Y}$ by termwise inverting all arrows, then we have:

$$\mathsf{Hom}_{\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}}(\mathsf{QCoh}_{\mathcal{Y}}, \mathsf{QCoh}_{\mathcal{Y}}) = \mathsf{QCoh}(\mathcal{Y}^{inv}).$$

Here the induced functor $\mathsf{QCoh}(\mathcal{Y}^{inv}) \to \mathsf{QCoh}(\mathcal{Y})$ is given by pullback along $\mathcal{Y} \to \mathcal{Y}^{inv}$, and is fully-faithful.

Remark 11.17.2. We will give a precise construction of the above in what follows. The reader who can take the above on faith may safely skip ahead to §11.20.

11.18. **Lax functors.** Given a category[28] $\mathcal{C}$ and a 2-category $\mathcal{D}$, there is a 1-category $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$, the *category of lax functors* $\mathcal{C} \to \mathcal{D}$, described as follows. Objects of $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ are functors $F : \mathcal{C} \to \mathcal{D}$. Morphisms (alias: *lax natural transformations*) $\eta : F \to G$ are given by data of natural maps $\eta_X : F(X) \to G(X)$ defined for every $X \in \mathcal{C}$, plus for every $f : X \to Y$ in $\mathcal{C}$, we are given a 2-morphism in $\mathcal{D}$ between the compositions:

$$\begin{gathered}\Big(F(X) \overset{\eta_X}{\to} G(X) \overset{G(f)}{\to} G(Y)\Big) \\ \Big\Downarrow \\ \Big(F(X) \overset{F(f)}{\to} F(Y) \overset{\eta_Y}{\to} G(Y)\Big).\end{gathered} \tag{11.18.1}$$

For the identity map $\mathrm{id}_X : X \to X$, this natural transformation should be the tautological 2-isomorphism. Of course, the data above are required to be natural in all variables, compatible with categorical operations (e.g., composition), all understood in the natural meaning given by higher category theory.

Let $\mathcal{D}^{1-cat}$ denote the 1-category underlying $\mathcal{D}$, in which we only allow invertible 2-morphisms. Note that $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ contains $\mathsf{Hom}(\mathcal{C}, \mathcal{D}^{1-cat})$ as a 1-full subcategory, where objects are the same but morphisms require the 2-morphism (11.18.1) to be invertible.

Remark 11.18.1. If the morphism $f : X \to Y \in \mathcal{C}$ above is invertible, then the natural transformation (11.18.1) is necessarily invertible. Therefore, $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D}) = \mathsf{Hom}(\mathcal{C}, \mathcal{D}^{1-cat})$ if $\mathcal{C}$ is a groupoid.

[28]More generally, a 2-category can be allowed, but we will not use the construction in this generality.

Remark 11.18.2. Formation of $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ is appropriately functorial in $\mathcal{C}$ and $\mathcal{D}$. The best way to say this precisely is to use the definition for $\mathcal{C}$ allowed to be a 2-category, and to say that we have a certain 2-category of 2-categories where the category of functors $\mathcal{C} \to \mathcal{D}$ is taken to be $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$.

Remark 11.18.3. More generally, suppose that $\mathcal{I}$ is an indexing category and consider objects $i \mapsto \mathcal{C}_i$ and $i \mapsto \mathcal{D}_i$ of $\mathsf{Hom}(\mathcal{I}, 2\text{–}\mathsf{Cat})$. Then we have a category $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ constructed in the same way as above, where roughly, objects of $\mathsf{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ are compatible functors $\mathcal{C}_i \to \mathcal{D}_i$, and morphisms are compatible systems of lax natural transformations.

One can alternatively recover this notion from the one presented above (in the case $\mathcal{I} = *$) by using the Grothendieck construction; we do not pursue this here.

11.19. In the framework of Remark 11.18.3, for $\mathcal{Y}$ a lax prestack, we define $\mathsf{ShvCat}_{/\mathcal{Y}}$ as the category of lax morphisms $\mathcal{Y} \to \mathsf{ShvCat}_{/-}$, where $\mathsf{ShvCat}_{/-}$ is the functor $\mathsf{AffSch}^{op} \to 2\text{–}\mathsf{Cat}$ sending S to $\mathsf{ShvCat}_{/S}$.

We define $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}$ as the category of usual functors $\mathcal{Y} \to \mathsf{ShvCat}_{/-}$.

Remark 11.19.1. Tautologically, $\mathsf{ShvCat}_{/\mathcal{Y}}$ contains $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}$ as a 1-full subcategory with the same underlying groupoid, and therefore we may speak without hesitation about a sheaf of categories on $\mathcal{Y} \in \mathsf{PreStk}^{lax}$: the only ambiguity is in speaking of morphisms of sheaves of categories. Of course, if $\mathcal{Y}$ is a usual prestack then this issue disappears.

Example 11.19.2. We have the obvious sheaf of categories $\mathsf{QCoh}_{\mathcal{Y}}$ on $\mathcal{Y}$.

Remark 11.19.3. Note that both $\mathsf{ShvCat}_{/\mathcal{Y}}$ and $\mathsf{ShvCat}_{/\mathcal{Y}}^{naive}$ admit obvious 2-categorical enhancements, and we will sometimes abuse notation by denoting the corresponding 2-categories by the same notation.

Even better, they both are enriched over DGCat_{cont}. We abuse notation in letting Hom also denote the enriched Hom over DGCat_{cont}.

By Example 11.17.1, for $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$, we define $\Gamma(\mathcal{Y}, \mathsf{C}) \in \mathsf{DGCat}_{cont}$ as:

$$\Gamma(\mathcal{Y}, \mathsf{C}) := \mathsf{Hom}_{\mathsf{ShvCat}_{/\mathcal{Y}}}(\mathsf{QCoh}_{\mathcal{Y}}, \mathsf{C}).$$

11.20. For every lax prestack $\mathcal{Y}$, recall that $\mathcal{Y}^{\mathsf{PreStk}}$ denotes the (non-lax) prestack underlying $\mathcal{Y}$.

We have the following obvious lemma:

Lemma 11.20.1. *The functors:*

$$\mathsf{QCoh}(\mathcal{Y}) \to \mathsf{QCoh}(\mathcal{Y}^{\mathsf{PreStk}})$$

$$\mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Y}^{\mathsf{PreStk}}}$$

of restriction along the map:

$$\mathcal{Y}^{\mathsf{PreStk}} \to \mathcal{Y}$$

are conservative.

11.21. **Ran space for prestacks.** If $\mathcal{X}$ is a prestack, then we obtain the prestack $\mathrm{Ran}_{\mathcal{X}}$ defined by

$$\mathrm{Ran}_{\mathcal{X}}(S) := \mathrm{Ran}_{\mathcal{X}(S)} \in \mathsf{Gpd}$$

for $S \in \mathsf{AffSch}$, and similarly, we have the prestack $\mathrm{Ran}_{\mathcal{X},\varnothing} = \mathrm{Ran}_{\mathcal{X}} \coprod *$ and the lax prestack $\mathrm{Ran}_{\mathcal{X}}^{un}$.

Each of $\mathrm{Ran}_{\mathcal{X},\varnothing}$ and $\mathrm{Ran}_{\mathcal{X}}^{un}$ admits a commutative monoid structure defined by add, and $\mathrm{Ran}_{\mathcal{X}}$ admits a commutative semigroup structure.

Note that the prestack $\mathrm{Ran}_{\mathcal{X}}^{un,\mathsf{PreStk}}$ underlying $\mathrm{Ran}_{\mathcal{X}}^{un}$ is $\mathrm{Ran}_{\mathcal{X},\varnothing}$.

Remark 11.21.1. We obtain prestacks $\mathrm{Ran}_{\mathcal{X}}^{\rightarrow}$ and $\mathrm{Ran}_{\mathcal{X},\varnothing}^{\rightarrow}$ by the same procedure, referring to §11.12 for the corresponding construction for groupoids. We use the notations $\mathrm{Oblv}^{\leftarrow}$ and $\mathrm{Oblv}^{\rightarrow}$ in the same way as in *loc. cit.*

We recall that $\mathrm{Ran}_{\mathcal{X}}^{\to}$ should be thought of as parametrizing pairs $S \subseteq T \subseteq \mathcal{X}$ of finite sets, and that $\mathrm{Oblv}^{\leftarrow}$ is the forgetful map corresponding to the S-variable, while $\mathrm{Oblv}^{\to}$ is the forgetful map corresponding to the T-variable.

11.22. By definition, a *unital quasi-coherent sheaf* on $\mathrm{Ran}_{\mathcal{X}}$ is a quasi-coherent sheaf on $\mathrm{Ran}_{\mathcal{X}}^{un}$. Similarly, we have the notion of *unital sheaf of categories* over $\mathrm{Ran}_{\mathcal{X}}$.

For X a scheme of finite type, we say a *unital D-module* on Ran_X is a quasi-coherent sheaf on $\mathrm{Ran}_{X_{dR}}^{un} = (\mathrm{Ran}_X^{un})_{dR}$, and similarly for *unital crystal of categories* on $\mathrm{Ran}_{\mathcal{X}}$.

Notation 11.22.1. For, say, C a unital sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$, we generally do not differentiate in our notation between the underlying sheaves of categories on $\mathrm{Ran}_{\mathcal{X}}^{un}$ and $\mathrm{Ran}_{\mathcal{X},\varnothing}$, leaving the distinction to context or to some explicit signifier where necessary.

11.23. We will need the following general constructions with unital sheaves of categories on Ran space.

For such C a unital sheaf of categories, we have a canonical *unit* or *fusion* morphism:

$$\mathfrak{Fus} = \mathfrak{Fus}_{\mathsf{C}} : \mathrm{Oblv}^{\leftarrow,*}(\mathsf{C}) \to \mathrm{Oblv}^{\to,*}(\mathsf{C}) \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X},\varnothing}^{\to}} \tag{11.23.1}$$

where the relevant notation was introduced in Remark 11.21.1.

Remark 11.23.1. Of course, such a map exists for unital quasi-coherent sheaves, D-modules, etc.

The following hypothesis is natural to require on the unit of a chiral category.

Definition 11.23.2. The sheaf of categories C is *adj-unital* if the unit map $\mathfrak{Fus}_{\mathsf{C}}$ admits a right adjoint in the 2-category $\mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X},\varnothing}^{\to}}$.

11.24. For C as above, let $\mathsf{C}_{\varnothing} \in \mathsf{DGCat}_{cont}$ denote the fiber of C along the map $\mathrm{Spec}(k) \xrightarrow{\varnothing} \mathrm{Ran}_{\mathcal{X}}^{un}$. Suppose that we are given an identification $\mathsf{C}_{\varnothing} \simeq \mathsf{Vect}$.

Applying the restriction functor for sheaves of categories on $\mathrm{Ran}_{\mathcal{X}}^{un}$ to $\mathrm{Ran}_{\mathcal{X}}$, the map $\mathfrak{Fus}_{\mathsf{C}}$ produces a canonical map:

$$\mathsf{QCoh}_{\mathrm{Ran}_{\mathcal{X}}} \to \mathsf{C} \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}}$$

or equivalently, an object $\mathrm{unit}_{\mathsf{C}}$ of $\Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathsf{C})$.

Definition 11.24.1. The resulting object $\mathrm{unit}_{\mathsf{C}}$ is called the *unit object* of the unital sheaf of categories C.

Terminology 11.24.2. According to Corollary 11.6.2, a unital sheaf of categories is equivalent to a system (in the homotopical sense) of sheaves of categories $\mathsf{C}_{\mathcal{X}^I} \in \mathsf{ShvCat}_{/\mathcal{X}^I}$ defined for every finite set I, plus compatible morphisms:

$$\Delta_f^*(\mathsf{C}_{\mathcal{X}^I}) \to \mathsf{C}_{\mathcal{X}^J}$$

for every $f : I \to J$, with $\Delta_f : \mathcal{X}^J \to \mathcal{X}^I$ the induced map, and such that when f is a surjection this map is an equivalence.

For a pair of finite sets I and J, the inclusion $I \hookrightarrow I \coprod J$ therefore defines a map:

$$\mathsf{C}_{\mathcal{X}^I} \boxtimes \mathsf{QCoh}_{\mathcal{X}^J} \to \mathsf{C}_{\mathcal{X}^{I \coprod J}}$$

that we will also refer to as a *unit functor*.

11.25. Let $\mathcal{Y} \in \mathsf{PreStk}^{lax}$ be fixed. As in §19.4, we say that a functor $\mathsf{D} \to \mathsf{C}$ in $\mathsf{ShvCat}_{/\mathcal{Y}}$ is *(locally) fully-faithful* if for every affine scheme S and map $f : S \to \mathcal{Y}$ the corresponding functor $\Gamma(S, f^*(\mathsf{D})) \to \Gamma(S, f^*(\mathsf{C}))$ is fully-faithful.

The following lemma records the immediate consequences of the definition.

Lemma 11.25.1. *Let $\mathcal{Y}$ be a lax prestack.*

(1) A morphism $\mathsf{D} \to \mathsf{C}$ in $\mathsf{ShvCat}_{/\mathcal{Y}}$ is fully-faithful if and only if its restriction to $\mathcal{Y}^{\mathsf{PreStk}}$ is.

(2) *Every fully-faithful functor is a monomorphism in the category* $\mathsf{ShvCat}_{/\mathcal{Y}}$. *Moreover, given* $\mathsf{D} \to \mathsf{C}$ *fully-faithful and a map* $\varphi : \mathsf{E} \to \mathsf{C}$, *to see if* φ *factors through* D *it suffices to check this after restriction to* $\mathcal{Y}^{\mathsf{PreStk}}$.

(3) *For* $\mathsf{D} \to \mathsf{C}$ *fully-faithful, the induced functor:*

$$\Gamma(\mathcal{Y}, \mathsf{D}) \to \Gamma(\mathcal{Y}, \mathsf{C})$$

is fully-faithful.

(4) *Fully-faithful functors are preserved under pullbacks* $\mathcal{Y}' \to \mathcal{Y}$.

(5) *Given* $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ *with restriction* $\overline{\mathsf{C}} \in \mathsf{ShvCat}_{/\mathcal{Y}^{\mathsf{PreStk}}}$, *the datum of a fully-faithful functor* $\mathsf{D} \to \mathsf{C}$ *in* $\mathsf{ShvCat}^{naive}_{/\mathcal{Y}}$ *is equivalent to the datum of a fully-faithful embedding:*

$$\overline{\mathsf{D}} \hookrightarrow \overline{\mathsf{C}} \in \mathsf{ShvCat}_{/\mathcal{Y}^{\mathsf{PreStk}}}$$

such that, for every test scheme S *and pair of morphisms* $f, g : S \to \mathcal{Y}$ *with a 2-morphism* $\varepsilon : f \to g \in \mathcal{Y}(S)$, *the induced functor:*

$$\Gamma\big(S, f^*(\overline{\mathsf{C}})\big) \to \Gamma\big(S, g^*(\overline{\mathsf{C}})\big)$$

maps $\Gamma(S, f^*(\overline{\mathsf{D}}))$ *to* $\Gamma(S, g^*(\overline{\mathsf{D}}))$.

11.26. Next, we give a general construction of unital sheaves of categories that is useful, for example, in dealing with the geometric Whittaker models. The reader without interest in such applications may safely skip this material and go ahead to §11.27.

The following result is somewhat technical and perhaps difficult to interpret. We present it in a more down-to-earth way in Remark 11.26.2.

Proposition-Construction 11.26.1. *Suppose that* C *is an adj-unital sheaf of categories on* $\mathrm{Ran}_{\mathfrak{X}}$, D *is a sheaf of categories on* $\mathrm{Ran}_{\mathfrak{X},\varnothing}$, *and we are given a fully-faithful functor:*

$$\mathsf{D} \hookrightarrow \mathsf{C} \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X},\varnothing}}.$$

Suppose that we have $\mathsf{D}_\varnothing \xrightarrow{\simeq} \mathsf{C}_\varnothing \simeq \mathsf{Vect}$*, where the former is induced by the fully-faithful functor and the latter is an extra piece of structure.*

Let:

$$\mathfrak{Fus}_{\mathsf{C}}^R : \mathrm{Oblv}^{\rightarrow,*}(\mathsf{C}) \to \mathrm{Oblv}^{\leftarrow,*}(\mathsf{C}) \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X},\varnothing}^{\rightarrow}}$$

denote the right adjoint to the functor $\mathfrak{Fus}_{\mathsf{C}}$ *from* (11.23.1).

Suppose that $\mathfrak{Fus}_{\mathsf{C}}^R$ *sends* $\mathrm{Oblv}^{\rightarrow,*}(\mathsf{D})$ *into* $\mathrm{Oblv}^{\leftarrow,*}(\mathsf{D}) \subseteq \mathrm{Oblv}^{\leftarrow,*}(\mathsf{C})$.

Suppose, moreover, that the corresponding functor:

$$\mathrm{Oblv}^{\rightarrow,*}(\mathsf{D}) \to \mathrm{Oblv}^{\leftarrow,*}(\mathsf{D}) \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X},\varnothing}^{\rightarrow}}$$

admits a left adjoint $\mathfrak{Fus}_{\mathsf{D}}$*.*

Then D *inherits a canonical unital structure such that the functor* $\mathsf{D} \to \mathsf{C}$ *upgrades to a functor of unital sheaves of categories on* $\mathrm{Ran}_{\mathcal{X}}$*. The unit for this structure is given by* $\mathfrak{Fus}_{\mathsf{D}}$*.*

Remark 11.26.2. We use the notation of §10.5 to speak about unital sheaves of categories. For compatibility with *loc. cit.*, we use the notation X in place of $\mathcal{X}$, and $\mathcal{C}$ and $\mathcal{D}$ in place of C and D.

The question Proposition-Construction 11.26.1 addresses is, given $\mathcal{C}$ a unital sheaf of categories and a (non-unital) subcategory $\mathcal{D}$, when does $\mathcal{D}$ inherit a unital structure?

One easy answer: if the unit maps preserve $\mathcal{D}$. I.e., in our heuristic, this says that for every embedding:

$$\{x_1, \ldots, x_n\} \subseteq \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\} \subseteq X$$

we have:

$$
\begin{tikzcd}
\mathcal{D}_{x_1,\ldots,x_n} \arrow[r, dotted] \arrow[d, hook] & \mathcal{D}_{x_1,\ldots,x_m} \arrow[d, hook] \\
\mathcal{C}_{x_1,\ldots,x_n} \arrow[r] & \mathcal{C}_{x_1,\ldots,x_m}.
\end{tikzcd}
\tag{11.26.1}
$$

Proposition-Construction 11.26.1 gives a less obvious situation in which $\mathcal{D}$ still inherits a unit structure.

It asks the following:

- The functors $\mathcal{C}_{x_1,\ldots,x_n} \to \mathcal{C}_{x_1,\ldots,x_m}$ should admit right adjoints.
- The right adjoints $\mathcal{C}_{x_1,\ldots,x_m} \to \mathcal{C}_{x_1,\ldots,x_n}$ should take $\mathcal{D}_{x_1,\ldots,x_m}$ to $\mathcal{D}_{x_1,\ldots,x_n}$, i.e., we ask for the mirror image of the diagram (11.26.1).
- The resulting functors $\mathcal{D}_{x_1,\ldots,x_m} \to \mathcal{D}_{x_1,\ldots,x_n}$ should admit left adjoints.

In this case, $\mathcal{D}$ will admit a unit structure with unit maps:

$$\mathcal{D}_{x_1,\ldots,x_n} \to \mathcal{D}_{x_1,\ldots,x_m}$$

given by these left adjoints.

We emphasize that this does not at all force the diagram (11.26.1) to commute (and it will not for Whittaker sheaves!): this is exactly the difference between $\mathsf{ShvCat}_{/-}$ and $\mathsf{ShvCat}_{/-}^{naive}$.

Warning 11.26.3. The heuristic of Remark 11.26.2 sweeps an important point under the rug: it is not enough to check these properties pointwise — one needs to verify them as the points move and are allowed to collide. In fact, §7 exists expressly to make such a verification that is obvious pointwise.

Proof of Proposition-Construction 11.26.1. We freely use the description of unital Ran space from §11.7. We also assume the 2-categorical formalism of [GR14], which allows us to functorially pass to adjoints.

Let $\mathrm{Ran}_{\mathfrak{X}}^{un,op}$ denote the lax prestack in which we take opposite categories at every point.

The adj-unital condition on C produces a sheaf of categories $\widetilde{\mathsf{C}}$ on $\mathrm{Ran}_{\mathfrak{X}}^{un,op}$ with "fusion" given by (11.26.1).

Then Lemma 11.25.1 produces a sheaf of categories $\widetilde{\mathsf{D}}$ on $\mathrm{Ran}_{\mathfrak{X}}^{un,op}$ with a fully-faithful functor:

$$\widetilde{\mathsf{D}} \to \widetilde{\mathsf{C}} \in \mathsf{ShvCat}^{naive}_{/\mathrm{Ran}_{\mathfrak{X}}^{un,op}}.$$

Finally, passing to left adjoints, we obtain the desired result.

$\square$

11.27. We define $[\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj}$ and $[\mathrm{Ran}_{\mathfrak{X}}^{un} \times \mathrm{Ran}_{\mathfrak{X}}^{un}]_{disj}$ as prestacks termwise by §11.13.

Tautologically, the morphisms:

$$\begin{gathered} [\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj} \to \mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}} \\ [\mathrm{Ran}_{\mathfrak{X}}^{un} \times \mathrm{Ran}_{\mathfrak{X}}^{un}]_{disj} \to \mathrm{Ran}_{\mathfrak{X}}^{un} \times \mathrm{Ran}_{\mathfrak{X}}^{un} \end{gathered} \tag{11.27.1}$$

are termwise fully-faithful.

11.28. Let PreStk_{corr} and $\mathsf{PreStk}_{corr}^{lax}$ denote the categories of correspondences associated with the complete categories PreStk and PreStk^{lax}. We regard these categories as equipped with the usual symmetric monoidal structures computed objectwise by Cartesian products.

Because the morphisms (11.27.1) are monomorphisms, and similarly for the variant for n-fold products of Ran space, we have canonical non-unital commutative algebra structures on $\mathrm{Ran}_{\mathfrak{X}}$ in PreStk_{corr} and $\mathrm{Ran}_{\mathfrak{X}}^{un}$ in $\mathsf{PreStk}_{corr}^{lax}$, where the multiplication maps are defined by the correspondences:

$$\begin{tikzcd}[column sep=small] & {[\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj}} \arrow[dl] \arrow[dr] & & & {[\mathrm{Ran}^{un}_{\mathfrak{X}} \times \mathrm{Ran}^{un}_{\mathfrak{X}}]_{disj}} \arrow[dl] \arrow[dr] & \\ \mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}} & & \mathrm{Ran}_{\mathfrak{X}} & \mathrm{Ran}^{un}_{\mathfrak{X}} \times \mathrm{Ran}^{un}_{\mathfrak{X}} & & \mathrm{Ran}^{un}_{\mathfrak{X}}\,. \end{tikzcd}$$

For $\mathrm{Ran}^{un}_{\mathfrak{X}}$, this commutative algebra structure is unital, with the obvious unit.

We let add_{disj} denote each of the right arrows in the correspondences above.

For emphasis, we will write $\mathrm{Ran}^{ch}_{\mathfrak{X}}$ and $\mathrm{Ran}^{un,ch}_{\mathfrak{X}}$ for the resulting commutative algebras, referring to the multiplication as the *chiral product*.

We will also denote by $\mathrm{Ran}^{*}_{\mathfrak{X},\varnothing}$ the commutative monoid in PreStk given by $\mathrm{Ran}_{\mathfrak{X},\varnothing}$ with the multiplication add, and similarly for $\mathrm{Ran}^{*}_{\mathfrak{X}} \in \mathsf{ComAlg}_{non-unital}(\mathsf{PreStk})$ and $\mathrm{Ran}^{un,*}_{\mathfrak{X}} \in \mathsf{ComAlg}(\mathsf{PreStk}^{lax})$.

Remark 11.28.1. For a more detailed approach on the construction of the chiral product, see §14.7.

12. Multiplicative sheaves and correspondences

12.1. In this section, we provide a general language that we will apply in §13 to the Ran space to obtain the theory of chiral categories.

12.2. The material of this section is mostly a matter of organization of the type that is not typically needed outside of homotopical algebra.

Therefore, we give an extended introduction to its contents in §12.3-12.8.

12.3. **Algebras under correspondences.** Our basic format is a (lax) prestack $\mathcal{S}$ with a *commutative* algebra structure under correspondences.

Concretely, this means that we are given *multiplication* and *unit* correspondences:

$$\begin{tikzcd}[column sep=small] & \mathrm{mult}_{\mathcal{S}} \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & & & & \mathrm{unit}_{\mathcal{S}} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ \mathcal{S} \times \mathcal{S} & & \mathcal{S} & \text{and} & * & & \mathcal{S} \end{tikzcd}$$

satisfying various associativity and commutativity conditions. E.g., commutativity here says that $\mathrm{mult}_{\mathcal{S}}$ is given a $\mathbb{Z}/2\mathbb{Z}$-action with m_1 being $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to switching the two factors of the target, and m_2 being $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to the trivial action on $\mathcal{S}$.

Example 12.3.1. As in §10.9, $\mathrm{Ran}_{\mathfrak{X},\varnothing}$ and $\mathrm{Ran}_{\mathfrak{X}}^{un}$ admit this structure using the loci of disjoint pairs of s.

12.4. **Multiplicative sheaves of categories.** Given such a datum, we define in §12.21 the notion of *multiplicative sheaf of categories* on $\mathcal{S}$.

Up to homotopic problems, this means that we give a sheaf of categories Ψ on $\mathcal{S}$ along with isomorphisms:

$$m_1^*(\Psi) \simeq m_2^*(\Psi) \in \mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}}$$

$$\mathsf{QCoh}_{\mathrm{unit}_{\mathcal{S}}} \simeq e_2^*(\Psi) \in \mathsf{ShvCat}_{/\,\mathrm{unit}_{\mathcal{S}}}$$

with these isomorphisms satisfying associativity and commutativity.

Remark 12.4.1. We also introduce a notion of *weakly* multiplicative sheaf of categories, where e.g. we are only required to specify a morphism:

$$m_1^*(\Psi) \to m_2^*(\Psi)$$

12.5. **Multiplicative sheaves.** Given Ψ a multiplicative sheaf of categories on $\mathcal{S}$, there is a notion of *multiplicative object* ψ of Ψ.

This is an object:

$$\psi \in \Gamma(\mathcal{S}, \Psi)$$

with isomorphisms:

$$m_1^*(\psi) \simeq m_2^*(\psi) \in \Gamma(\mathrm{mult}_{\mathcal{S}}, m_1^*(\Psi)) \simeq \Gamma(\mathrm{mult}_{\mathcal{S}}, m_2^*(\Psi))$$

$$\mathcal{O}_{\mathrm{unit}_{\mathcal{S}}} \simeq e_2^*(\psi) \in \mathsf{QCoh}(\mathrm{unit}_{\mathcal{S}}) \simeq \Gamma(\mathrm{mult}_{\mathcal{S}}, e_2^*(\Psi)).$$

Remark 12.5.1. As in Remark 12.4.1, there is a similar notion of weakly multiplicative object of a weakly multiplicative sheaf of categories.

12.6. **Modules.** There are variants of the above notions for modules. Let $\mathcal{S}$, Ψ, and ψ be as above.

A *module space* for $\mathcal{S}$ is a (lax) prestack $\mathcal{M}$ which is a module for $\mathcal{S}$ under correspondences, so we are in particular given an *action* correspondence:

$$\begin{tikzcd} & \mathrm{act}_{\mathcal{M}} \arrow[dl, "\mathrm{act}_1"'] \arrow[dr, "\mathrm{act}_2"] & \\ \mathcal{S} \times \mathcal{M} & & \mathcal{M} \end{tikzcd}$$

defining an associative and unital action of $\mathcal{S}$ in the sense of correspondences.

We can then speak about Ψ-module categories on $\mathcal{M}$: this is the datum of a sheaf of categories Φ being a *module for* Ψ. This means that we are given isomorphisms:

$$\mathrm{act}_1^*(\Psi \boxtimes \Phi) \simeq \mathrm{act}_2^*(\Phi) \in \mathsf{ShvCat}_{/\,\mathrm{act}_{\mathcal{M}}}$$

satisfying associativity and unitality.

In this case, we can also speak about *modules for* ψ. Such a datum is an object $\varphi \in \Gamma(\mathcal{M}, \Phi)$ equipped with associative and unital isomorphisms:

$$\mathrm{act}_1^*(\psi \boxtimes \phi) \simeq \mathrm{act}_2^*(\phi) \in \Gamma(\mathrm{act}_{\mathcal{M}}, \mathrm{act}_1^*(\Psi \boxtimes \Phi))) \simeq \Gamma(\mathrm{act}_{\mathcal{M}}, \mathrm{act}_2^*(\Phi)).$$

Remark 12.6.1. The above is an indication that multiplicative sheaves can be defined in much more generality: they can be defined for any colored operad. Then, e.g., taking the colored operad of choice to be the operad for a commutative algebra and a module over it, one recovers the above.

12.7. Finally, in §12.31-12.32 we mention that subcategories and quotients of multiplicative sheaves of categories inherit such structures when certain obvious conditions are satisfied: for subcategories, the multiplicative isomorphisms should induce an isomorphism between the subcategories, and for quotient categories, there is an ideal-type condition to be satisfied.

We refer to *loc. cit.*, where these conditions are spelled out completely (and in a way that should be easy to read given the above).

12.8. At this point, the reader may safely skip ahead to §13.

12.9. **A Grothendieck construction among correspondences.** The major technical tool we will use is the following construction:

Given a functor[29] $F : \mathcal{I}^{op} \to \mathsf{Cat}_{pres}$, we will define a certain category $\mathsf{Groth}_{corr}(F)$, described below.

This construction will play a key role in setting up the theory of multiplicative sheaves in the correspondence setting. With that said, the reader should be fine understanding the heuristic description below and skipping ahead to §12.18 to see how it is actually used (which we do not to explain presently).

$\mathsf{Groth}_{corr}(F)$ has the following properties:

- Objects of $\mathsf{Groth}_{corr}(F)$ are pairs $i \in \mathcal{I}$ and $X_i \in F(i)$.
- Morphisms $(i, X_i) \to (j, X_j)$ in $\mathsf{Groth}_{corr}(F)$ are given by the data of a correspondence:

[29]The covariance of the functor F is for convenience: it is what occurs in practice for us, and the author personally finds the notation easier to follow this way.

$$\begin{tikzcd} & h \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ i & & j \end{tikzcd}$$

in $\mathcal{I}$, and a morphism:[30]

$$\varphi_{ij} : \alpha(X_i) \to \beta(X_j) \in F(h).$$

- To compute compositions, we compose the correspondences in $\mathcal{I}$ in the usual way:

$$\begin{tikzcd} & & h \underset{j}{\times} h' \arrow[dl, "\varepsilon"'] \arrow[dr, "\eta"] & & \\ & h \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & h' \arrow[dl, "\Upsilon"'] \arrow[dr, "\delta"] & \\ i & & j & & k \end{tikzcd}$$

and take the induced map:

$$\varepsilon\alpha(X_i) \xrightarrow{\varepsilon(\varphi_{ij})} \varepsilon\beta(X_j) = \eta\Upsilon(X_j) \xrightarrow{\eta(\varphi_{jk})} \eta\delta(X_k)$$

in $F(h \times_j h')$.

Remark 12.9.1. In 12.15-12.16, we will explain that if $\mathcal{I}$ is equipped with a symmetric monoidal structure and F is lax symmetric monoidal, then $\mathsf{Groth}_{corr}(F)$ inherits a natural symmetric monoidal structure.

12.10. Suppose that $\mathcal{I}$ is a category equipped with a functor $F : \mathcal{I}^{op} \to \mathsf{Cat}_{pres}$, where we recall that Cat_{pres} denotes the category of cocomplete categories under functors commuting with all colimits.

[30]Our notation follows the convention of §2.10 here.

Lemma 12.10.1. *If $\mathcal{I}$ admits fiber products, then the category* $\mathsf{Groth}(F)$ *admits pushouts. The functor* $\mathsf{Groth}(F) \to \mathcal{I}^{op}$ *commutes with pushouts.*

Proof. This follows from the results in [Lur09] §4.3.1.

For completeness, we note that pushouts can be computed in the following manner. For a diagram:

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_i \\ X_j & \end{tikzcd}$$

in $\mathsf{Groth}(F)$, one forms the pushout of the diagram:

$$\begin{tikzcd} \Upsilon(X_k) \arrow[r] \arrow[d] & \beta(X_j) \\ \alpha(X_i) & \end{tikzcd}$$

in $\mathcal{I}_{i \times_k j}$, where α, β and Υ are the maps $i \times_k j \to i$, $i \times_k j \to j$ and $i \times_k j \to k$ in $\mathcal{I}$.

$\square$

Remark 12.10.2. The above can be generalized to any class of diagrams in place of pushouts. Moreover, we only need to require that F is a functor to the category of categories admitting colimits for these diagrams under functors preserving such.

12.11. For a category $\mathcal{C}$ with pushouts, we let $\mathcal{C}_{op-corr}$ denote the category of correspondences for $\mathcal{C}^{op}$. We represent morphisms $X \to Y$ in $\mathcal{C}_{op-corr}$ by diagrams:

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & H. & \end{tikzcd}$$

Remark 12.11.1. The category $\mathcal{C}_{op-corr}$, being a category of correspondences, admits a canonical 2-category enhancement $\mathcal{C}^{2-cat}_{op-corr}$. For clarity the sake of clarity, we note that this construction is normalized so that a 2-morphism:

$$\left(\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & H_1 & \end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & H_2 & \end{tikzcd}\right)$$

is equivalent to a commutative diagram:

$$\begin{tikzcd} X \arrow[dr] \arrow[ddr] & & Y \arrow[dl] \arrow[ddl] \\ & H_1 & \\ & H_2 \arrow[u] & \end{tikzcd}$$

12.12. Suppose that $\mathcal{I}$ admits fiber products and $F : \mathcal{I}^{op} \to \mathsf{Cat}_{pres}$ is a functor.

By Lemma 12.10.1, we may form the category $\mathsf{Groth}(F)_{op-corr}$.

12.13. The category $\mathsf{Groth}(F)_{op-corr}$ may be described explicitly as follows.

The objects of $\mathsf{Groth}(F)_{op-corr}$ are pairs $i \in \mathcal{I}$, $X_i \in F(i)$. Morphisms $X_i \to X_j$ are given by the data of a hat:

$$\begin{tikzcd} & h \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ i & & j \end{tikzcd} \tag{12.13.1}$$

in $\mathcal{I}$, an object $H_h \in F(h)$, and a diagram:

$$\begin{tikzcd} \alpha(X_i) \arrow[dr] & & \beta(X_j) \arrow[dl] \\ & H_h & \end{tikzcd} \tag{12.13.2}$$

in $F(h)$. Composition of two morphisms $X_i \to X_j \to X_k$ is defined by forming the fiber product:

$$\begin{tikzcd}
 & & h'' \arrow[rr, equal] \arrow[dl, "\varepsilon"'] & & h \underset{j}{\times} h' \arrow[dr, "\eta"] & & \\
 & h \arrow[dl, "\alpha"'] \arrow[drr, "\beta"] & & & & h' \arrow[dll, "\gamma"'] \arrow[dr, "\delta"] & \\
i & & & j & & & k
\end{tikzcd} \tag{12.13.3}$$

and then taking the induced diagram:

$$\begin{tikzcd}
\varepsilon\alpha(X_i) \arrow[dr] & & \varepsilon\beta(X_j) \arrow[dl] \arrow[rr, equal] & & \eta\gamma(X_j) \arrow[dr] & & \eta\delta(X_k) \arrow[dl] \\
 & \varepsilon(H_h) \arrow[drr] & & & & \eta(H_{h'}) \arrow[dll] & \\
 & & & H_{h''}. & & &
\end{tikzcd}$$

12.14. Define the 1-full subcategory $\mathsf{Groth}_{corr}(F) \subseteq \mathsf{Groth}(F)_{op-corr}$ by allowing the same objects, but only allowing morphisms (12.13.2) in which the map $\beta(X_j) \to H_h$ is an equivalence in $F(h)$.

Note that $\mathsf{Groth}_{corr}(F)$ is equipped with a functor to $\mathcal{I}_{corr}$ and the fiber of $\mathsf{Groth}_{corr}(F)$ over the 1-full subcategory $\mathcal{I}^{op}$ of $\mathcal{I}_{corr}$ is equivalent to $\mathsf{Groth}(F)$. Moreover, the fiber of $\mathsf{Groth}_{corr}(F)$ over any object $i \in \mathcal{I}$ is equivalent to $F(i)$.

Variant 12.14.1. As in Remark 12.11.1, $\mathsf{Groth}(F)_{op-corr}$ admits a canonical 2-categorical enhancement $\mathsf{Groth}(F)^{2-cat}_{op-corr}$. We will define a similar 2-categorical structure $\mathsf{Groth}_{corr}(F)^{2-cat}$ on $\mathsf{Groth}_{corr}(F)$.

In the explicit terms used above, 2-morphisms in $\mathsf{Groth}(F)^{2-cat}_{op-corr}$ between morphisms in $\mathsf{Groth}_{corr}(F)$ are represented by pairs of commutative diagrams:

$$
\begin{tikzcd}
 & h \arrow[d, "\Upsilon"] \arrow[ddl, "\alpha"'] \arrow[ddr, "\beta"] & \\
 & h' \arrow[dl, "\alpha'"] \arrow[dr, "\beta'"'] & \\
i & & j
\end{tikzcd}
$$

and:

$$
\begin{tikzcd}
\alpha(X_i) \arrow[dr] \arrow[ddr] & & \beta(X_j) \arrow[dl] \arrow[ddl] \\
 & \Upsilon(H_{h'}) \arrow[d] & \\
 & H_h. &
\end{tikzcd}
$$

We will take the corresponding 2-categorical structure $\mathsf{Groth}_{corr}(F)^{2-cat}$ on $\mathsf{Groth}_{corr}(F)$ where we also require that the corresponding morphism $\Upsilon(H_{h'}) \to H_h$ is an equivalence.

Note that the corresponding morphism $\mathsf{Groth}_{corr}(F) \to \mathcal{I}_{corr}$ upgrades to a functor $\mathsf{Groth}_{corr}(F)^{2-cat} \to \mathcal{I}_{corr}^{2-cat}$ of 2-categories, because $\mathsf{Groth}(F)_{op-corr} \to \mathcal{I}_{corr}$ obviously does.

Remark 12.14.2. The reason for only allowing certain 2-morphisms in Variant 12.11.1 is so that the fiber product:

$$\mathsf{Groth}_{corr}(F)^{2-cat} \underset{\mathcal{I}_{corr}^{2-cat}}{\times} \mathcal{I}_{corr}$$

identifies with $\mathsf{Groth}_{corr}(F)$. Of course, here $\mathcal{I}_{corr} \to \mathcal{I}_{corr}^{2-cat}$ is the embedding of the 2-full subcategory where we only allow invertible 2-morphisms.

12.15. We digress to give a general construction from category theory.

Suppose that $\mathcal{C}$ is a category equipped with a functor:

$$\Phi : \mathcal{C} \to \mathsf{Cat}.$$

Recall that objects of the base of the coCartesian fibration $\mathsf{Groth}(\Phi) \to \mathcal{C}$ may be described as pairs (Y, Z) consisting of $Y \in \mathcal{C}$ and $Z \in \Phi(Y)$.

Now suppose that $\mathcal{C}$ is equipped with a symmetric monoidal structure $\otimes$ and Φ is lax symmetric monoidal. For $Y_1, Y_2 \in \mathcal{C}$ we let $\varepsilon_{Y_1,Y_2} : F(Y_1) \times \Phi(Y_2) \to \Phi(Y_1 \otimes Y_2)$ denote the corresponding functor.

In this case, $\mathsf{Groth}(\Phi)$ is equipped with a canonical symmetric monoidal structure as well so that $\mathsf{Groth}(\Phi) \to \mathcal{C}$ is symmetric monoidal. E.g., the product is given pointwise by the formula:

$$(Y_1, Z_1) \otimes (Y_2, Z_2) = (Y_1 \otimes Y_2, \varepsilon_{Y_1,Y_2}(Z_1, Z_2)).$$

Remark 12.15.1. This construction generalizes to any colored operad. In particular, the above generalizes the the non-unital symmetric monoidal case and there is an obvious variant in the presence of a module category for $\mathcal{C}$ with a (lax) compatible functor to Cat.

Remark 12.15.2. In the above setting, let $\mathsf{coGroth}(\Phi) \to \mathcal{C}^{op}$ denote the corresponding Cartesian fibration. By duality, in the above setting $\mathsf{coGroth}(\Phi)$ carries a canonical (resp. non-unital) symmetric monoidal structure such that $\mathsf{coGroth}(\Phi) \to \mathcal{C}$ is symmetric monoidal.

12.16. Suppose now that $\mathcal{I}$ is equipped with a symmetric monoidal structure and $F : \mathcal{I}^{op} \to \mathsf{Cat}_{pres}$ is lax symmetric monoidal for the Cartesian monoidal structure on Cat_{pres}.

As in §12.15, $\mathsf{Groth}_{corr}(F)$ carries a canonical symmetric monoidal structure such that the forgetful functor $\mathsf{Groth}_{corr}(F) \to \mathcal{I}_{corr}$ is symmetric monoidal.

The same holds true with any operad replacing the commutative operad.

12.17. As in §19.3 and 11.19, we have a functor:

$$\mathsf{ShvCat}_{/-} : \mathsf{PreStk}^{lax,op} \to \mathsf{Cat}_{pres}$$

that assigns to every lax prestack $\mathcal{Y}$ the category $\mathsf{ShvCat}_{/\mathcal{Y}}$ of sheaves of categories on $\mathcal{Y}$.

The functor $\mathsf{ShvCat}_{/-}$ is lax symmetric monoidal relative to the Cartesian product monoidal structures, where for lax prestacks $\mathcal{Y}$ and $\mathcal{Z}$ the corresponding structure map is:

$$\boxtimes : \mathsf{ShvCat}_{/\mathcal{Y}} \times \mathsf{ShvCat}_{/\mathcal{Z}} \to \mathsf{ShvCat}_{/\mathcal{Y}\times\mathcal{Z}}.$$

Remark 12.17.1. Note that for any lax prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$ we have:

$$\mathsf{QCoh}_{\mathcal{Y}} \boxtimes \mathsf{QCoh}_{\mathcal{Z}} \xrightarrow{\simeq} \mathsf{QCoh}_{\mathcal{Y}\times\mathcal{Z}}.$$

The failure of Γ to send $\boxtimes$ to $\otimes$ accounts for the failure of the map $\mathsf{QCoh}(\mathcal{Y})\otimes\mathsf{QCoh}(\mathcal{Z}) \to \mathsf{QCoh}(\mathcal{Y}\times\mathcal{Z})$ to be an isomorphism in general.

12.18. We apply the above formalism to $\mathcal{I} = \mathsf{PreStk}^{lax}$ and $F = \mathsf{ShvCat}_{/-}$.

We obtain the symmetric monoidal category $\mathsf{Groth}_{corr}(\mathsf{ShvCat}_{/-})$ that we will denote by the shorthand $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$. We consider objects of $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$ as pairs of $\mathcal{Y}$ a lax prestack and C a sheaf of categories on $\mathcal{Y}$.

We let $\mathsf{PreStk}_{corr}^{\mathsf{ShvCat}}$ denote the subcategory of $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$ in which we only allow usual prestacks, not lax prestacks.

Remark 12.18.1. Note that PreStk_{corr} and its relatives are not locally small categories. This fact will not cause any difficulties for us below.

12.19. **Digression: The 2-categorical structure on 2-categorical correspondences.** The following discussion will be used implicity in the text, but may be skipped by the reader at first.

Let $\mathcal{C}$ be a 2-category and let $\mathcal{C}^{1-cat}$ denote its underlying 1-category. We propose a canonical 2-categorical enhancement $\mathcal{C}_{corr}$ of $\mathcal{C}^{1-cat}_{corr} := (\mathcal{C}^{1-cat})_{corr}$.

Note that there are two flavors of 2-morphism present: one coming from the correspondence structure, and one coming from $\mathcal{C}$.

Exactly as in [GR14], one can construct a 2-category structure $\mathcal{C}_{corr}$ on $\mathcal{C}^{1-cat}_{corr}$ so that objects are $X \in \mathcal{C}$, 1-morphisms $X \to Y$ in $\mathcal{C}_{corr}$ are given by correspondences $(X \leftarrow H \rightarrow Y)$, and 2-morphisms:

$$\left(\begin{tikzcd} & H_1 \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd} & H_2 \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\right)$$

are given by diagrams:

$$\begin{tikzcd} & H_1 \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & H_2 \arrow[dl] \arrow[dr] \arrow[r, phantom, "\Swarrow"] & \\ X & & Y. \end{tikzcd} \tag{12.19.1}$$

Here the notation indicates that we specify a 2-morphism:

$$(H_1 \to Y) \to (H_1 \to H_2 \to Y)$$

and that the left triangle of (12.19.1) is honestly commutative (i.e., there is an implicit *invertible* 2-morphism).

Remark 12.19.1. The purpose of imposing this restriction on 2-morphisms is so that the 1-full subcategory $\mathcal{C}^{1-cat}$ of $\mathcal{C}^{1-cat}_{corr}$ inherits the 2-categorical structure $\mathcal{C}$.

When discussing the 2-categorical structure of $\mathsf{PreStk}^{lax}_{corr}$, we will be implicitly referring to the 2-categorical structure coming from the above.

Remark 12.19.2. This discussion can be integrated with the discussion of Variant 12.14.1 in the obvious way. This is relevant for describing the 2-categorical structure on $\mathsf{PreStk}^{lax,\mathsf{ShvCat}}_{corr}$.

Note that in the framework above, there were two types of 2-morphisms; in this setting, there are three. There are those of correspondence nature, those that reflect the 2-categorical structure of the base of the "fibration," and those that reflect the fact that the functor "F" takes values in 2-categories.

12.20. Let $\mathcal{S}$ be a commutative algebra in $\mathsf{PreStk}^{lax}_{corr} := (\mathsf{PreStk}^{lax})_{corr}$.

Definition 12.20.1. A *weakly multiplicative sheaf of categories* on $\mathcal{S}$ is a commutative algebra in $\mathsf{PreStk}^{lax,\mathsf{ShvCat}}_{corr}$ mapping to $\mathcal{S}$ as a commutative algebra under the forgetful functor.

We let $\mathsf{MultCat}^w(\mathcal{S})$ denote the category of weakly multiplicative sheaves of categories on $\mathcal{S}$, i.e., the appropriate category of commuative algebras.

Every weakly multiplicative sheaf of categories on $\mathcal{S}$ has an underlying sheaf of categories $\Psi \in \mathsf{ShvCat}_{/\mathcal{S}}$. We sometimes abuse terminology in saying that $\Psi \in \mathsf{ShvCat}_{/\mathcal{S}}$ itself is a multiplicative sheaf of categories.

12.21. Let $\mathcal{S}$ be a commutative algebra in $\mathsf{PreStk}^{lax}_{corr}$ with correspondences:

$$\begin{tikzcd} & \mathrm{mult}_{\mathcal{S}} \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & \\ \mathcal{S}\times\mathcal{S} & & \mathcal{S} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathrm{unit}_{\mathcal{S}} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ * & & \mathcal{S} \end{tikzcd} \tag{12.21.1}$$

defining the multiplication and unit operations for $\mathcal{S}$. Then a weakly multiplicative sheaf of categories $\Psi \in \mathsf{ShvCat}_{/\mathcal{S}}$ has a "multiplication" map:

$$\eta_m : m_1^*(\Psi \boxtimes \Psi) \to m_2^*(\Psi) \in \mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}} \tag{12.21.2}$$

and a "unit" map:

$$\eta_e : \mathsf{QCoh}_{\mathrm{unit}_{\mathcal{S}}} = e_1^*(\mathsf{Vect}) \to e_2^*(\Psi) \in \mathsf{ShvCat}_{/\,\mathrm{unit}_{\mathcal{S}}}. \tag{12.21.3}$$

We have similar maps for the n-ary multiplications for all n.

Definition 12.21.1. A weakly multiplicative sheaf of categories Ψ is *multiplicative* if, for every $n \geqslant 0$, the corresponding structure map as above is an equivalence.

We let $\mathsf{MultCat}(\mathcal{S}) \subseteq \mathsf{MultCat}^w(\mathcal{S})$ denote the category of multiplicative sheaves of categories on $\mathcal{S}$.

Example 12.21.2. $\mathsf{QCoh}_{\mathcal{S}}$ carries a canonical structure of multiplicative sheaf on any $\mathcal{S}$.

Remark 12.21.3. We made a choice earlier by using $\mathsf{ShvCat}_{/-}$ in place of $\mathsf{ShvCat}_{/-}^{naive}$. Had we used $\mathsf{ShvCat}_{/-}^{naive}$ instead of $\mathsf{ShvCat}_{/-}$, we would end up with different weakly multiplicative sheaves, because e.g the morphism (12.21.2) would have to be a morphism in $\mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}}$. However, we would have the same multiplicative sheaves of categories, because the underlying groupoids of $\mathsf{ShvCat}_{/-}$ and $\mathsf{ShvCat}_{/-}^{naive}$ are the same.

However, while the objects would be the same, the morphisms allowed in $\mathsf{MultCat}(\mathcal{S})$ are different by virtue of choosing $\mathsf{ShvCat}_{/-}$.

12.22. More generally, for any colored operad $\mathfrak{O}$ and any $\mathfrak{O}$-algebra $\mathcal{S}$ in $\mathsf{PreStk}_{corr}^{lax}$, we have the category $\mathsf{MultCat}_{\mathfrak{O}}^w(\mathcal{S})$, and the full subcategory $\mathsf{MultCat}_{\mathfrak{O}}(\mathcal{S})$ where the morphisms analogous to (12.21.2) corresponding to all operations are equivalences.

In particular, for $\mathcal{S}$ a non-unital commutative algebra in $\mathsf{PreStk}_{corr}^{lax}$, we have $\mathsf{MultCat}_{non-unital}(\mathcal{S})$ the category of non-unital multiplicative sheaves of categories on $\mathcal{S}$.

12.23. Let $\mathcal{C}$ be a symmetric monoidal 2-category and let $X, Y \in \mathcal{C}$ be commutative algebras.

Recall that in this case we have a notion *lax morphism of commutative algebras* $X \to Y$, which gives rise in particular to a morphism $X \to Y$ and a natural transformation between the compositions:

$$\begin{array}{c} (X \otimes X \to Y \otimes Y \to Y) \\ \Downarrow \\ (X \otimes X \to X \to Y) \end{array}$$

When $\mathcal{C}$ is the 2-category of categories, this gives rise to the usual notion of lax symmetric monoidal functor between symmetric monoidal categories.

12.24. Note that $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$ carries a canonical structure of 2-category as in §12.19. We see that the symmetric monoidal structure lifts to this enhancement as well.

Therefore, we obtain the category $\mathsf{MultCat}^{w,lax}(\mathcal{S})$ where we allow lax morphisms (lying over the identity for $\mathcal{S}$). Then $\mathsf{MultCat}^{w,lax}(\mathcal{S})$ contains $\mathsf{MultCat}^{w}(\mathcal{S})$ as a 1-full subcategory with the same underlying groupoid.

Remark 12.24.1. We emphasize that the use of the term *lax* here is of different nature from that of lax prestack, and rather reflect a general categorical notion applied in two different circumstances. In particular, for a non-lax prestack $\mathcal{S}$ with commutative algebra structure in PreStk_{corr}, there is a significant difference between the categories $\mathsf{MultCat}^{w,lax}(\mathcal{S})$ and $\mathsf{MultCat}^{w}(\mathcal{S})$.

Remark 12.24.2. Recall from Remark 12.19.2 that there are essentially three types of 2-morphisms in $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$. Only the third from the list of *loc. cit.* plays a role in the above discussion: the two coming from the discussion in the beginning of §12.19 are irrelevant.

12.25. Let Ψ be a weakly multiplicative sheaf of categories on on a commutative algebra $\mathcal{S} \in \mathsf{PreStk}_{corr}^{lax}$.

Definition 12.25.1. A *weakly multiplicative object* ψ in Ψ is a morphism:

$$\mathsf{QCoh}_{\mathcal{S}} \to \Psi \tag{12.25.1}$$

in the category $\mathsf{MultCat}^{w,lax}(\mathcal{S})$.

We denote the category of weakly multiplicative objects in Ψ by $\mathsf{Mult}^{w}(\Psi)$.

Notation 12.25.2. Any weakly multiplicative object ψ in Ψ has an underlying morphism $\mathsf{QCoh}_{\mathcal{S}} \to \Psi$ in $\mathsf{ShvCat}_{/\mathcal{S}}$, i.e., it defines an object of $\Gamma(\mathcal{S}, \Psi)$.

We denote this object also by ψ, and summarize the situation by saying that *the object* ψ is a weakly multiplicative object in Ψ.

12.26. Here is a convenient reformulation of the definition of weakly multiplicative object. The reader may skip this material and return to it where needed.

Recall that $\mathsf{Groth}(\mathsf{ShvCat}_{/-})$ denotes the coCartesian fibration over $\mathsf{PreStk}^{lax,op}$ defined by the functor $\mathsf{ShvCat}_{/-}$. We have the canonical functor:

$$\Gamma(-,-) : \mathsf{Groth}(\mathsf{ShvCat}_{/-}) \to \mathsf{DGCat}_{cont}$$

$$(\mathcal{Y}, \mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}) \mapsto \Gamma(\mathcal{Y}, \mathsf{C}).$$

As in §12.14, a variant of the Grothendieck construction defines a category for this section simply by $\mathcal{G}$, whose objects are triples:

$$\Big(\mathcal{Y} \in \mathsf{PreStk}^{lax}, \mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}, \mathcal{F} \in \Gamma(\mathcal{Y}, \mathsf{C})\Big) \tag{12.26.1}$$

and where morphisms:

$$\Big(\mathcal{Y}_1 \in \mathsf{PreStk}^{lax}, \mathsf{C}_1 \in \mathsf{ShvCat}_{/\mathcal{Y}_1}, \mathcal{F}_1 \in \Gamma(\mathcal{Y}_1, \mathsf{C}_1)\Big) \to \Big(\mathcal{Y}_2 \in \mathsf{PreStk}^{lax}, \mathsf{C}_2 \in \mathsf{ShvCat}_{/\mathcal{Y}_2}, \mathcal{F}_2 \in \Gamma(\mathcal{Y}_2, \mathsf{C}_2)\Big)$$

are defined by the data of a correspondence:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathcal{Y}_1 & & \mathcal{Y}_2 \end{tikzcd}$$

in PreStk^{lax}, a morphism:

$$\eta : \alpha^*(\mathsf{C}_1) \to \beta^*(\mathsf{C}_2) \in \mathsf{ShvCat}_{/\mathcal{H}}$$

and a morphism in $\Gamma(\mathcal{H}, \beta^*(\mathsf{C}_2))$ from the image of $\mathcal{F}_1$ to the image of $\mathcal{F}_2$ under the two morphisms:

$$\Gamma(\mathcal{Y}_1, \mathsf{C}_1) \to \Gamma(\mathcal{H}, \alpha^*(\mathsf{C}_1)) \xrightarrow{\Gamma(\eta)} \Gamma(\mathcal{H}, \beta^*(\mathsf{C}_2))$$
$$\text{and } \Gamma(\mathcal{Y}_2, \mathsf{C}_2) \to \Gamma(\mathcal{H}, \beta^*(\mathsf{C}_2)).$$

The category $\mathcal{G}$ is canonically symmetric monoidal in the obvious way, and we have a symmetric monoidal functor:

$$\mathcal{G} \to \mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}} \tag{12.26.2}$$

given by forgetting the third term in (12.26.1).

Then, tautologically, a weakly multiplicative object in a weakly multiplicative sheaf of categories $\Psi \in \mathsf{MultCat}^w(\mathcal{S})$ is equivalent to a commutative algebra in $\mathcal{G}$ mapping to Ψ under the forgetful functor (12.26.2).

12.27. In the notation of §12.21, a weakly multiplicative object $\psi \in \Psi$ defines a morphism:

$$\eta_m\big(m_1^*(\psi \boxtimes \psi)\big) \to m_2^*(\psi) \in \Gamma(\mathrm{mult}_{\mathcal{S}}, m_2^*(\Psi))$$

and similarly for the unit operation, and general n-ary multiplication operations.

Definition 12.27.1. The object ψ is a *multiplicative object* in Ψ if these morphisms are isomorphisms.

Remark 12.27.2. Tautologically, one can rephrase the definition by asking that the morphism (12.25.1) be a morphism of commutative algebras and not a lax morphism, i.e., it should be a morphism in $\mathsf{MultCat}^{w}(\mathcal{S})$.

We denote the resulting full subcategory of $\mathsf{Mult}^{w}(\Psi)$ by $\mathsf{Mult}(\Psi)$.

Example 12.27.3. In the setting of Example 12.21.2, the object $\mathcal{O}_{\mathcal{S}}$ carries a canonical multiplicative structure.

Remark 12.27.4. By Remark 12.21.3, the choice to use $\mathsf{ShvCat}_{/-}$ in place of $\mathsf{ShvCat}_{/-}^{naive}$ gives a different definition of multiplicative objects.

The key difference is explained in Example 11.17.1: we would not have "interesting" multiplicative sheaves, i.e., they would be insensitive to the non-invertibility of morphisms in the categories taken as values of $\mathcal{S}$.

Remark 12.27.5. The category $\mathsf{Mult}(\mathcal{S})$ admits sifted colimits

12.28. In the setting of §12.22, for $\Psi \in \mathsf{Mult}^{w}_{\mathfrak{O}}$ we obtain the categories $\mathsf{Mult}^{w}_{\mathfrak{O}}(\Psi)$ and its full subcategory $\mathsf{Mult}_{\mathfrak{O}}(\Psi)$.

For the sake of clarity: let us denote the category of colors underlying $\mathfrak{O}$ by $\mathfrak{O}^{\blacklozenge}$. For an $\mathfrak{O}$-algebra in PreStk^{lax}, we have in particular a rule assigning to $\xi \in \mathfrak{O}^{\blacklozenge}$ a lax prestack $\mathcal{S}_{\xi}$. Then the role of $\mathsf{QCoh}_{/\mathcal{S}}$ from the symmetric monoidal case is played by the rule assigning to each $\mathcal{S}_{\xi}$ the sheaf of categories $\mathsf{QCoh}_{/\mathcal{S}_{\xi}}$.

12.29. **Variant: Coalgebraic description.** Let $\mathcal{S}$ be as above.

For any category $\mathcal{C}$ with fiber products, we have the canonical equivalence $(\mathcal{C}_{corr})^{op} \simeq \mathcal{C}_{corr}$ given by "flipping" the correspondence. This construction allows us to view $\mathcal{S}$ as a cocommutative coalgebra in $\mathsf{PreStk}^{lax}_{corr}$.

We have the category $\mathsf{MultCat}^{op-w}(\mathcal{S})$ of *op-weakly multiplicative sheaves of categories*: these are coalgebras in $\mathsf{PreStk}_{corr}^{lax,\mathsf{ShvCat}}$ lying over $\mathcal{S}$.

Any op-weakly multiplicative sheaf of categories has structure maps:

$$\begin{gathered}\widetilde{\eta}_m : m_2^*(\Psi) \to m_1^*(\Psi \boxtimes \Psi) \in \mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}} \text{ and} \\ \widetilde{\eta}_e : e_2^*(\Psi) \to \mathsf{QCoh}_{\mathrm{unit}_{\mathcal{S}}} = e_1^*(\mathsf{Vect}) \in \mathsf{ShvCat}_{/\,\mathrm{unit}_{\mathcal{S}}}.\end{gathered} \tag{12.29.1}$$

By general principles from [GR14], the subcategory of $\mathsf{MultCat}^{op-w}(\mathcal{S})$ where the maps in (12.29.1) are equivalences is canonically equivalent to $\mathsf{MultCat}(\mathcal{S})$.

More generally, we have the following general result.

Proposition 12.29.1. *Let* $\mathsf{MultCat}^{w,l.adj}(\mathcal{S}) \subseteq \mathsf{MultCat}^{w}(\mathcal{S})$ *denote the full subcategory in which the arrows* (12.21.2) *and* (12.21.3) *admit left adjoint in the 2-category* $\mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}}$ *and* $\mathsf{ShvCat}_{/\,\mathrm{unit}_{\mathcal{S}}}$ *respectively (equivalently: the analogous result for all n-ary operations in the commutative operad).*

Similarly, define $\mathsf{MultCat}^{op-w,r.adj}(\mathcal{S})$ *to be the full subcategory of* $\mathsf{MultCat}^{op-w}(\mathcal{S})$ *in which the morphisms* (12.29.1) *admit right adjoints.*

Then there is a canonical equivalence:

$$\mathsf{MultCat}^{w,l.adj}(\mathcal{S}) \simeq \mathsf{MultCat}^{op-w,r.adj}(\mathcal{S})$$

commuting with forgetful functors to $\mathsf{ShvCat}_{/\mathcal{S}}$*, defined by passing to the appropriate adjoints for all operations.*

Remark 12.29.2. The roles of left and right could be interchanged in the statement of this proposition, but we will apply it with the normalizations above.

12.30. Similarly, we have the notion of op-weakly multiplicative object of an op-multiplicative sheaf of categories $\Psi \in \mathsf{MultCat}^{op-w}(\mathcal{S})$. We denote the resulting category by $\mathsf{Mult}^{op-w}(\Psi)$. In a multiplicative sheaf of categories Ψ, considered as an op-weakly mutliplicative sheaf of categories as above, the corresponding notion of multiplicative object canonically identifies with the category $\mathsf{Mult}(\Psi)$ as defined in the "covariant" setting above.

The op-multiplicative setting has the following advantages:

Lemma 12.30.1. *The categories* $\mathsf{MultCat}^{op-w}(\mathcal{S})$ *and* $\mathsf{Mult}^{op-w}(\Psi)$ *are cocomplete (even presentable) and the corresponding functors:*

$$\mathsf{MultCat}^{op,w}(\mathcal{S}) \to \mathsf{ShvCat}(\mathcal{S})$$

$$\mathsf{Mult}^{op,w}(\Psi) \to \Gamma(\mathcal{S}, \Psi)$$

commute with colimits.

12.31. **Subcategories.** Suppose that $\mathcal{S}$ is a commutative monoid in $\mathsf{PreStk}^{lax}_{corr}$, Ψ is a weakly multiplicative sheaf of categories on $\mathcal{S}$, and $\Phi \hookrightarrow \Psi$ is a fully-faithful functor in $\mathsf{ShvCat}_{/\mathcal{S}}$, in the sense of §11.26.

We say that Φ is *weakly compatible* with the weakly multiplicative structure on Ψ if the morphism η_m from (12.21.2) maps $m_1^*(\Phi \boxtimes \Phi)$ into $m_2^*(\Phi) \subseteq m_2^*(\Psi)$, and η_e from (12.21.3) factors through $e_2^*(\Phi) \subseteq e_2^*(\Psi)$.

In this case, Φ inherits a unique weakly multiplicative structure such that the morphism $\Phi \to \Psi$ upgrades to a morphism of weakly multiplicative sheaves of categories.

We say that Φ is *compatible* if the induced weakly multiplicative structure is multiplicative.

A variant of this discussion holds for general colored operads.

12.32. **Localizations.** Suppose that $\mathcal{S}$ is a commutative monoid in $\mathsf{PreStk}^{lax}_{corr}$, Ψ is an op-weakly multiplicative sheaf of categories on $\mathcal{S}$, and $\Phi \subseteq \Psi$ is a full subcategory.

As in §19.6, we can form the quotient sheaf of categories $\Psi/\Phi \in \mathsf{ShvCat}_{/\mathcal{S}}$.

We say that Φ is a *weak ideal* subcategory of Ψ if the compositions:

$$m_2^*(\Phi) \hookrightarrow m_2^*(\Psi) \xrightarrow{\widetilde{\eta}_m} m_1^*(\Psi \boxtimes \Psi) \to m_1^*\Big((\Psi/\Phi) \boxtimes (\Psi/\Phi)\Big) \text{ and}$$

$$e_2^*(\Phi) \hookrightarrow e_2^*(\Psi) \xrightarrow{\widetilde{\eta}_e} \mathsf{QCoh}_{\mathrm{unit}_{\mathcal{S}}}$$

are zero. Here the notations $\widetilde{\eta}_m$ and $\widetilde{\eta}_e$ are taken from (12.29.1).

In this case, the quotient Ψ/Φ inherits a canonical structure of op-weakly multiplicative sheaf of categories on $\mathcal{S}$.

If Ψ is a (non-weakly) multiplicative sheaf of categories on $\mathcal{S}$, we say that Φ is an *ideal* subcategory if induced op-weakly multiplicative structure on the quotient Ψ/Φ is multiplicative.

Again, this material generalizes in the appropriate way to an arbitrary colored operad.

12.33. **Functoriality.** Before discussing functoriality of multiplicative sheaves, we return to the general framework of §12.16, so $\mathcal{I}$ is a symmetric monoidal category that admits fiber products and $F : \mathcal{I}^{op} \to \mathsf{Cat}_{pres}$ is a lax symmetric monoidal functor.

Lemma 12.33.1. *Let $\mathfrak{O}$ be a colored operad, and denote also by $\mathfrak{O}^{\diamond}$ the underlying category in which we only allow 1-ary operations.*

Then the functor:

$$\mathsf{Alg}_{\mathfrak{O}}\Big(\mathsf{Groth}_{corr}(F)\Big) \underset{\mathsf{Hom}(\mathfrak{O}^{\diamond}, \mathcal{I}_{corr})}{\times} \mathsf{Hom}(\mathfrak{O}^{\diamond}, \mathcal{I}^{op}) \to \mathsf{Alg}_{\mathfrak{O}}(\mathcal{I}_{corr}) \underset{\mathsf{Hom}(\mathfrak{O}^{\diamond}, \mathcal{I}_{corr})}{\times} \mathsf{Hom}(\mathfrak{O}^{\diamond}, \mathcal{I}^{op})$$

is a coCartesian fibration.

This result follows from the following more general categorical lemma.

Lemma 12.33.2. *Suppose that $\mathcal{C}$ and $\mathcal{J}$ are symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{J}$ is a symmetric monoidal functor.*

Suppose that $\mathcal{J}^0$ is a symmetric monoidal 1-full subcategory of $\mathcal{J}$ such that $\mathcal{C} \times_{\mathcal{J}} \mathcal{J}^0 \to \mathcal{J}^0$ is a coCartesian fibration, and arrows in $\mathcal{C}$ coCartesian over $\mathcal{J}^0$ are coCartesian over all

$\mathcal{J}$. *Suppose moreover that arrows in* $\mathcal{C}$ *coCartesian over* $\mathcal{J}^0$ *are preserved under tensor products in* $\mathcal{C}$.

Suppose that we are given a symmetric monoidal category $\mathcal{D}$, *symmetric monoidal functors* $G_i : \mathcal{D} \to \mathcal{J}$, $i = 1, 2$ *and morphism* $\eta : G_1 \to G_2$ *of symmetric monoidal functors, such that for every* $X \in \mathcal{D}$ *the morphism* $G_1(X) \to G_2(X)$ *is a morphism in* $\mathcal{J}^0$.

Then the functor:

$$\mathsf{Hom}^{\otimes}(\mathcal{D}, \mathcal{C}) \underset{\mathsf{Hom}^{\otimes}(\mathcal{D},\mathcal{J})}{\times} \Delta^1 \to \Delta^1$$

is coCartesian, where the fiber is taken over η. *Here* $\mathsf{Hom}^{\otimes}$ *denotes the category of symmetric monoidal functors. An arrow in* $\mathsf{Hom}^{\otimes}(\mathcal{D}, \mathcal{C}) \times_{\mathsf{Hom}^{\otimes}(\mathcal{D},\mathcal{J})} \Delta^1$ *is coCartesian if and only if, for every* $X \in \mathcal{D}$, *the induced arrow in* $\mathcal{C}$ *is coCartesian over* $\mathcal{J}^0$.

Remark 12.33.3. That we can reduce Lemma 12.33.1 to the symmetric monoidal case follows from the theory of monoidal envelopes in [Lur12]. However, this is not a serious point.

Proof (sketch). Using the description of symmetric monoidal categories in terms of coCartesian fibrations, reduce to the case where we deal with with non-symmetric monoidal categories and functors, where it follows by an appropriate generalization of [Lur09] Proposition 3.1.2.1.

□

Remark 12.33.4. The above material is stated in a somewhat abstract way. It amounts to the following. Suppose we are in the setting of Lemma 12.33.2, but let us omit the words "symmetric monoidal" everywhere. The lemma then says that, given $G_1 \to G_2$ as in *loc. cit.*, and a $\widetilde{G}_1$ a lift of G_1 to a functor $\mathcal{D} \to \mathcal{C}$, then we obtain a functor $\widetilde{G}_2$ lifting G_2 and equipped with a morphism $\widetilde{G}_1 \to \widetilde{G}_2$.

Naively: for $X \in \mathcal{D}$, define $\widetilde{G}_2(X)$ as the tip of the coCartesian arrow in $\mathcal{C}$ with source $\widetilde{G}_1(X)$, and lying over the morphism $G_1(X) \to G_2(X)$ (which, by assumption, is an arrow in $\mathcal{J}^0$). Then, for a morphism $X \to Y$ in $\mathcal{D}$, we have the square:

$$\begin{tikzcd} \widetilde{G}_1(X) \arrow[r] \arrow[d] & \widetilde{G}_2(X) \arrow[d, dotted] \\ \widetilde{G}_1(Y) \arrow[r] & \widetilde{G}_2(Y). \end{tikzcd}$$

The dotted arrow comes from the fact that $\widetilde{G}_1(X) \to \widetilde{G}_2(X)$ is a coCartesian arrow in $\mathcal{C}$, and from the morphism $\widetilde{G}_1(X) \to \widetilde{G}_2(Y)$ given by tracing out the lower edge of the diagram.

Variant 12.33.5. In the setting of Lemma 12.33.2, suppose that $\mathcal{C}$ and $\mathcal{J}$ are taken to be symmetric monoidal 2-categories instead, and $\mathcal{J}^0$ is again a 1-full subcategory with the same compatibility. Then the conclusion of Lemma 12.33.2 again holds, but in the 2-categorical sense. In fact, there are two formulations: we can allow lax or strict morphisms of symmetric monoidal functors, and the result holds in either setting.

Therefore, by Remark 12.14.2, we have a variant of Lemma 12.33.1 in which we use the 2-categorical enhancements $\mathsf{Groth}_{corr}(F)^{2-cat}$ and $\mathcal{I}^{2-cat}_{corr}$.

12.34. Suppose that $f : \mathcal{S} \to \mathcal{T}$ is a morphism of commutative algebras (or $\mathfrak{O}$-algebras) in $\mathsf{PreStk}^{lax}_{corr}$ such that the underlying morphism in $\mathsf{PreStk}^{lax}_{corr}$ is a morphism in the 1-full subcategory PreStk^{lax}.

By Lemma 12.33.1 we obtain pullback functors:

$$\begin{gathered} f^* : \mathsf{MultCat}^w(\mathcal{T}) \to \mathsf{MultCat}^w(\mathcal{S}) \\ \mathsf{Mult}^w(\Psi) \to \mathsf{Mult}^w(f^*(\Psi)) \end{gathered} \tag{12.34.1}$$

where $\Psi \in \mathsf{MultCat}^w(\mathcal{T})$. These functors preserve the full subcategories $\mathsf{MultCat}$ and Mult respectively.

Moreover, the 2-categorical version of Lemma 12.33.1, applied to account for the 2-categorical struture on PreStk^{lax}, implies that if $\eta : f \to g$ is a 2-morphism of maps $f, g : \mathcal{S} \to \mathcal{T}$ of commutative algebras as above, then we obtain natural transformations of the corresponding functors (12.34.1).

12.35. **A variant.** We have the following variant of these definitions as well. Let $\mathcal{S}$ be a commutative algebra in $\mathsf{PreStk}^{lax}_{corr}$ as above.

Suppose that $\mathfrak{F} : \mathsf{PreStk}^{lax,op} \to \mathsf{Cat}$ (or valued in Cat_{pres}) is a lax symmetric monoidal functor. Then, exactly as in the definition of multiplicative sheaf of categories, we have a notion of *multiplicative sheaf on $\mathcal{S}$ with values in $\mathfrak{F}$*.

Example 12.35.1. If $\mathfrak{F} = \mathsf{ShvCat}_{/-}$, then we recover the notion of multiplicative sheaf of categories on $\mathcal{S}$.

If $\mathfrak{F} = \mathsf{QCoh}(-)$ with the exterior product defining the lax symmetric monoidal structure, then we recover the notion of multiplicative object in the multiplicative sheaf of categories $\mathsf{QCoh}_{/\mathcal{S}}$.

Example 12.35.2. If $\mathcal{C}$ is a symmetric monoidal category, then we may view $\mathcal{C}$ as a lax symmetric monoidal functor $* \to \mathsf{Cat}$ and therefore we obtain a lax symmetric monoidal functor:

$$\mathsf{PreStk}^{op} \to * \to \mathsf{Cat}.$$

Taking this composition as the functor $\mathfrak{F}$, we recover a notion of multiplicative sheaf with values in the symmetric monoidal category $\mathcal{C}$.

Example 12.35.3. One can use this framework to make sense of a factorizable monoidal category.

Again, this discussion carries over to a general colored operad.

13. Chiral categories and factorization algebras

13.1. In this section, we give the formalism of chiral categories and factorization algebras in them by applying the material of §12 to Ran space.

We fix a prestack $\mathcal{X}$ throughout this section.

13.2. **Chiral categories and factorization algebras.** Here are the main definitions of this section.

Definition 13.2.1. A *chiral category* or *factorization category* C on $\mathcal{X}$ is a non-unital multiplicative category on the non-unital commutative algebra $\mathrm{Ran}_{\mathcal{X}}^{ch} \in \mathsf{PreStk}_{corr} \subseteq \mathsf{PreStk}_{corr}^{lax}$.

A *factorization algebra* $\mathcal{A}$ in a factorization category C is a multiplicative object of C.

A *unital chiral category* or *unital factorization category* C on $\mathcal{X}$ is a multiplicative category on $\mathrm{Ran}_{\mathcal{X}}^{un,ch} \in \mathsf{PreStk}_{corr}^{lax}$.

A *unital factorization algebra* $\mathcal{A}$ in a unital factorization category is a multiplicative object of C.

We denote the respective categories by:

$$\mathsf{Cat}^{ch}(\mathcal{X}) \qquad \mathsf{Cat}_{un}^{ch}(\mathcal{X})$$

$$\mathsf{Alg}^{\mathrm{fact}}(\mathsf{C}) \qquad \mathsf{Alg}_{un}^{\mathrm{fact}}(\mathsf{C})$$

for C a (resp. unital) chiral category. We have forgetful functors:

$$\mathsf{Cat}_{un}^{ch}(\mathcal{X}) \to \mathsf{Cat}^{ch}(\mathcal{X})$$

$$\mathsf{Alg}_{un}^{\mathrm{fact}}(\mathsf{C}) \to \mathsf{Alg}^{\mathrm{fact}}(\mathsf{C}).$$

for C a unital factorization category.

Remark 13.2.2. We refer to §10 for more concrete descriptions of factorization categories.

Remark 13.2.3. One immediately sees that e.g. factorization categories on $\mathcal{X}$ are equivalent to *unital* multiplicative categories on $\mathrm{Ran}_{\mathcal{X},\varnothing}$.

Terminology 13.2.4. We will frequently abuse language by saying that $\mathsf{C} \in \mathsf{ShvCat}_{\mathrm{Ran}_{\mathcal{X}}}$ is a chiral category, or $\mathcal{A} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathsf{C})$ is a factorization algebra in C, and so on.

Notation 13.2.5. For $\mathsf{C} = \mathsf{QCoh}_{\mathrm{Ran}_{\mathcal{X}}}$, we write $\mathsf{Alg}^{\mathrm{fact}}(\mathcal{X})$ and $\mathsf{Alg}^{\mathrm{fact}}_{un}(\mathcal{X})$ in place of the notation above, and refer to objects of these categories merely as (unital) factorization algebras on $\mathcal{X}$.

Terminology 13.2.6. We refer to morphisms in $\mathsf{Cat}^{ch}(\mathcal{X})$ and $\mathsf{Cat}^{ch}_{un}(\mathcal{X})$ as *factorization functors* and *unital factorization functors* respectively.

Remark 13.2.7. The comparison with the theory of [FG12] is indirect, and therefore postponed to Remark 13.19.5.

Remark 13.2.8. By definition of multiplicative sheaf, given a factorization functor $\mathsf{C} \to \mathsf{D}$ we obtain a canonical morphism:

$$\mathsf{Alg}^{\mathrm{fact}}(\mathsf{C}) \to \mathsf{Alg}^{\mathrm{fact}}(\mathsf{D})$$

compatible with forgetful functors. The same holds in the unital setting.

Variant 13.2.9. A *weak chiral category* is a weakly multiplicative sheaf of categories on $\mathrm{Ran}^{ch}_{\mathcal{X}}$. We let $\mathsf{Cat}^{w,ch}(\mathcal{X})$ denote the category of weak chiral categories on $\mathcal{X}$. Similarly, we have the unital variant $\mathsf{Cat}^{w,ch}(\mathcal{X})$. Recall that $\mathsf{Cat}^{ch}(\mathcal{X})$ (resp. $\mathsf{Cat}^{ch}_{un}(\mathcal{X})$) is tautologically a full subcategory of $\mathsf{Cat}^{w,ch}(\mathcal{X})$ (resp. $\mathsf{Cat}^{w,ch}_{un}(\mathcal{X})$).

13.3. **The unit.** Therefore, we may apply the discussion of §11.24, and we will use the terminology of *loc. cit.* freely.

We will show that $\mathrm{unit}_{\mathsf{C}}$ admits a canonical unital factorization algebra structure.

The chiral product on $\mathrm{Ran}_{\mathcal{X}}^{un}$ induces commutative algebra structures on $\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}$ and $[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}$.

Moreover, one sees first that the maps:

$$
[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj} \underset{\mathrm{add}}{\overset{p_2}{\rightrightarrows}} \mathrm{Ran}_{\mathcal{X}}^{un}
$$

are morphisms of commutative algebras in $\mathsf{PreStk}_{corr}^{lax}$, and that the obvious 2-morphism:

$$
\begin{tikzcd}
{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \arrow[r, bend left, "p_2"] \arrow[r, bend right, "\mathrm{add}"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Ran}_{\mathcal{X}}^{un}
\end{tikzcd}
. \tag{13.3.1}
$$

is compatible with the commutative algebra structures.

Restricting to $\mathrm{Ran}_{\mathcal{X}}^{un} \times \{\varnothing\}$ and applying the discussion from §12.34 we see that $\mathrm{unit}_{\mathsf{C}}$ inherits the canonical structure of unital factorization algebra.

Furthermore, we see that any $\mathcal{A} \in \mathsf{Alg}_{un}^{\mathrm{fact}}(\mathsf{C})$ admits a canonical map:

$$
\mathrm{unit}_{\mathsf{C}} \to \mathcal{A} \tag{13.3.2}
$$

of unital factorization algebras. We refer to this map as the *unit map* for $\mathcal{A}$.

Remark 13.3.1. Given a unital factorization functor $F : \mathsf{C} \to \mathsf{D}$, there is not necessarily an identification $F(\mathrm{unit}_{\mathsf{C}}) \simeq \mathrm{unit}_{\mathsf{D}}$, but rather there is only a morphism:

$$
\mathrm{unit}_{\mathsf{D}} \to F(\mathrm{unit}_{\mathsf{C}}) \tag{13.3.3}
$$

of unital factorization algebras in D.

Definition 13.3.2. A unital factorization functor is *strictly unital* if (13.3.3) is an equivalence.

We let $\mathsf{Cat}_{un,str}^{ch}(\mathcal{X})$ denote the 1-full subcategory of $\mathsf{Cat}_{un}^{ch}(\mathcal{X})$ consisting of unital chiral categories on $\mathcal{X}$ under strictly unital morphisms.

Remark 13.3.3. We will sometimes say a general unital factorization functor is *lax unital* to emphasize that it may not be (or is not) strictly unital, but the word "lax" should be taken as redundant here.

Recalling that unital factorization algebras in C are by definition unital factorization functors $\mathsf{QCoh}_{\mathcal{X}} \to \mathsf{C}$, we see that this construction generalizes the construction of (13.3.2) presented above.

Remark 13.3.4. Remark 13.3.1 is a manifestation of the following general philosophy: under the analogy between chiral categories and monoidal categories, chiral functors correspond to *lax* monoidal functors (recall that in the setting of (unital) monoidal categories, it is natural to assume that lax monoidal functors are merely lax unital).

13.4. We now discuss a construction of unital factorization structures useful in §6.

Suppose that C is a unital factorization category and $\mathsf{D} \hookrightarrow \mathsf{C}$ is a fully-faithful functor in $\mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}^{un}}$.

Suppose that D is compatible with the factorization structure in the sense that we have a (necessarily unique) factorization:

$$\begin{tikzcd}
\Big(\mathsf{D} \boxtimes \mathsf{D}\Big)|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \arrow[r, dotted] \arrow[d] & \mathrm{add}^*(\mathsf{D})|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \arrow[d, hook] \\
\Big(\mathsf{C} \boxtimes \mathsf{C}\Big)|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \arrow[r, equal] & \mathrm{add}^*(\mathsf{C})|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}}
\end{tikzcd}$$

that is an equivalence, and moreover, the map:

$$\mathsf{D}_{\varnothing} \to \mathsf{C}_{\varnothing} \simeq \mathsf{Vect}$$

is an equivalence as well.

In this case, the discussion of §12.31 implies that D inherits a canonical unital factorization structure.

Remark 13.4.1. Note that there is an analogous version of this discussion for non-unital factorization categories.

Moreover, in the unital setting, we observe that for factorization category C and $\mathsf{D} \subseteq \mathsf{C} \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}^{un}}$ as above, it suffices to check the compatibility with the unital factorization structure by checking compatibility with the non-unital factorization structure by restriction to $\mathrm{Ran}_{\mathcal{X},\varnothing}$ (viewing non-unital factorization categories via 13.2.3).

Combining this discussion with Proposition-Construction 11.26.1, we obtain the following result:

Proposition 13.4.2. *Suppose that C is a unital factorization category on $\mathcal{X}$ that is adj-unital (as a mere unital sheaf of categories, i.e., ignoring the factorization structure).*

Suppose that D is a factorization category on $\mathcal{X}$ equipped with a factorization functor $G : \mathsf{D} \to \mathsf{C}$ such that the underlying morphism in $\mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}}$ is fully-faithful.

Let D also denote the corresponding sheaf of categories on $\mathrm{Ran}_{\mathcal{X},\varnothing} = \mathrm{Ran}_{\mathcal{X}} \coprod \mathrm{Spec}(k)$ where $\mathsf{D}_{\varnothing} := \mathsf{Vect}$.

Now suppose that the hypotheses of Proposition-Construction 11.26.1 are satisfied.

Then D with its unital structure from Proposition-Construction 11.26.1 inherits a unique unital factorization structure such that the functor $\mathsf{D} \to \mathsf{C} \in \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}^{un}}$ upgrades to a functor of unital factorization categories.

13.5. **Localizations.** We now render the material of §12.32 to the setting of factorization categories.

Suppose that C is a unital factorization category on $\mathcal{X}$ and $\mathsf{D} \hookrightarrow \mathsf{C} \in \mathsf{ShvCat}_{\mathrm{Ran}_{\mathcal{X}}^{un}}$ is a unital subcategory with $\mathsf{D}_{\varnothing} = 0$ and such that the composition:

$$\mathrm{add}^*(\mathsf{D})|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \hookrightarrow \mathrm{add}^*(\mathsf{C})|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \xrightarrow{\simeq}$$
$$\Big(\mathsf{C} \boxtimes \mathsf{C}\Big)|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}} \to \Big(\mathsf{C}/\mathsf{D} \boxtimes \mathsf{C}/\mathsf{D}\Big)|_{[\mathrm{Ran}_{\mathcal{X}}^{un} \times \mathrm{Ran}_{\mathcal{X}}^{un}]_{disj}}$$

is zero, and the induced map:

$$\mathrm{add}^*(\mathsf{C}/\mathsf{D})|_{[\mathrm{Ran}_{\mathfrak{X}}^{un}\times\mathrm{Ran}_{\mathfrak{X}}^{un}]_{disj}} \to \Big(\mathsf{C}/\mathsf{D}\boxtimes\mathsf{C}/\mathsf{D}\Big)|_{[\mathrm{Ran}_{\mathfrak{X}}^{un}\times\mathrm{Ran}_{\mathfrak{X}}^{un}]_{disj}}$$

is an equivalence.

Then C/D inherits a canonical structure of unital factorization category. Moreover, the structure morphism $\mathsf{C}\to\mathsf{C}/\mathsf{D}$ is a morphism of unital factorization categories. Note that C/D satisfies a universal property: to give a unital factorization functor $\mathsf{C}/\mathsf{D}\to\mathsf{C}'$ is equivalent to give a functor $\mathsf{C}\to\mathsf{C}'$ sending D to 0.

This material renders to the non-unital setting with the appropriate changes in notation.

13.6. **Module spaces.** Next, we discuss factorization modules. We begin with the non-unital setting.

Definition 13.6.1. A *factorization module space* $\mathcal{Z}$ for $\mathrm{Ran}_{\mathfrak{X}}$ is a (by necessity: non-unital) $\mathrm{Ran}_{\mathfrak{X}}^{ch}$-module in PreStk_{corr}. An *augmented factorization module space* (over $\mathrm{Ran}_{\mathfrak{X}}$) is a factorization module space equipped with a morphism:

$$\varpi:\mathcal{Z}\to\mathrm{Ran}_{\mathfrak{X}}$$

of prestacks (not merely a correspondence), with ϖ equipped with a structure of morphism of $\mathrm{Ran}_{\mathfrak{X}}^{ch}$-modules in PreStk_{corr}, where $\mathrm{Ran}_{\mathfrak{X}}^{ch}$ acts on itself by the chiral action.

Remark 13.6.2. To unwind this definition somewhat: a factorization module space $\mathcal{Z}$ is, in particular, equipped with an *action* correspondence:

$$\begin{tikzcd} & \mathcal{H}_{\mathcal{Z}} \arrow[dl] \arrow[dr] & \\ \mathrm{Ran}_{\mathfrak{X}}\times\mathcal{Z} & & \mathcal{Z}. \end{tikzcd}$$

For an augmented factorization module space $\mathcal{Z}$, the morphism ϖ induces a map:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{H}_{\mathcal{Z}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{Ran}_{\mathfrak{X}} \times \mathcal{Z} \arrow[d, "\mathrm{id} \times \varpi"'] & {[\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj}} \arrow[dl] \arrow[dr] & \mathcal{Z} \arrow[d, "\varpi"] \\
\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}} & & \mathrm{Ran}_{\mathfrak{X}}
\end{tikzcd}$$

with the left square Cartesian.

Note that this means that if we are trying to define the structure of augmented factorization module space on $\mathcal{Z} \to \mathrm{Ran}_{\mathfrak{X}}$ over $\mathrm{Ran}_{\mathfrak{X}}$, we already know what $\mathcal{H}_{\mathcal{Z}}$ must be, and the content lies in defining the map:

$$\mathcal{H}_{\mathcal{Z}} = \Big(\mathrm{Ran}_{\mathfrak{X}} \times \mathcal{Z}\Big) \underset{\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}}{\times} [\mathrm{Ran}_{\mathfrak{X}} \times \mathrm{Ran}_{\mathfrak{X}}]_{disj} \to \mathcal{Z}$$

and its higher compatibilities.

Example 13.6.3. Suppose that $\mathcal{Z} \in \mathsf{PreStk}$ admits an action (in PreStk) by $\mathrm{Ran}_{\mathfrak{X}}^{*} = (\mathrm{Ran}_{\mathfrak{X}}, \mathrm{add})$, and a $\mathrm{Ran}_{\mathfrak{X}}$-equivariant morphism:

$$\mathcal{Z} \to \mathrm{Ran}_{\mathfrak{X}}.$$

Then we claim that $\mathcal{Z}$ admits a canonical structure of augmented factorization module space. Indeed, this follows in the same way that $\mathrm{Ran}_{\mathfrak{X}}$ inherits its chiral multiplication.

13.7. **Examples of factorization module spaces.** We have two key examples of factorization module spaces: $\mathrm{Ran}_{\mathfrak{X},I}$ introduced below for I a finite set, and $\mathrm{Ran}_{\mathfrak{X}}^{\rightarrow}$.

Let fSet_I denote the category whose objects are arbitrary maps $I \to J$ and where morphisms are commutative diagrams:

$$\begin{tikzcd} & J \arrow[dd] \\ I \arrow[ur] \arrow[dr] & \\ & J'. \end{tikzcd}$$

We define the $\mathcal{X}^I$*-marked Ran space* $\mathrm{Ran}_{\mathcal{X},I}$ as the colimit:

$$\mathrm{Ran}_{\mathcal{X},I} := \underset{(I\to J)\in \mathsf{fSet}_I^{op}}{\mathrm{colim}} \mathcal{X}^J \in \mathsf{PreStk}.$$

There is a canonical map $\mathrm{Ran}_{\mathcal{X},I} \to \mathcal{X}^I$.

Remark 13.7.1. The reader should think of $\mathrm{Ran}_{\mathcal{X},I}$ as the parameter space of a map $I \xrightarrow{i\mapsto x_i} \mathcal{X}$ and an embedding $\{x_i\} \subseteq J \subseteq \mathcal{X}$ of finite subsets.

Then $\mathrm{Ran}_{\mathcal{X},I}$ admits an obvious structure of $\mathrm{Ran}^*_{\mathcal{X}}$-module space, and therefore, by Example 13.6.3, $\mathrm{Ran}_{\mathcal{X},I}$ obtains a canonical structure of augmented factorization module space.

Similarly, $\mathrm{Ran}^{\rightarrow}_{\mathcal{X}}$ admits a canonical $\mathrm{Ran}^*_{\mathcal{X}}$-module space structure.

Here we introduce the category $\mathsf{fSet}^{\rightarrow}$ whose objects are arbitrary maps $I \to J$ of non-empty finite sets, and where morphisms are commutative diagrams with termwise surjective maps. We remark that $\mathsf{fSet}^{\rightarrow}$ was introduced in 11.8 under the notation $\mathsf{fSet}^{\rightarrow}_{[1]}$.

Recall that we have:

$$\mathrm{Ran}^{\rightarrow}_{\mathcal{X}} = \underset{(I\to J)\in \mathsf{fSet}^{\rightarrow,op}}{\mathrm{colim}} \mathcal{X}^J \in \mathsf{PreStk}.$$

The action of $\mathrm{Ran}_{\mathcal{X}}$ on $\mathrm{Ran}^{\rightarrow}_{\mathcal{X}}$ is then defined using the maps:

$$\mathsf{fSet} \times \mathsf{fSet}^{\rightarrow} \to \mathsf{fSet}$$

$$\Big(K, (\curlyvee : I \to J)\Big) \mapsto (I \to J \coprod K).$$

Notation 13.7.2. We use the notation:

$$\sigma_{\mathcal{X}^I} : \mathcal{X}^I \to \mathrm{Ran}_{\mathcal{X},I}$$

$$\sigma_{\mathrm{Ran}_{\mathcal{X}}} : \mathrm{Ran}_{\mathcal{X}} \to \mathrm{Ran}_{\mathcal{X}}^{\to}$$

for the obvious sections.

13.8. **Factorization modules.** Let $\mathcal{Z}$ be a factorization $\mathrm{Ran}_{\mathcal{X}}$-module space.

Definition 13.8.1. As in §12.6, for C a chiral category on $\mathcal{X}$, we have a notion of chiral C-module category M over $\mathcal{Z}$. We denote the resulting category by $\mathsf{ModCat}^{ch}_{/\mathcal{Z}}(\mathsf{C})$.

Moreover, for $\mathcal{A}$ a factorization algebra in C and $\mathsf{M} \in \mathsf{ModCat}^{ch}_{/\mathcal{Z}}(\mathsf{C})$, §12.6 gives a notion of *factorization $\mathcal{A}$-module* in M. We denote the resulting category by $\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M})$.

Remark 13.8.2. Our notation will frequently identify $\mathsf{M} \in \mathsf{ModCat}^{ch}_{/\mathcal{Z}}(\mathsf{C})$ with its underlying sheaf of categories on $\mathcal{Z}$, and $M \in \mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M})$ with the underlying object of $\Gamma(\mathcal{Z}, \mathsf{M})$.

Remark 13.8.3. Using the general stability results in [Lur09], one readily sees that $\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M})$ is a cocomplete DG category.

Remark 13.8.4. Let $\mathcal{Z}$ be a factorization $\mathrm{Ran}_{\mathcal{X}}$-module space. Suppose that we have C and D chiral categories on $\mathcal{X}$ with chiral module categories $\mathsf{M} \in \mathsf{ModCat}^{ch}_{/\mathcal{Z}}(\mathsf{C})$, $\mathsf{N} \in \mathsf{ModCat}^{ch}_{/\mathcal{Z}}(\mathsf{D})$. Suppose that we have a morphism of factorization module data[31] from (C, M) to (D, N) with underlying functors:

$$\psi : \mathsf{C} \to \mathsf{D}$$

$$\varphi : \mathsf{M} \to \mathsf{N}.$$

By Remark 13.2.8, there is an induced functor $\psi : \mathsf{Alg}^{\mathrm{fact}}(\mathsf{C}) \to \mathsf{Alg}^{\mathrm{fact}}(\mathsf{D})$, and as in *loc. cit.*, for $\mathcal{A} \in \mathsf{Alg}^{\mathrm{fact}}(\mathsf{C})$ we obtain a canonical functor:

[31]Really, we mean a morphism of multiplicative sheaves of categories with respect to the colored operad controlling non-unital algebras with a left module.

$$\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M}) \to \varphi(\mathcal{A})\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{N}). \tag{13.8.1}$$

Notation 13.8.5. When $\mathcal{Z} = \mathrm{Ran}_{\mathcal{X},I}$, we use the notation $\mathsf{ModCat}^{ch}_{/\mathcal{X}^I}(\mathsf{C})$ in place of $\mathsf{ModCat}^{ch}_{/\mathrm{Ran}_{\mathcal{X},I}}(\mathsf{C})$, and $\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\sigma^*_{\mathcal{X}^I}(\mathsf{M}))$ in place of $\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M})$ when there is no risk for confusion. We refer to e.g. such chiral module categories as chiral module categories on $\mathcal{X}^I$ (for C). Note that in this setting, $\mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}(\mathsf{M})$ is a $\mathsf{QCoh}(\mathcal{X}^I)$-module category.

We remark that these notions were defined previously in the $I = *$ case in [BD04], and for higher order I in [Roz10] and [FG12].

Example 13.8.6. The restriction $\mathsf{C}_{\mathcal{X}^I}$ of C to $\mathcal{X}^I$ can be regarded as a factorization module category over C on $\mathcal{X}^I$.

13.9. **Unital modules.** Next, we discuss the unital setting. The definitions are largely parallel to those in the non-unital setting, and therefore we indicate them only briefly.

13.10. A *unital factorization module space* for $\mathrm{Ran}_{\mathcal{X}}$ is a lax prestack $\mathcal{Z}^{un}$ with an action of $\mathrm{Ran}^{un,ch}_{\mathcal{X}}$ in $\mathsf{PreStk}^{lax}_{corr}$. Similarly, we have the notion of *augmented* unital factorization module space: we ask in addition for a $\mathrm{Ran}^{un,ch}_{\mathcal{X}}$-equivariant map $\varpi : \mathcal{Z}^{un} \to \mathrm{Ran}^{un}_{\mathcal{X}}$ that is a morphism in the 1-full subcategory PreStk^{lax} of $\mathsf{PreStk}^{lax}_{corr}$.

Remark 13.10.1. Understanding these conditions explicitly works exactly as in the non-unital setting of Remark 13.6.2.

For $\mathcal{Z}^{un}$ a unital factorization module space, we define $\mathcal{Z} := \mathcal{Z}^{un,\mathsf{PreStk}} \in \mathsf{PreStk}$ to be the underlying prestack. Clearly $\mathcal{Z}$ carries a canonical structure of factorization module space for $\mathrm{Ran}_{\mathcal{X}}$.

Remark 13.10.2. We alert the reader to a potential source of confusion in this notation: $\mathcal{Z}$ is constructed from $\mathcal{Z}^{un}$, and not the other way around.

Terminology 13.10.3. We will sometimes abbreviate the situation by simply saying that $\mathcal{Z}$ is a unital factorization module space for $\mathrm{Ran}_{\mathcal{X}}$, with the structure of $\mathcal{Z}^{un}$ being implicit.

As in Example 13.6.3, we can produce augmented unital factorization module spaces from augmented $\mathrm{Ran}_{\mathcal{X}}^{un,*}$-modules in PreStk^{lax}.

Example 13.10.4. From this construction, one obtains lax prestacks $\mathrm{Ran}_{\mathcal{X}}^{un,\to}$ and $\mathrm{Ran}_{\mathcal{X},I}^{un}$ with unital factorization module space structures, and with underlying prestacks $\mathrm{Ran}_{\mathcal{X}}^{\to}$ and $\mathrm{Ran}_{\mathcal{X},I}$ respectively.

13.11. For $\mathcal{Z}^{un}$ a unital factorization module space for $\mathrm{Ran}_{\mathcal{X}}$, we define unital chiral module category M for a unital chiral category C as in the non-unital case.[32]

Similarly, we define unital factorization modules for a unital factorization algebra $\mathcal{A}$ in a specified unital factorization module category.

We denote the resulting categories by:

$$\mathsf{ModCat}^{ch}_{/\mathcal{Z},un}(\mathsf{C}) \text{ and } \mathcal{A}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\mathsf{M}).$$

The latter is a cocomplete DG category.

Notation 13.11.1. We will allow notations parallel to those from Notation 13.8.5 when $\mathcal{Z} = \mathrm{Ran}_{\mathcal{X},I}$.

Remark 13.11.2. The obvious counterpart to Remark 13.8.4 holds in the unital setting just as well.

13.12. **External fusion.** Next, we discuss the *external fusion* construction. For definiteness, we take $\mathcal{X} = X_{dR}$. Let C be a chiral category on $\mathcal{X}$ and let $\mathcal{A}$ be a factorization algebra in C.

[32]However, we emphasize that the colored operad we use is that controlling unital commutative algebras equipped with a unital module.

We give a description of what is expected from external fusion in this section, postponing its construction to 13.22.

For I a finite set, let $\mathsf{C}_{X^I_{dR}}$ denote the corresponding sheaf of categories on X^I_{dR}. As in Example 13.8.6, $\mathsf{C}_{X^I_{dR}}$ is a chiral module category for C. Therefore, we obtain the category $\mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^I_{dR}})$ of chiral modules for $\mathcal{A}$ on X^I_{dR}.

For I and J two finite sets, we form $[X^I_{dR} \times X^J_{dR}]_{disj}$ and let:

$$\mathsf{C}_{I,J,disj} \in \mathsf{ShvCat}_{/[X^I_{dR} \times X^J_{dR}]_{disj}}$$

denote the restriction of $\mathsf{C}_{X^{I \coprod J}_{dR}}$, considered as a C-chiral module category in the natural way.

The *external fusion* construction is a canonical functor:

$$\mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^I_{dR}}) \otimes \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^J_{dR}}) \to \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{I,J,disj}). \tag{13.12.1}$$

of $D(X^I) \otimes D(X^J)$-module categories.

At the level of global sections on X^I_{dR}, X^J_{dR} and $[X^I_{dR} \times X^J_{dR}]_{disj}$, this construction is given by external product. We describe it completely at the module level in §13.22.

Remark 13.12.1. We do not expect (13.12.1) to be an equivalence in general: rather, we expect this only after an appropriate *renormalization*, and this depends on the specific factorization algebra under consideration. For the Kac-Moody factorization algebra, the appropriate notion of renormalization is explained over a point in [FG09].

Remark 13.12.2. The functoriality of this construction will be enhanced in §14.14.

13.13. **Modules for the unit factorization algebra.** A key slogan in the unital setting is that a unital module structure for the unit is no extra data. We make this precise below.

Let C be a unital factorization category on $\mathcal{X}$ and let I be a finite set.

Construction 13.13.1. Form the diagram:

$$\begin{tikzcd} \mathrm{Ran}_{\mathcal{X},I,\varnothing} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{Ran}_{\mathcal{X}} \\ \mathcal{X}^I \end{tikzcd}$$

As in §13.3, the map $\mathfrak{Fus}_{\mathsf{C}}$ induces a functor:

$$p_1^*(\mathsf{C}_{\mathcal{X}^I}) \to p_2^*(\mathsf{C}).$$

As in *loc. cit.*, the material of §12.34 shows that the functor upgrades to give:

$$\Gamma(\mathcal{X}^I, \mathsf{C}_{\mathcal{X}^I}) \to \mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{\mathcal{X}^I}).$$

This functor is easily seen to be left adjoint to the obvious restriction functor.

Theorem 13.13.2. *For $\mathcal{X} = X_{dR}$ with X a finite type scheme, the restriction functor:*

$$\mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{X_{dR}^I}) \to \Gamma(X_{dR}^I, \mathsf{C}_{X_{dR}^I})$$

is an equivalence with inverse given by Construction 13.13.1.

Proof. The composition:

$$\Gamma(X_{dR}^I, \mathsf{C}_{X_{dR}^I}) \to \mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{X_{dR}^I}) \to \Gamma(X_{dR}^I, \mathsf{C}_{X_{dR}^I})$$

is obviously the identity functor.

One easily constructs (for general $\mathcal{X}$) a canonical natural transformation:

$$\begin{tikzcd} \mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{X_{dR}^I}) \to \Gamma(X_{dR}^I, \mathsf{C}_{X_{dR}^I}) \to \mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{X_{dR}^I}) \arrow[d, Rightarrow] \\ \mathrm{id}_{\mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(\mathsf{C}_{X_{dR}^I})} \end{tikzcd}$$

using fusion.

But this natural transformation is immediately seen to be an equivalence over strata in $\mathrm{Ran}_{X_{dR},I}$ by exploiting factorization, and then the fact that we are dealing with D-modules means that this map is an equivalence.

$\square$

13.14. In §13.14-13.20, we compare our definition of factorization algebra with that of [FG12] in the case $\mathcal{X} = X_{dR}$.

This material is a bit digressive, and the reader may safely skip it and refer back to it as necessary.

We fix X a separated scheme of finite type through §13.20.

Remark 13.14.1. We follow [FG12] closely in our definitions here.

Remark 13.14.2. What follows is, by necessity, entirely in the non-unital setting.

13.15. We begin with a construction in the general framework as in §12: let $\mathcal{S}$ be a commutative algebra in PreStk_{corr}. We use the notation (12.21.1) for the correspondences defining the multiplication and unit operations.

Under this hypothesis, Corollary 19.11.1 implies that $\mathsf{ShvCat}_{/\mathcal{S}}$ carries a canonical symmetric monoidal structure with monoidal product the composition:

$$\mathsf{ShvCat}_{/\mathcal{S}} \times \mathsf{ShvCat}_{/\mathcal{S}} \xrightarrow{-\boxtimes-} \mathsf{ShvCat}_{/\mathcal{S}\times\mathcal{S}} \xrightarrow{m_1^*} \mathsf{ShvCat}_{/\,\mathrm{mult}_{\mathcal{S}}} \xrightarrow{m_{2,*}} \mathsf{ShvCat}_{/\mathcal{S}}.$$

We will denote the tensor product for this symmetric monoidal structure by:

$$\Psi \ast \Phi := m_{2,*}m_1^*(\Psi \boxtimes \Phi).$$

Remark 13.15.1. Observe that the functor:

$$\Gamma(\mathcal{S}, -) : \mathsf{ShvCat}_{/\mathcal{S}} \to \mathsf{DGCat}_{cont}$$

is lax symmetric monoidal relative to the symmetric monoidal structure $*$ and the tensor product of cocomplete DG categories, respectively. The structure maps are given by the tautological map:

$$\Gamma(\mathcal{S}, \Psi) \otimes \Gamma(\mathcal{S}, \Phi) \to \Gamma(\mathcal{S} \times \mathcal{S}, \Psi \boxtimes \Phi) \to \Gamma\big(\,\mathrm{mult}_{\mathcal{S}}, m_1^*(\Psi \boxtimes \Phi)\big) =$$
$$\Gamma\big(\mathcal{S}, m_{2,*}m_1^*(\Psi \boxtimes \Phi)\big) =: \Gamma(\mathcal{S}, \Psi * \Phi).$$

Recall that we have defined $\mathsf{MultCat}^{op-w}(\mathcal{S})$ in §12.29. The following result follows from the theory of correspondences.

Proposition 13.15.2. *There is a canonical equivalence of categories:*

$$\mathsf{MultCat}^{op-w}(\mathcal{S}) \simeq \mathsf{ComCoalg}^{lax}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big).$$

Here the right hand side of the equality is the category of commutative coalgebras under lax morphisms, as in §12.23.

13.16. We will need the following material about the equivalence of Proposition 13.15.2.

Let:

$$\mathsf{ComCoalg}^{r.adj}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big) \subseteq \mathsf{ComCoalg}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big)$$

denote the full subcategory consisting of commutative coalgebras C for which the maps:

$$\Psi \to \Psi * \Psi \text{ and } \Psi \to \mathsf{QCoh}_{\mathcal{S}}$$

admit right adjoints in the category $\mathsf{ShvCat}_{/\mathcal{S}}$ (equivalently: we can ask this for all n-ary operations). Define the full subcategory:

$$\mathsf{ComAlg}^{l.adj}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big) \subseteq \mathsf{ComAlg}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big)$$

similarly, with left adjoints replacing the role of right adjoints.

By the theory [GR14] of 2-categories, we obtain an equivalence:

$$\mathsf{ComAlg}^{l.adj}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big) \simeq \mathsf{ComCoalg}^{r.adj}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big) \tag{13.16.1}$$

given by passing to adjoints in our operations.

Observe that, by Proposition 19.9.1 (3), if m_2 and e_2 are quasi-compact quasi-separated schematic morphisms, then the category $\mathsf{ComCoalg}^{r.adj}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big)$ contains

$$\mathsf{MultCat}^{op-w,r.adj}(\mathcal{S}) \subseteq \mathsf{MultCat}^{op-w}(\mathcal{S})$$

under the equivalence of Proposition 13.15.2. In particular, it contains $\mathsf{MultCat}(\mathcal{S})$.

13.17. We now give a version of Proposition 13.15.2 for multiplicative sheaves.

Given $\Psi \in \mathsf{ComAlg}\Big((\mathsf{ShvCat}_{/\mathcal{S}}, *)\Big)$, the category $\Gamma(\mathcal{S}, \Psi)$ inherits a canonical symmetric monoidal structure, coming from the lax symmetric monoidal structure of Remark 13.15.1.

Suppose that m_2 and e_2 are quasi-compact quasi-separated schematic morphisms. Proposition 13.15.2, the conclusion of §13.16, and (13.16.1) imply that for $\Psi \in \mathsf{MultCat}^{op-w,r.adj}(\mathcal{S})$, $\Gamma(\mathcal{S}, \Psi)$ inherits a canonical symmetric monoidal structure. We will denote the symmetric monoidal product here by $*$ as well.

Using the perspective of §12.26, we obtain the following counterpart to Proposition 13.15.2.

Proposition 13.17.1. *For $\Psi \in \mathsf{MultCat}^{op-w,r.adj}$, there is a canonical equivalence of categories:*

$$\mathsf{Mult}^{op-w}(\Psi) \simeq \mathsf{ComAlg}\Big(\Gamma(\mathcal{S}, \Psi), *\Big).$$

13.18. We now specialize to the case of Ran space.

We have the following lemma.

Lemma 13.18.1. *The morphism:*

$$\mathrm{add} : [\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}}]_{disj} \to \mathrm{Ran}_{X_{dR}}$$

is schematic and a quasi-compact étale morphism.

Proof. First, note that tautologically we have $\mathrm{Ran}_{X_{dR}} = (\mathrm{Ran}_X)_{dR}$.

Let S be an affine test scheme. As in Example 11.3.2, a morphism $\varphi : S \to [\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}}]_{disj}$ is equivalent to giving two finite sets:

$$\{\varphi_1^1, \ldots, \varphi_n^1\} \text{ and } \{\varphi_1^2, \ldots, \varphi_m^2\}$$

where each φ_i^j is a map $S^{cl,red} \to X$, and such that, for every $1 \leqslant i \leqslant n$ and $1 \leqslant i' \leqslant m$, the map $\varphi_i^1 \times \varphi_{i'}^2 : S^{cl,red} \to X \times X$ factors through the open $X \times X \backslash \Delta(X)$.

Moreover, a map $\psi : S \to \mathrm{Ran}_{X_{dR}}$ is equivalent to giving a finite collection of maps $\psi_1, \ldots, \psi_r : S^{cl,red} \to X$. Therefore, we see that the fiber over such a map is the coproduct of spaces:

$$S \underset{X_{dR}^r}{\times} [X_{dR}^n \times X_{dR}^m]_{disj}$$

with the coproduct taken over positive integers with $n + m = r$. This evidently gives the result.

$\square$

13.19. By Lemma 13.18.1, $\mathcal{S} := \mathrm{Ran}_{X_{dR},\varnothing}$ satisfies the requirements of the discussion in §13.15-13.17. Therefore, for $\mathsf{C} \in \mathsf{Cat}^{ch}(X_{dR})$, the category:

$$\Gamma(\mathrm{Ran}_{X_{dR},\varnothing}, \mathsf{C}) = \mathsf{Vect} \oplus \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C})$$

inherits a symmetric monoidal structure. More precisely, $\Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C})$ carries a non-unital commutative algebra structure in DGCat_{cont}, and this unital symmetric monoidal structure arises by formally adding a unit (in DGCat_{cont}).

We refer to this (resp. non-unital) symmetric monoidal structure as the *chiral tensor product* on $\Gamma(\mathrm{Ran}_{X_{dR},\varnothing}, \mathsf{C})$ (resp. $\Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C})$). We denote the resulting binary product by $-\overset{ch}{\otimes}-$.

Definition 13.19.1. A *chiral coalgebra* in C is a non-unital commutative coalgebra in $\left(\Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}), \overset{ch}{\otimes}\right)$. We denote the resulting category by $\mathsf{Coalg}^{\mathsf{ch}}(\mathsf{C})$.

Remark 13.19.2. The category $\mathsf{Coalg}^{\mathsf{ch}}(\mathsf{C})$ is cocomplete.

Remark 13.19.3. We can identify $\mathsf{Coalg}^{\mathsf{ch}}(\mathsf{C})$ with the full subcategory of unital coalgebras in $\left(\Gamma(\mathrm{Ran}_{X_{dR},\varnothing}, \mathsf{C}), \overset{ch}{\otimes}\right)$ consisting of those coalgebras such that the counit map becomes an isomorphism after applying the projection:

$$\left(\Gamma(\mathrm{Ran}_{X_{dR},\varnothing}, \mathsf{C}), \overset{ch}{\otimes}\right) = \mathsf{Vect} \oplus \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}) \to \mathsf{Vect}.$$

The following results from Proposition 13.17.1.

Proposition 13.19.4. *There is a canonical equivalence:*

$$\mathsf{Mult}^{op-w}_{non-unital}(\mathsf{C}) \simeq \mathsf{Coalg}^{\mathsf{ch}}(\mathsf{C}).$$

Here, as in §13.2, the subscript "non-unital" indicates that we take the operad controlling non-unital commutative algebras.

Remark 13.19.5. This proposition implies that, for X a separated scheme of finite type, the category $\mathsf{Alg}^{\mathrm{fact}}(X_{dR})$ coincides with the category of factorization algebras as defined in [FG12]. A variant of the above material with general colored operads allows us to put the theory of chiral modules from [FG12] into our framework as well.

13.20. Let C be a factorization category on X_{dR}.

Definition 13.20.1. We define the category $\mathsf{LieAlg}^{\mathsf{ch}}(\mathsf{C})$ of *chiral Lie algebras* in C as the category of Lie algebras in $\left(\Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}), \overset{ch}{\otimes}\right)$.

We define the full subcategory $\mathsf{Alg}^{ch}(\mathsf{C}) \subseteq \mathsf{LieAlg}^{\mathsf{ch}}(\mathsf{C})$ of *chiral algebras* in C to consist of those chiral Lie algebras whose underlying object lies in the full subcategory:

$$\Gamma(X_{dR}, \mathsf{C}|_{X_{dR}}) \subseteq \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}).$$

13.21. Fix $\mathsf{C} \in \mathsf{Cat}^{ch}(X_{dR})$, and let $\mathcal{C} = \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C})$ be considered a non-unital algebra in DGCat_{cont} through the chiral tensor product.

As in [FG12], we have the following result:

Theorem 13.21.1. *The Koszul duality functor:*

$$\mathsf{LieAlg}^{\mathsf{ch}}(\mathsf{C}) := \mathsf{LieAlg}(\mathcal{C}) \to \mathsf{ComCoalg}(\mathcal{C}) =: \mathsf{Coalg}^{\mathsf{ch}}(\mathsf{C})$$

is an equivalence.

This equivalence identifies the full subcategories $\mathsf{Alg}^{ch}(\mathsf{C})$ *and* $\mathsf{Alg}^{\mathrm{fact}}(\mathsf{C})$.

Warning 13.21.2. We remind that this functor does not commute with forgetful functors to $\mathcal{C}$: rather, the composition $\mathsf{LieAlg}(\mathcal{C}) \to \mathsf{ComCoalg}(\mathcal{C}) \xrightarrow{\mathrm{Oblv}} \mathcal{C}$ is given by the (reduced) homological Chevalley complex.

13.22. **Construction of external fusion.** As promised in §13.12, we now carefully describe the external fusion construction.

Remark 13.22.1. The construction imitates the construction of the tensor product of modules as the geometric realization of the bar construction.

Recall the prestack $\mathrm{Ran}_{X_{dR},I}$ (resp. $\mathrm{Ran}_{X_{dR},J}$) from Example 13.7. Let σ_I (resp. σ_J) denote the structure map to $\mathrm{Ran}_{X_{dR}}$. Let $\mathrm{Ran}_{X_{dR},I,J,disj}$ denote the variant of $\mathrm{Ran}_{X_{dR},I\coprod J}$ where we require our points in $X_{dR}^I \times X_{dR}^J$ to lie $[X_{dR}^I \times X_{dR}^J]_{disj} \xrightarrow{\mathrm{add}} \mathrm{Ran}_{X_{dR}}$.

Let $M \in \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^I_{dR}})$ and $N \in \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^J_{dR}})$. Let $\widetilde{M} \in \Gamma(\mathrm{Ran}_{X_{dR},I}, \sigma_I^*(\mathsf{C}))$ be the object defining the factorization module structure for M, and let $\tilde{N}$ be defined similarly.

We form the augmented simplicial object:

$$\ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} [\mathrm{Ran}_{X_{dR},I} \times \mathrm{Ran}_{X_{dR},\varnothing} \times \mathrm{Ran}_{X_{dR},J}]_{disj} \rightrightarrows [\mathrm{Ran}_{X_{dR},I} \times \mathrm{Ran}_{X_{dR},J}]_{disj} \longrightarrow \mathrm{Ran}_{X_{dR},I,J,disj}$$

where e.g. $[\mathrm{Ran}_{X_{dR},I} \times \mathrm{Ran}_{X_{dR},J}]_{disj}$ denotes the locus where the corresponding points of $\mathrm{Ran}_{X_dR} \times \mathrm{Ran}_{X_{dR}}$ are disjoint, and $[\mathrm{Ran}_{X_{dR},I} \times \mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR},J}]_{disj}$ denotes the locus where the triple of points of $\mathrm{Ran}_{X_{dR}}$ are pairwise disjoint, etc. The two horizontal maps in the above simplicial object are given by the action maps for $\mathrm{Ran}_{X_{dR},I}$ and $\mathrm{Ran}_{X_{dR},J}$ respectively.

We form a compatible sheaf of categories on this simplicial diagram by pullback of C from $\mathrm{Ran}_{X_{dR}}$. Indeed, the factorization of C allows us to form this construction.

Then the structure of module on M and N allows us to form a compatible system of global sections here, where on the first term we take $\widetilde{M} \boxtimes \tilde{N}$ (i.e., its restriction to the disjoint locus), and on the second term we take $\widetilde{M} \boxtimes \mathcal{A} \boxtimes \tilde{N}$, $\widetilde{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \boxtimes \tilde{N}$, etc.

Observe that our augmented simplicial object above is an étale hypercovering of $\mathrm{Ran}_{X_{dR},I,J,disj}$ (c.f. Lemma 13.18.1). Therefore, by étale hyperdescent, this defines an object $\widetilde{M \boxtimes N}$ on $\mathrm{Ran}_{X_{dR},I,J,disj}$. One easily verifies that it carries a canonical structure of $\mathcal{A}$-module as desired.

Remark 13.22.2. The above works in the unital setting as well, showing that the external fusion of unital modules is naturally a unital module as well.

14. Chiral categories via partitions

14.1. In this section, we give an alternative approach to the theory of chiral categories and factorization algebras using categories of partitions.

This approach is a much more faithful realization of the heuristic of §1.13. In particular, it gives a theory of chiral categories on a finite type scheme that uses only finite-dimensional geometry, i.e., the Ran space is not explicitly mentioned.

After developing this material, the author found that the main idea of this approach independently appears already in a preprint of [Rei12].

We fix a prestack $\mathcal{X}$ throughout this section.

Remark 14.1.1. In this section, we prove a result that says that giving a factorization category is equivalent to giving data:

$$\mathsf{C}_{\mathcal{X}^I} \in \mathsf{ShvCat}_{/\mathcal{X}^I}$$

and equivalences:

$$\mathsf{C}_{\mathcal{X}^I} \boxtimes \mathsf{C}_{\mathcal{X}^J}\big|_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \simeq \mathsf{C}_{\mathcal{X}^{I \coprod J}}\big|_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \in \mathsf{ShvCat}_{/[\mathcal{X}^I \times \mathcal{X}^J]_{disj}}$$

satisfying further compatibilities.

The reader willing to take such statements on faith, or who believes this to be a tautology given our earlier material, is advised to skip this section entirely.

Remark 14.1.2. For the reader who has continued reader past Remark 14.1.1, we note what technical issues occur.

By definition, a multiplicative sheaf on a prestack with a multiplicative structure in the correspondence category (say, associative but not assumed commutative, for simplicity of terminology) is an algebra in a certain correspondence category.

Roughly, in higher algebra, an algebra somewhere is something like a simplicial object. A priori, if one thinks out what a simplicial object in a correspondence category is in terms of the original category, it appears to be a very large quantity of data.

This is exactly what we are trying to do here: to give a definition of chiral category that does not mention Ran space or correspondences, we need to give an alternative description of algebras in correspondence categories.

This is exactly what is done in the appendix §20: we give a workable perspective on simplicial objects in correspondence categories, or more generally, on any functor into a correspondence category.

This is the main technique that is exploited in this section; the remainder consists of details.

14.2. We begin by defining certain combinatorial categories of partitions.

Define the (1,1)-category Part of partitions as the category with objects surjections $(p : I \twoheadrightarrow J)$ of non-empty finite sets and with morphisms from $(p_1 : I_1 \twoheadrightarrow J_1)$ to $(p_2 : I_2 \twoheadrightarrow J_2)$ defined by commutative diagrams:

$$\begin{tikzcd} I_1 \arrow[r, two heads] \arrow[d, "p_1"] & I_2 \arrow[d, "p_2"] \\ J_1 & J_2 \arrow[l, two heads] \end{tikzcd} \tag{14.2.1}$$

under the obvious compositions.

Similarly, define Part_{un} as the category whose objects are (arbitrary) maps $p : I \to J$ of (possibly empty) finite sets and in which morphisms $(p_1 : I_1 \to J_1) \to (p_2 : I_2 \to J_2)$ are commutative diagrams:

$$\begin{tikzcd} I_1 \arrow[r] \arrow[d, "p_1"] & I_2 \arrow[d, "p_2"] \\ J_1 & J_2 \arrow[l] \end{tikzcd}$$

Remark 14.2.1. One can think of such a map $p : I \twoheadrightarrow J$ as a partition of I indexed by J, where the associated partition is $I = \coprod_{j \in J} I_j$, $I_j := p^{-1}(j)$. Allowing non-surjective maps in Part_{un} then translates into allowing partitions into possibly empty sets.

Remark 14.2.2. Note that Part contains fSet as the full subcategory of partitions indexed by a singleton set. The functor $\mathsf{fSet}^{op} \to \mathsf{Part}^{op}$ is cofinal. There is a canonical splitting $\mathsf{Part} \to \mathsf{fSet}$ of this functor sending $(p : I \twoheadrightarrow J) \in \mathsf{Part}$ to I.

The same remarks hold with Part_{un} replacing Part and $\mathsf{Set}_{<\infty}$ replacing fSet.

We have a non-unital symmetric monoidal structure on Part given by disjoint union of (pairs of) sets. We denote the corresponding product by $\coprod$, although it is not the coproduct on this category.

Remark 14.2.3. In the notation of §20, we have $\mathsf{Part} = \mathsf{Tw}(\mathsf{fSet})$ and $\mathsf{Part}_{un} = \mathsf{Tw}(\mathsf{Set}_{<\infty})$, compatibly with (non-unital for Part) symmetric monoidal structures.

14.3. Define the prestack $[\mathcal{X} \times \mathcal{X}]_{disj}$ as in (11.13.1). That is, it is the open subprestack of $\mathcal{X}^2$ defined by the condition that a pair of maps $\varphi = (\varphi_1, \varphi_2) : S \to \mathcal{X}^2$ factors through $[\mathcal{X} \times \mathcal{X}]_{disj}$ if the diagram:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & S \arrow[d, "\varphi_2"] \\ S \arrow[r, "\varphi_1"] & \mathcal{X} \end{tikzcd}$$

is Cartesian.

For $(p : I \twoheadrightarrow J) \in \mathsf{Part}_{un}$, define $U(p) \in \mathsf{PreStk}$ as the open subprestack of $\mathcal{X}^I$ defined for an affine test scheme S by:

$$U(p)(S) = \left\{ \varphi = (\varphi_i)_{i \in I} : S \to \mathcal{X}^I \;\middle|\; \begin{array}{l} \text{for every } i_1, i_2 \in I \text{ with } p(i_1) \neq p(i_2) \text{ the map} \\ (\varphi_{i_1}, \varphi_{i_2}) : S \to \mathcal{X}^2 \text{ factors through } [\mathcal{X} \times \mathcal{X}]_{disj} \end{array} \right\}. \tag{14.3.1}$$

Example 14.3.1. For p the identity map $\{1, 2\} \to \{1, 2\}$ we have $U(p) = [\mathcal{X} \times \mathcal{X}]_{disj}$.

Given a map $\varepsilon : (p_1 : I_1 \twoheadrightarrow J_1) \to (p_2 : I_2 \twoheadrightarrow J_2)$ in Part_{un}, we obtain a map $U(\varepsilon) : U(p_2) \to U(p_1)$ induced by the diagram:

$$\begin{tikzcd}
U(p_2) \arrow[r, dotted, "U(\varepsilon)"] \arrow[d, hook] & U(p_1) \arrow[d, hook] \\
\mathfrak{X}^{I_2} \arrow[r] & \mathfrak{X}^{I_1}.
\end{tikzcd}$$

This gives a functor:

$$U : \mathsf{Part}_{un}^{op} \to \mathsf{PreStk} \tag{14.3.2}$$

sending p to $U(p)$. It is naturally colax symmetric monoidal relative to the Cartesian product on the target, i.e., we have natural maps:

$$U(p \coprod q) \hookrightarrow U(p) \times U(q). \tag{14.3.3}$$

Remark 14.3.2. We will also denote the restriction of the functor (14.3.2) to Part^{op} by U.

14.4. **Main result.** We imitate the earlier constructions to obtain the lax symmetric monoidal functor:

$$\mathsf{ShvCat}_{/U} : \mathsf{Part}_{un} \to \mathsf{Cat}_{pres}$$
$$\Big(p : I \to J\Big) \mapsto \mathsf{ShvCat}_{/U(p)}.$$

and thereby (c.f. §12.15) the symmetric monoidal functor of symmetric monoidal categories:

$$\mathsf{Groth}(\mathsf{ShvCat}_{/U}) \to \mathsf{Part}_{un}.$$

The main construction of this section is given by the following.

Proposition-Construction 14.4.1. *(1) The category* $\mathsf{Cat}^{ch}(\mathcal{X})$ *is equivalent to the category of symmetric monoidal*[33] *sections:*

$$\begin{tikzcd} & \mathsf{Groth}(\mathsf{ShvCat}_{/U}) \arrow[d] \\ \mathsf{Part} \arrow[ur, dotted] \arrow[r, hook] & \mathsf{Part}_{un} \end{tikzcd} \tag{14.4.1}$$

sending all arrows in Part *to coCartesian arrows.*

(2) The category $\mathsf{Cat}^{ch}_{un,str}(\mathcal{X})$ *(see Remark 13.3.1 for the notation) is canonically equivalent to the category of symmetric monoidal sections:*

$$p \mapsto \mathsf{C}_{U_{un}(p)} \in \mathsf{ShvCat}_{/U_{un}(p)} \tag{14.4.2}$$

of $\mathsf{Groth}(\mathsf{ShvCat}_{/U}) \to \mathsf{Part}_{un}$ *such that:*

(a) Arrows in Part *map to coCartesian arrows.*

(b) Arrows in:

$$\mathsf{Set}^{op}_{<\infty} = \{p : \varnothing \to I\} \subseteq \mathsf{Part}_{un}$$

map to coCartesian arrows.

Remark 14.4.2. It will follow from the construction that Proposition-Construction 14.4.1 satisfies the following compatibilities.

- The non-unital symmetric monoidal functor $\mathsf{Part} \hookrightarrow \mathsf{Part}_{un}$ induces the restriction functor:

$$\mathsf{Cat}^{ch}_{un,str}(\mathcal{X}) \to \mathsf{Cat}^{ch}(\mathcal{X}).$$

- Restricting a functor (14.4.1) to:

[33] Necessarily understood in the sense of non-unital symmetric monoidal categories and functors.

$$\mathsf{fSet} = \{I \to *\} \subseteq \mathsf{Part}$$

(necessarily forgetting the symmetric monoidal structure), we obtain a compatible system of sheaves of categories on the $\mathcal{X}^I$ for $I \in \mathsf{fSet}$, i.e., a sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$. This corresponds to the restriction:

$$\mathsf{Cat}^{ch}(\mathcal{X}) \to \mathsf{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}}.$$

- Restricting a functor (14.4.2) to:

$$\mathsf{Set}_{<\infty} = \{I \to *\} \subseteq \mathsf{Part}_{un}$$

we obtain a lax compatible system of sheaves of categories on the $\mathcal{X}^I$ for $I \in \mathsf{Set}_{<\infty}$, and that is strictly compatible with respect to morphisms in fSet. By Corollary 11.6.2, this amounts to a sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}^{un}$.

This construction then corresponds to the restriction:

$$\mathsf{Cat}^{ch}_{un,str}(\mathcal{X}) \to \mathsf{ShvCat}_{/\mathrm{Ran}^{un}_{\mathcal{X}}}.$$

Remark 14.4.3. The reader who runs through the definitions should be convinced that Proposition-Construction 14.4.1 is essentially tautological. The only difficulties arising below are of the usual sort in higher category theory: we just provide the necessary categorical language for the obvious constructions.

Remark 14.4.4. The technical perspective on chiral categories provided by Proposition-Construction 14.4.1 differs from the one provided in §13 in that Ran space is not explicitly mentioned. This is somewhat convenient for constructing chiral categories from geometry, but is somewhat complicates developing the theory of §13. Moreover, working with non-strict unital chiral functors is not technically convenient in the partition framework.

One can readily develop much of the language of (unital and non-unital) chiral algebras and their modules in this framework.

14.5. We will develop a minimal working theory of operadic right Kan extensions, similar to the theory of operadic left Kan extensions in [Lur12] §2. This material can be significantly generalized, but we take a more pedestrian approach.

The main result here is the following.

Proposition 14.5.1. *Suppose that we are given a symmetric monoidal functor $\Psi : \mathcal{C}_1 \to \mathcal{C}_2$ of symmetric monoidal categories such that for every $X, Y \in \mathcal{C}_2$ the tensor product functor:*

$$\mathcal{C}_{1,X/} \times \mathcal{C}_{1,Y/} \to \mathcal{C}_{1,X\otimes Y/}$$

is op-cofinal. Here, e.g., $\mathcal{C}_{1,X/}$ is the associated undercategory.

Suppose that $\mathcal{D}$ is a symmetric monoidal category that is complete as a category.

Then the functor:

$$\mathsf{Hom}^{\otimes,\mathrm{lax}}(\mathcal{C}_2, \mathcal{D}) \to \mathsf{Hom}^{\otimes,\mathrm{lax}}(\mathcal{C}_1, \mathcal{D})$$

admits a right adjoint. At the level of mere functors, this right adjoint is computed as the right Kan extension.

Proof. Suppose that F is a lax symmetric monoidal functor $\mathcal{C}_1 \to \mathcal{D}$. Let $F^{\otimes} : \mathcal{C}_1^{\otimes} \to \mathcal{D}^{\otimes}$ denote the corresponding functor of categories coCartesian over Segal's category $\mathit{\Gamma}$, in the notation of [Lur12].

Standard arguments show that our hypotheses imply that the relative right Kan extension of $F^{\otimes}$, taken relative to $\mathit{\Gamma}$, exists, and preserves the appropriate coCartesian arrows to define a lax symmetric monoidal functor. This functor obviously computes the desired right adjoint. Moreover, by [Lur09] Corollary 4.3.1.16, we see that this relative right Kan extension restricts to the usual right Kan extension over $\{*\} \in \mathit{\Gamma}$, as desired.

$\square$

Remark 14.5.2. As the proof shows, we do not need to assume that $\mathcal{D}$ is complete: for a fixed lax symmetric monoidal functor $F : \mathcal{C}_1 \to \mathcal{D}$ we only need to assume that the relevant limits exist for the right adjoint to be defined on F.

Remark 14.5.3. In more down-to-earth terms, let $\widetilde{F} : \mathcal{C}_2 \to \mathcal{D}$ be the right Kan extension of a lax symmetric monoidal functor $F : \mathcal{C}_1 \to \mathcal{D}$. For $X, Y \in \mathcal{C}_2$, we have a diagram:

$$\begin{tikzcd}
\widetilde{F}(X) \otimes \widetilde{F}(Y) := \lim\limits_{\substack{X' \in \mathcal{C}_1 \\ X \to \Psi(X')}} F(X') \otimes \lim\limits_{\substack{Y' \in \mathcal{C}_1 \\ Y \to \Psi(Y')}} F(Y') \arrow[r] & \lim\limits_{\substack{X' \in \mathcal{C}_1, X \to \Psi(X') \\ Y' \in \mathcal{C}_1, Y \to \Psi(Y')}} F(X') \otimes F(Y') \\
& \lim\limits_{\substack{Z \in \mathcal{C}_1 \\ X \otimes Y \to \Psi(Z)}} F(Z) =: \widetilde{F}(X \otimes Y). \arrow[u]
\end{tikzcd}$$

Moreover, the left arrow at the end is an equivalence by the cofinality assumption. Therefore, we obtain a canonical map:

$$\widetilde{F}(X) \otimes \widetilde{F}(Y) \to \widetilde{F}(X \otimes Y)$$

as desired.

14.6. Let $\mathcal{C}$ be a symmetric monoidal category. Then a unital commutative algebra in $\mathcal{C}$ is equivalent to a symmetric monoidal functor $\mathsf{Set}_{<\infty} \to \mathcal{C}$, and a non-unital commutative algebra is equivalent to a non-unital symmetric monoidal functor $\mathsf{fSet} \to \mathcal{C}$ (see [Lur12] §2.2.4).

Therefore, a unital commutative algebra in $\mathcal{C}_{corr}$ is equivalent to a symmetric monoidal functor $\mathsf{Set}_{<\infty} \to \mathcal{C}_{corr}$. By §20, this data is equivalent to a symmetric monoidal functor $\mathsf{Part}_{un} = \mathsf{Tw}(\mathsf{Set}_{<\infty}) \to \mathcal{C}$ such that, for $I \xrightarrow{p} J \xrightarrow{q} K$ in $\mathsf{Set}_{<\infty}$ the diagram:

$$
\begin{tikzcd}
(I \xrightarrow{q\circ p} K) \arrow[r] \arrow[d] & (J \xrightarrow{q} K) \arrow[d] \\
(I \xrightarrow{p} J) \arrow[r] & (J \xrightarrow{\mathrm{id}_J} J)
\end{tikzcd}
\tag{14.6.1}
$$

maps to a Cartesian diagram.

Similarly, a non-unital commutative algebra in $\mathcal{C}_{corr}$ is equivalent to a non-unital symmetric monoidal functor $\mathsf{Part} \to \mathcal{C}$ sending the appropriate squares to Cartesian squares.

More explicitly: suppose we are given a non-unital symmetric monoidal functor $F : \mathsf{Part} \to \mathcal{C}$. We have the following correspondence in Part:

$$
\begin{tikzcd}
& (\{1,2\} \to *) \arrow[dl] \arrow[dr] & \\
(\{1,2\} \to \{1,2\}) & & (* \to *)
\end{tikzcd}
$$

and its image under F defines a correspondence:

$$
\begin{tikzcd}
& F(\{1,2\} \to *) \arrow[dl] \arrow[dr] & \\
A \otimes A & & A
\end{tikzcd}
$$

for $A := F(* \to *)$, and this correspondence defines the multiplication for A in $\mathcal{C}_{corr}$. The condition on fiber squares is relevant for considering associativity.

14.7. As in the framework of §11, let $\mathcal{G}$ be a groupoid.

For $(p : I \twoheadrightarrow J) \in \mathsf{Part}$, define the full subgroupoid $\mathrm{Ran}^I_{\mathcal{G},p-disj} \subseteq \mathrm{Ran}^I_{\mathcal{G}}$ by only allowing objects in $\pi_0(\mathrm{Ran}^I_{\mathcal{G}}) = \mathrm{Ran}^I_{\pi_0(\mathcal{G})}$ corresponding to I-tuples:

$$(S_i \subseteq \pi_0(\mathcal{G}) \text{ non-empty and finite})_{i\in I}$$

such that, for every $i_1 \neq i_2 \in I$ with $p(i_1) = p(i_2)$, the point $(S_{i_1}, S_{i_2}) \in \mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}$ lies in $[\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}]_{disj}$.

Example 14.7.1. For $I = \{1, 2\}$ and $J = *$, we have:

$$\mathrm{Ran}^{I}_{\mathcal{G},p-disj} = [\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G}}]_{disj}.$$

On the other hand, for p a bijection we have $\mathrm{Ran}^{I}_{\mathcal{G},p-disj} = \mathrm{Ran}^{I}_{\mathcal{G}}$.

In general, one writes I as the disjoint union of the sets $I_j := p^{-1}(j)$ and then $\mathrm{Ran}^{I}_{\mathcal{G},p-disj}$ is the product over J of the loci $[\mathrm{Ran}^{I_j}_{\mathcal{G}}]_{disj}$ in $\mathrm{Ran}^{I_j}_{\mathcal{G}}$ where all collections of points in $\mathcal{G}$ are pairwise disjoint.

We claim that this construction extends to a symmetric monoidal functor:

$$\begin{gathered} \mathsf{Part} \to \mathsf{Gpd} \\ (p : I \to J) \mapsto \mathrm{Ran}^{I}_{\mathcal{G},p-disj}\,. \end{gathered} \tag{14.7.1}$$

Indeed, first note that we have a canonical symmetric monoidal functor $\mathsf{Part} \to \mathsf{Gpd}$ sending $I \to \mathrm{Ran}^{I}_{\mathcal{G}}$ factoring through the projection $\mathsf{Part} \to \mathsf{fSet}$ and encoding the non-unital commutative algebra structure from §11.3. One immediately verifies that this induces the functor (14.7.1) in the obvious way.

Remark 14.7.2. The functor (14.7.1) has the following special property: given morphisms $I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K$ in fSet, the square (14.6.1) maps to a Cartesian square of groupoids. Therefore, our functor defines the structure of non-unital commutative algebra on $\mathrm{Ran}_{\mathcal{G}}$ in Gpd_{corr}, and this is exactly the chiral product.

By functoriality of the above construction in $\mathcal{G}$, it applies just as well in the setting in which prestacks replace groupoids.

14.8. We will need the following combinatorial digression.

Define the category Trip (of "triples") to consist of diagrams:

$$I \twoheadrightarrow J \twoheadrightarrow K$$

of non-empty finite sets. For morphisms, we take surjective morphisms that are contravariant in the I and K-variables and covariant in J. That is, morphisms are given by commutative diagrams:

$$\begin{tikzcd} I_1 \arrow[r, two heads, "p_1"] & J_1 \arrow[r, two heads, "q_1"] \arrow[d, two heads, "\beta"] & K_1 \\ I_2 \arrow[u, two heads, "\alpha"] \arrow[r, two heads, "p_2"] & J_2 \arrow[r, two heads, "q_2"] & K_2 \arrow[u, two heads, "\curlyvee"'] \end{tikzcd} \tag{14.8.1}$$

Note that Trip is a non-unital symmetric monoidal category under disjoint unions.

Notation 14.8.1. For $(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) \in \mathsf{Trip}$ and $k \in K$, we define:

$$I_k := (q \circ p)^{-1}(k)$$

$$J_k := q^{-1}(k)$$

$$(p_k : I_k \twoheadrightarrow J_k) := p|_{I_k}.$$

Similarly, for $j \in J$ we let $I_j := p^{-1}(j)$.

Suppose that we are given a morphism (14.8.1) in Trip. Fix $k' \in K_2$ and let $k := \curlyvee(k') \in K_1$. We will construct a canonical map:

$$U(p_{1,\curlyvee(k')}) = U(p_{1,k}) \to U(p_{2,k'}). \tag{14.8.2}$$

First, note that we can write $p_{1,k} : I_{1,k} \to J_{1,k}$ as a disjoint union of terms $p_{1,\kappa} : I_{1,\kappa} \twoheadrightarrow J_{1,\kappa}$ over $\kappa \in \curlyvee^{-1}(k)$, where e.g. $I_{1,\kappa}$ is the fiber over κ of the map $I_1 \twoheadrightarrow K_2$ defined by the diagram (14.8.2).

Therefore, by the colax symmetric monoidal structure on (14.3.2), we obtain a canonical morphism:

$$U(p_{1,k}) \to \prod_{\kappa \in \Upsilon^{-1}(k)} U(p_{1,\kappa}) \to U(p_{1,k'})$$

where this second morphism is the projection.

Now the commutative diagram:

$$\begin{tikzcd} I_{1,k'} \arrow[r, two heads] & J_{1,k'} \arrow[d] \\ I_{2,k'} \arrow[u, two heads] \arrow[r, two heads] & J_{2,k'} \end{tikzcd}$$

gives a morphism:

$$U(p_{1,k'}) \to U(p_{2,k'})$$

inducing (14.8.2) as desired.

This defines a symmetric monoidal functor:

$$\begin{aligned} \Psi^{\mathsf{Trip}} : \mathsf{Trip} &\to \mathsf{PreStk} \\ (I \twoheadrightarrow J \twoheadrightarrow K) &\mapsto \prod_{k \in K} U(p_k). \end{aligned} \tag{14.8.3}$$

where for (14.8.1), the functoriality is defined by the morphism:

$$\prod_{k \in K_1} U(p_{1,k}) \to \prod_{k' \in K_2} U(p_{2,k'})$$

given on a coordinate $k' \in K_2$ by:

$$\prod_{k \in K_1} U(p_{1,k}) \to U(p_{1,\Upsilon(k')}) \xrightarrow{(14.8.2)} U(p_{2,k'}).$$

Remark 14.8.2. Trip^{op} is the non-unital monoidal envelope of Part in the sense of [Lur12], and the functor Ψ^{Trip} is induced by the functor $U : \mathsf{Part}^{op} \to \mathsf{PreStk}$ in this way.

14.9. We have a symmetric monoidal functor:

$$\begin{gathered}\mathsf{Trip} \to \mathsf{Part} \\ (I \twoheadrightarrow J \twoheadrightarrow K) \mapsto (J \twoheadrightarrow K).\end{gathered} \tag{14.9.1}$$

Therefore, we obtain a second symmetric functor:

$$\Phi^{\mathsf{Trip}} : \mathsf{Trip} \to \mathsf{PreStk}$$

by composing (14.7.1) with (14.9.1).

We have a canonical natural transformation of symmetric monoidal functors:

$$\eta^{\mathsf{Trip}} : \Psi^{\mathsf{Trip}} \to \Phi^{\mathsf{Trip}}$$

evaluated termwise at $(I \twoheadrightarrow J \twoheadrightarrow K) \in \mathsf{Trip}$ as:

$$\begin{tikzcd}
\Psi^{\mathsf{Trip}}(I \twoheadrightarrow J \twoheadrightarrow K) \arrow[r, equal] & \prod_{k\in K} U(p_k) \arrow[r, dotted] \arrow[d, hook] & \prod_{j\in J}[\mathrm{Ran}_{\mathcal{X}}^{J_k}]_{disj} \arrow[r, equal] \arrow[d, hook] & \Phi^{\mathsf{Trip}}(I \twoheadrightarrow J \twoheadrightarrow K) \\
& \prod_{k\in K} \mathcal{X}^{I_k} \arrow[d, equal] & \prod_{j\in J} \mathrm{Ran}_{\mathcal{X}}^{J_k} \arrow[d, equal] & \\
& \prod_{j\in J} \mathcal{X}^{I_j} \arrow[r] & \prod_{j\in J} \mathrm{Ran}_{\mathcal{X}} . &
\end{tikzcd}$$

Remark 14.9.1. We will revisit the construction of η^{Trip} is §14.11 below.

14.10. We will need the following technical observation in what follows.

Fix $(J_\varepsilon \twoheadrightarrow K_\varepsilon) \in \mathsf{Part}$, $\varepsilon = 1, 2$. Form the overcategory:

$$\mathsf{Trip}_{/(J_1 \coprod J_2 \twoheadrightarrow K_1 \coprod K_2)}$$

with respect to (14.9.1).

We claim that the functor of disjoint union:

$$\mathsf{Trip}_{/(K_1 \twoheadrightarrow J_1)} \times \mathsf{Trip}_{/(K_2 \twoheadrightarrow J_2)} \to \mathsf{Trip}_{/(J_1 \coprod J_2 \twoheadrightarrow K_1 \coprod K_2)}$$

is an equivalence.

By definition, $\mathsf{Trip}_{/(K_1 \coprod K_2 \twoheadrightarrow J_1 \coprod J_2)}$ is the category of diagrams:

$$\big(I' \twoheadrightarrow J' \twoheadrightarrow K'\big) \in \mathsf{Trip}, \text{ plus } \begin{tikzcd} J_1 \coprod J_2 \arrow[r, two heads] \arrow[d, two heads] & K_1 \coprod K_2 \\ J' \arrow[r, two heads] & K' \arrow[u, two heads] \end{tikzcd}$$

under appropriate functoriality. Given such a datum, for $\varepsilon = 1, 2$ we define $I'_\varepsilon, J'_\varepsilon, K'_\varepsilon$ as the inverse images of K_ε under the map to $K_1 \coprod K_2$. This functor defines the desired inverse.

14.11. Given §14.10, we can apply the dual version of Proposition 14.5.1 to see that the left Kan extension of Ψ^{Trip} along $\mathsf{Trip} \xrightarrow{(14.9.1)} \mathsf{Part}$ is a colax symmetric monoidal functor. Moreover, one immediately verifies that this left Kan extension is actually a symmetric monoidal functor and that it is computed as the functor (14.7.1).

Moreover, the natural transformation η^{Trip} now arises via the universal property from Proposition 14.5.1.

14.12. We can now give Proposition-Construction 14.4.1 (1), i.e., the non-unital case of *loc. cit.*

By definition, a chiral category on $\mathcal{X}$ is a multiplicative sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}^{ch}$. Therefore, we will prove the following variant of *loc. cit.*

(∗): There is a canonical equivalence of categories between $\mathsf{MultCat}^{w}(\mathrm{Ran}_{\mathcal{X}}^{w})$ and the category of lax symmetric monoidal functors (14.4.1) sending all arrows in Part to coCartesian arrows.

It will follow from the construction that this equivalence identifies the subcategory of chiral categories with the subcategory of usual (i.e., non-lax) symmetric monoidal functors.

Step 1. First, recall from Variant 13.2.9 that *weak* chiral categories (alias: weakly multiplicative sheaves of categories on $\mathrm{Ran}_{\mathfrak{X}}^{ch}$) are defined as commutative algebras in $\mathsf{PreStk}_{corr}^{\mathsf{ShvCat}}$ lifting $\mathrm{Ran}_{\mathfrak{X}}^{ch} \in \mathsf{PreStk}_{corr}$. Here the notation $\mathsf{PreStk}_{corr}^{\mathsf{ShvCat}}$ was defined in §12.18. We recall that it is defined as a certain 1-full subcategory of:

$$\Big(\big(\mathsf{Groth}(\mathsf{ShvCat}_{/-})\big)^{op}\Big)_{corr}. \tag{14.12.1}$$

By §14.6, such a datum is equivalent to a symmetric monoidal functor:

$$\mathsf{Part} \to \big(\mathsf{Groth}(\mathsf{ShvCat}_{/-})\big)^{op} \tag{14.12.2}$$

lifting the functor (14.7.1), sending squares (14.6.1) to Cartesian squares, and satisfying a certain property encoding that the corresponding functor to (14.12.1) should map into $\mathsf{PreStk}_{corr}^{\mathsf{ShvCat}}$.

Precisely, this last property is readily checked to say that every arrow in Part inducing isomorphisms on the J-terms (i.e., in (14.2.1), $J_2 \xrightarrow{\simeq} J_1$; in §20, such arrows were called horizontal) should map to a coCartesian arrow (that is, when considered as an arrow in $\mathsf{Groth}(\mathsf{ShvCat}_{/-})$).

We then see that the condition that squares (14.6.1) map to to Cartesian squares is actually redundant: it is subsumed by the condition that horizontal arrows map to coCartesian arrows by applying Remark 14.7.2 and (the proof of) Lemma 12.10.1.

Step 2. We will make implicit use of the following observation below:

We have a tautological Cartesian square:

$$\begin{tikzcd}
\mathsf{Groth}(\mathsf{ShvCat}_{/U}) \arrow[r] \arrow[d] & \mathsf{Groth}(\mathsf{ShvCat}_{/-}) \arrow[d] \\
\mathsf{Part} \arrow[r, "U^{op}"] & \mathsf{PreStk}^{op}.
\end{tikzcd}$$

Step 3. Suppose we are given a lax symmetric monoidal section (14.4.1) sending all arrows to coCartesian arrows.

As in Remark 14.8.2, we obtain a symmetric monoidal functor:

$$F : \mathsf{Trip} \to \mathsf{Groth}(\mathsf{ShvCat}_{/-})^{op}$$

lifting $\Psi^{\mathsf{Trip},op}$.

The fact that (14.4.1) sends all arrows to coCartesian arrows implies that the left Kan extension of F along $\mathsf{Trip} \to \mathsf{Part}$ exists, and by Proposition 14.5.1, it carries a canonical structure of colax symmetric monoidal functor.

One readily verifies that it is actually symmetric monoidal, lifts (14.7.1) and satisfies the conditions articulated in Step 1. Therefore, this functor defines a weakly multiplicative sheaf of categories as desired.

Step 4. Suppose we have a functor (14.12.2) defining a weakly multiplicative sheaf of categories. Restricting along $\mathsf{Trip} \to \mathsf{Part}$, we obtain a similar functor with source Trip.

Applying the coCartesian condition and the symmetric monoidal natural transformation η^{Trip}, we obtain a symmetric monoidal functor $\mathsf{Trip} \to \mathsf{Groth}(\mathsf{ShvCat}_{/-})$ lifting Ψ^{Trip}. Applying Remark 14.8.2 again, we obtain a lax symmetric monoidal functor $\mathsf{Part}^{op} \to \mathsf{Groth}(\mathsf{ShvCat}_{/-})$ of the desired type.

14.13. This completes the treatment of the non-unital case. The unital case is treated in exactly the same way, though the category Trip should of course be replaced with a category Trip_{un} with arbitrary maps of finite sets replacing surjections.

One may describe chiral module categories and factorization modules in similar terms. The formulation and the details of the comparison are left to the interested reader.

14.14. **External fusion redux.** Suppose that $\mathcal{X} = X_{dR}$ for X a scheme of finite type, as in §13.12. Let C be a chiral category on X_{dR} and let $\mathcal{A} \in \mathsf{Alg}^{\text{fact}}(\mathsf{C})$. As in *loc. cit.*, let $\mathsf{C}_{X^I_{dR}} \in \mathsf{ShvCat}(X^I_{dR})$ denote the sheaf of categories underlying C.

Enhancing[34] the external fusion construction of §13.22, one can upgrade the construction $I \mapsto \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C}_{X^I_{dR}})$ to a functor (14.12.2) satisfying the hypotheses spelled out in Step 1 (found in §14.12 above).

Therefore, by *loc. cit.*, we obtain a weak chiral category $\mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C})$ on X_{dR}, where the morphisms (12.21.2) and (13.12.1) identify (upon passing to the limit for the latter).

Similarly, if $\mathcal{A}$ and C are unital, then $\mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}_{un}(\mathsf{C})$ is a weak unital chiral category.

We can formulate this more precisely in the following proposition.

Proposition 14.14.1. *(1) External fusion defines functors:*

$$\{\mathsf{C} \in \mathsf{Cat}^{ch}(X_{dR}), \mathcal{A} \in \mathsf{Alg}^{\text{fact}}(\mathsf{C})\} \to \mathsf{Cat}^{w,ch}$$

$$\Big(\mathsf{C}, \mathcal{A} \in \mathsf{Alg}^{\text{fact}}(\mathsf{C})\Big) \mapsto \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}(\mathsf{C})$$

and:

$$\{\mathsf{C} \in \mathsf{Cat}^{ch}_{un}(X_{dR}), \mathcal{A} \in \mathsf{Alg}^{\text{fact}}_{un}(\mathsf{C})\} \to \mathsf{Cat}^{w,ch}_{un}$$

$$\Big(\mathsf{C}, \mathcal{A} \in \mathsf{Alg}^{\text{fact}}_{un}(\mathsf{C})\Big) \mapsto \mathcal{A}\text{–}\mathsf{mod}^{\text{fact}}_{un}(\mathsf{C}).$$

(2) The induced functor:

$$\mathsf{Cat}^{w,ch}_{un}(X_{dR}) \to \mathsf{Cat}^{w,ch}_{un}(X_{dR})$$

$$\mathsf{C} \mapsto \mathrm{unit}_{\mathsf{C}}\text{–}\mathsf{mod}^{\text{fact}}_{un}(\mathsf{C})$$

[34] We note an analogy to some constructions involved in [Lur12] Lemma 4.3.6.9.

is (canonically identified with) the canonical embedding of unital chiral categories into weak unital chiral categories, and in a manner compatible with Theorem 13.13.2.

Remark 14.14.2. In the above, for example the somewhat ambiguous notation $\{\mathsf{C} \in \mathsf{Cat}^{ch}_{un}(X_{dR}), \mathcal{A} \in \mathsf{Alg}^{\text{fact}}_{un}(\mathsf{C})\}$ is properly understood using the formalism of §12. We note that the category is designed so that morphisms:

$$(\mathsf{C}_1, \mathcal{A}_1) \to (\mathsf{C}_2, \mathcal{A}_2)$$

are given by pairs of a morphism $\varphi : \mathsf{C}_1 \to \mathsf{C}_2$ of chiral categories and a morphism $\eta : \varphi(\mathcal{A}_1) \to \mathcal{A}_2$ of factorization algebras, where $\varphi(\mathcal{A}_1)$ is understood as a factorization algebra in C_2 using the discussion of §12.34.

15. Commutative chiral categories

15.1. In this section, we develop a theory of commutative chiral categories and commutative factorization algebras, following [BD04].

15.2. Let $\mathcal{X}$ be a fixed prestack.

Recall that $\text{Ran}^*_{\mathcal{X}}$ denotes the prestack $\text{Ran}_{\mathcal{X}}$ considered with the non-unital commutative monoid structure of addition.

Definition 15.2.1. A *commutative weak chiral category* is a multiplicative sheaf of categories on $\text{Ran}^*_{\mathcal{X}}$.

The identity morphism for $\text{Ran}_{\mathcal{X}}$ obviously upgrades to a lax morphism:

$$\text{Ran}^{ch}_{\mathcal{X}} \to \text{Ran}^*_{\mathcal{X}}$$

of non-unital commutative algebras in the 2-category PreStk_{corr} (see §12.23 for the notion of lax morphism of monoids in a 2-category). Using this structure, one constructs a canonical restriction functor from commutative weak chiral categories to weak chiral categories.

Definition 15.2.2. A *commutative chiral category* is a commutative weak chiral category whose underlying weak chiral category is a chiral category.

Similarly, a *commutative factorization algebra* in a commutative chiral category C is a weakly multiplicative sheaf over $\mathrm{Ran}_{\mathcal{X}}^{*}$ whose underlying weakly multiplicative sheaf over $\mathrm{Ran}_{\mathcal{X}}^{ch}$ is a multiplicative sheaf.

Remark 15.2.3. Roughly, a commutative chiral category is a sheaf of categories C on $\mathrm{Ran}_{\mathcal{X}}$ with a morphism:

$$\kappa_{\mathsf{C}} : \mathsf{C} \boxtimes \mathsf{C} \to \mathrm{add}^{*}(\mathsf{C}) \in \mathsf{ShvCat}(\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}})$$

that is an isomorphism over the disjoint locus (and satisfying higher compatibilities).

A commutative factorization algebra in C is an object $\mathcal{A} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathsf{C})$ with morphisms:

$$\kappa_{\mathsf{C}}(\mathcal{A} \boxtimes \mathcal{A}) \to \mathrm{add}^{*}(\mathcal{A}) \in \Gamma(\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}, \mathrm{add}^{*}(\mathsf{C}))$$

that is an isomorphism over the disjoint locus.

Remark 15.2.4. It is obvious that $\mathsf{QCoh}_{\mathcal{X}}$ is a commutative chiral category. In this case, our notion of commutative factorization algebra contains as a special case the same-named notion from [BD04], and provides a derived version of the latter.

15.3. We now explain the combinatorial approach to commutative chiral categories, in the spirit of §14. We use the notation of *loc. cit.* freely.

We let $\mathcal{P}$ denote the symmetric functor $\mathsf{fSet}^{op} \to \mathsf{PreStk}$ given by $I \mapsto \mathcal{X}^{I}$. The Grothendieck construction produces a symmetric monoidal functor:

$$\begin{array}{c} \mathsf{Groth}(\mathsf{ShvCat}_{/\mathcal{P}(-)}) \\ \big\downarrow \\ \mathsf{fSet}. \end{array} \tag{15.3.1}$$

The next result follows in the same was as Proposition-Construction 14.4.1.

Proposition-Construction 15.3.1. *A* commutative weak chiral category *is equivalent to a commutative diagram of colax symmetric monoidal sections to* (15.3.1)*:*

$$\begin{tikzcd} \mathsf{Groth}(\mathsf{ShvCat}_{/\mathcal{P}(-)}) \arrow[d, shift left] \\ \mathsf{fSet} \arrow[u, dotted, shift left] \end{tikzcd}$$

sending all arrows to coCartesian arrows.

The induced lax symmetric monoidal functor:

$$\mathsf{Part} \to \mathsf{Groth}(\mathsf{ShvCat}_{/U})$$

obtained by using the coCartesian structure and the 2-commutative diagram:

$$\begin{tikzcd} \mathsf{Part}^{op} \arrow[d, "(I \twoheadrightarrow J) \mapsto I"'] \arrow[drr, "U"] \arrow[dr, phantom, "\Downarrow"] & & \\ \mathsf{fSet}^{op} \arrow[rr, "\mathcal{P}"] & & \mathsf{PreStk} \end{tikzcd}$$

corresponds (via §14.12 Step 1) to the underlying weak chiral category.

Remark 15.3.2. We leave to the reader the problem of finding a unital version of Proposition-Construction 15.3.1, imitating Proposition-Construction 14.4.1.

15.4. Suppose that $\mathcal{D}$ is a non-unital (resp. unital) commutative monoid in DGCat_{cont}, and let $\mathcal{X}$ be either a scheme of finite type or the de Rham space of such a scheme. We will associate to this data a commutative (resp. unital) factorization category $\mathsf{Loc}_{\mathcal{X}}(\mathcal{D})$ over $\mathcal{X}$. For convenience, we work in the non-unital setting.

The reader may be advised to skip ahead to Remark 15.4.1, where the constructions given below are spelled out in simple cases.

Step 1. For convenience, we will construct $\mathsf{Loc}_{\mathcal{X}}(\mathcal{D})$ using Proposition-Construction 15.3.1.

We will use the notation Trip and its associates from §14.

Step 2. For any prestack $\mathcal{Y}$, let $\mathcal{D}_{\mathcal{Y}}$ denote $\mathcal{D} \otimes_{\mathsf{Vect}} \mathsf{QCoh}_{\mathcal{Y}} \in \mathsf{ShvCat}_{/\mathcal{Y}}$.

The assignment:

$$\big(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\big) \mapsto \boxtimes_{k\in K} \mathcal{D}_{U(p_k)} \in \mathsf{ShvCat}(\prod_{k\in K} U(p_k)) = \mathsf{ShvCat}(\Psi^{\mathsf{Trip}}(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K))$$

defines a symmetric monoidal section:

$$\begin{tikzcd} \mathsf{Groth}(\mathsf{ShvCat}_{/\Psi^{\mathsf{Trip}}(-)}) \arrow[d] \\ \mathsf{Trip}^{op} \arrow[u, dotted, bend left] \end{tikzcd} \tag{15.4.1}$$

Indeed, one can easily produce this structure by viewing the non-unital symmetric monoidal structure on $\mathcal{D}$ as a symmetric monoidal functor $\mathsf{fSet} \to \mathsf{DGCat}_{cont}$.

Step 3. Define the colax symmetric monoidal functor:

$$\Xi^{\mathsf{Trip}} : \mathsf{Trip} \to \mathsf{PreStk}$$

$$\big(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\big) \mapsto U(p).$$

We have a natural transformation:

$$\Xi^{\mathsf{Trip}} \to \Psi^{\mathsf{Trip}}$$

of colax symmetric monoidal functors evaluated termwise as:

$$\Xi^{\mathsf{Trip}}(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) = U(p) \to \prod_{k\in K} U(p_k) = \Psi^{\mathsf{Trip}}(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K).$$

Combining this structure, the pullback structure on sheaves of categories, and (15.4.1), we obtain a symmetric monoidal section:

$$\begin{tikzcd} \mathsf{Groth}(\mathsf{ShvCat}_{/\Xi^{\mathsf{Trip}}(-)}) \arrow[d] \\ \mathsf{Trip}^{op} \arrow[u, bend left, dotted] \end{tikzcd} \tag{15.4.2}$$

given by:

$$\big(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\big) \mapsto \boxtimes_{k\in K} \mathcal{D}_{U(p_k)}|_{U(p)} \in \mathsf{ShvCat}_{/U(p)}.$$

Step 4. Next, define the symmetric monoidal functor:

$$\mathcal{P}^{\mathsf{Trip}} : \mathsf{Trip} \to \mathsf{PreStk}$$
$$\big(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\big) \mapsto \mathcal{X}^I.$$

The assumption on $\mathcal{X}$ from §15.4 and the material of §19 imply that we have a well-behaved theory of pushforwards of sheaves of categories. Therefore, using the natural transformation of colax symmetric monoidal functors:

$$\Xi^{\mathsf{Trip}} \to \mathcal{P}^{\mathsf{Trip}}$$

given termwise by the obvious maps $\jmath_p : U(p) \to \mathcal{X}^I$, we obtain from (15.4.2) the lax symmetric monoidal functor:

$$\begin{tikzcd} \mathsf{Groth}(\mathsf{ShvCat}_{/\mathcal{P}^{\mathsf{Trip}}(-)}) \arrow[d] \\ \mathsf{Trip}^{op} \arrow[u, bend left, dotted] \end{tikzcd} \tag{15.4.3}$$

given by:

$$\left(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\right) \mapsto \jmath_{p,*}\left(\boxtimes_{k\in K} \mathcal{D}_{U(p_k)}|_{U(p)}\right) \in \mathsf{ShvCat}_{\mathcal{X}^I}.$$

Step 5. We now apply Proposition 14.5.1 to the map:

$$\mathsf{Trip}^{op} \to \mathsf{fSet}$$
$$\left(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K\right) \mapsto I$$

to obtain a lax symmetric monoidal structure on the right Kan extension:

$$\mathsf{fSet} \to \mathsf{Groth}(\mathsf{ShvCat}_{/\mathcal{P}(-)}).$$

One immediately verifies that it satisfies the required hypotheses to define a commutative chiral category.

Remark 15.4.1. The above construction may appear somewhat inexplicit, so let us explain in concretely in some cases. It follows explicitly from the construction that $\mathsf{Loc}_{\mathcal{X}}(\mathcal{D})_{X^I}$ is given by a limit:

$$\lim_{I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K \in \mathsf{Trip}^{op}} \jmath_{p,*}(\jmath_{p,*}\left(\boxtimes_{k\in K} \mathcal{D}_{U(p_k)}|_{U(p)}\right)).$$

For I a singleton set, the indexing category is a singleton as well, and therefore $\mathsf{Loc}_{\mathcal{X}}(\mathcal{D})_X = \mathcal{D}_X = \mathcal{D} \otimes \mathsf{QCoh}_{\mathcal{X}}$.

For $I = \{1, 2\}$, we find that the indexing category is:

$$\begin{tikzcd} & (I \twoheadrightarrow * \twoheadrightarrow *) \arrow[d] \\ (I \overset{\mathrm{id}}{\twoheadrightarrow} I \overset{\mathrm{id}}{\twoheadrightarrow} I) \arrow[r] & (I \overset{\mathrm{id}}{\twoheadrightarrow} I \twoheadrightarrow *) \end{tikzcd}$$

and therefore $\mathsf{Loc}_{\mathcal{X}}(\mathcal{D})_{\mathcal{X}^2}$ fits into a Cartesian diagram:

$$\begin{array}{ccc} \mathsf{Loc}_{\mathcal{X}}(\mathcal{D})_{\mathcal{X}^2} & \longrightarrow & \mathcal{D}_{X^2} \\ \downarrow & & \downarrow \\ \jmath_*\jmath^*(\mathcal{D}_X \boxtimes \mathcal{D}_X) & \longrightarrow & \jmath_*\jmath^*(\mathcal{D}_{X^2}) \end{array}$$

where the lower arrow is induced by the tensor product in $\mathcal{D}$.

Variant 15.4.2. Given a commutative algebra $A \in \mathcal{D}$, the above procedure produces a factorization algebra $\mathrm{Loc}_{\mathcal{X}}(A) \in \mathsf{Loc}_{\mathcal{X}}(\mathcal{D})$, and similarly in the unital setting.

15.5. Next, we discuss the material from §13.14 in the case of a commutative chiral category.

15.6. We need some general material about crystalline sheaves of categories on *pseudo-indschemes.*

We follow [Gai11] in using the following (somewhat clunky) terminology:

Definition 15.6.1. A *pseudo-indscheme* Y is a pair of an indexing category $\mathcal{I}$ and a $\mathcal{I}$-diagram $i \mapsto Y_i$ of schemes of finite type such that all structure maps $Y_i \to Y_j$ are proper.

The *prestack underlying* Y is the colimit of this diagram $i \mapsto Y_i$ in PreStk. Where there is no risk for confusion, we denote this colimit also by Y.

Remark 15.6.2. The implicit notion of morphism:

$$Y = (\mathcal{I}, i \mapsto Y_i) \to Z = (\mathcal{J}, j \mapsto Z_j) \tag{15.6.1}$$

of pseudo-indschemes is that of a functor $F : \mathcal{I} \to \mathcal{J}$ and compatible morphisms $Y_i \to Z_{F(i)}$.

Remark 15.6.3. Our notion differs slightly from that of [Gai11]: in *loc. cit.*, pseudo-indschemes are defined as a full subcategory of PreStk obtained as colimits of diagrams of the above type. However, in many constructions in *loc. cit.*, pseudo-indschemes are assumed to be given by such a diagram and morphisms are assumed to be of the above type.

Definition 15.6.4. We say a morphism (15.6.1) of pseudo-indschemes is pseudo-indproper if each morphism $Y_i \to Z_{F(i)}$ is proper.

For a pseudo-indscheme Y, we let $Y_{dR} \in \mathsf{PreStk}$ denote the de Rham space of the prestack underlying Y.

Proposition-Construction 15.6.5. *Let $f : Y \to Z$ be a map of pseudo-indschemes and let C be a sheaf of categories on Z_{dR}. There is a canonical morphism:*

$$f_{*,dR,\mathsf{C}} : \Gamma(Y_{dR}, f^*(\mathsf{C})) \to \Gamma(Z_{dR}, \mathsf{C})$$

of de Rham pushforward*, and that is canonically left adjoint to the pullback map if f is pseudo-indproper, and functorial for morphisms of pseudo-indschemes over Z.*
admits a left adjoint

Proof. For $Z = \operatorname{colim}_j Z_j$, let ψ_j denote the structure map $Z_j \to Z$. Then we tautologically have:

$$\Gamma(Z_{dR}, \mathsf{C}) = \lim_{j \in \mathcal{J}^{op}} \Gamma(Z_{j,dR}, \psi_j^*(\mathsf{C})).$$

However, because the structure maps $Z_j \to Z_{j'}$ are proper, and because each $Z_{j,dR}$ is 1-affine, we see that the structure maps in this limit admit left adjoints (given by tensoring with the de Rham pushforward functors $D(Z_j) \to D(Z_{j'})$. Therefore, we obtain an expression:

$$\operatorname*{colim}_{j \in \mathcal{J}} \Gamma(Z_{j,dR}, \psi_j^*(\mathsf{C}))$$

for $\Gamma(Z_{dR}, \mathsf{C})$, the colimit taking place in DGCat_{cont}.

We have a similar expression for $\Gamma(Y_{dR}, f^*(\mathsf{C}))$, and the de Rham pushforward functor is then constructed using the compatible maps:

$$\Gamma(Y_{i,dR}, \varphi_i^*(\mathsf{C})) \to \Gamma(Z_{F(i),dR}, \psi_i^*(\mathsf{C}))$$

(with $\varphi_i : Y_i \to Y$ the structure map). This obviously satisfies the desired properties.

$\square$

15.7. Now observe that Ran_X is canonically a pseudo-indscheme, since $\mathrm{Ran}_X = \mathrm{colim}_{I \in \mathsf{fSet}^{op}} X^I$. Moreover, the map:

$$\mathrm{add} : \mathrm{Ran}_X \times \mathrm{Ran}_X \to \mathrm{Ran}_X$$

is canonically a morphism of pseudo-indschemes (considering the left hand side with the product pseudo-ind structure), using the maps:

$$\mathsf{fSet}^{op} \times \mathsf{fSet}^{op} \to \mathsf{fSet}^{op}$$

$$(I, J) \mapsto I \coprod J$$

$$X^I \times X^J \xrightarrow{\mathrm{id}} X^{I \coprod J}.$$

We immediately see that add is pseudo-indproper.

Of course, this discussion holds for higher products of Ran_X with itself and for higher operations in the non-unital commutative operad.

15.8. We fix C a commutative chiral category on X_{dR} in what follows, and let $\mathcal{C} := \Gamma(\mathrm{Ran}_{X,dR}, \mathsf{C})$.

Observe that $\mathcal{C}$ carries a canonical non-unital symmetric monoidal structure in DGCat_{cont} called the $*$-*tensor product*, and denoted $- \overset{*}{\otimes} -$. It is computed termwise as:

$$\Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}) \otimes \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}) \to \Gamma(\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}}, \mathsf{C} \boxtimes \mathsf{C}) \to$$

$$\Gamma(\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}}, \mathrm{add}^*(\mathsf{C})) \to \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C})$$

where the last arrow is the de Rham pushforward functor from Proposition-Construction 15.6.5. We note that this functor is left adjoint to the obvious map by *loc. cit.*

We leave the remaining details of this construction to the reader.

Note that the identity functor for $\mathcal{C}$ upgrades to a lax symmetric monoidal functor:

$$(\mathcal{C}, \overset{*}{\otimes}) \to (\mathcal{C}, \overset{ch}{\otimes}).$$

Remark 15.8.1. One easily sees that $\Gamma(X_{dR}, \mathcal{C})$ carries a canonical

Example 15.8.2. Suppose that $\mathcal{D}$ is a non-unital symmetric monoidal category, and let $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ denote the corresponding factorization category over X_{dR}.

Then the pushforward functor along $X \hookrightarrow \mathrm{Ran}_X$ defines a colax symmetric monoidal functor:

$$\mathcal{D} \xrightarrow{\mathrm{id} \otimes \omega_X} \mathcal{D} \otimes D(X) \to \Gamma(X_{dR}, \mathrm{Loc}_{X_{dR}}(\mathcal{D})_{X_{dR}}) \to \Gamma(\mathrm{Ran}_{X_{dR}}, \mathrm{Loc}_{X_{dR}}(\mathcal{D}))$$

where the latter is considered with its $\overset{*}{\otimes}$ symmetric monoidal structure.

15.9. We now observe that the theory of Lie-$*$ algebras from [FG12] generalizes to this general setting.

Definition 15.9.1. A *generalized Lie-$*$ algebra* in C is a Lie algebra object in $(\mathcal{C}, \overset{*}{\otimes})$. A *Lie-$*$ algebra* in C is a generalized Lie-$*$ algebra supported on X, i.e., that lives in the subcategory:

$$\Gamma(X_{dR}, \mathsf{C}|_{X_{dR}}) \subseteq \Gamma(\mathrm{Ran}_{X_{dR}}, \mathsf{C}) = \mathcal{C}.$$

There is an obvious forgetful functor from chiral Lie algebras to generalized Lie-$*$ algebras. As in [FG12] §6.4, it admits a left adjoint, and this left adjoint sends Lie-$*$ algebras to chiral algebras in $\mathcal{C}$. This functor is called *chiral enveloping algebra*.

Part 3. Appendices

16. D-modules in infinite dimensions

16.1. In this section, we develop the D-module formalism on indschemes of ind-infinite type.

16.2. The basic feature that we struggle against is that there are two types of infinite-dimensionality at play: pro-infinite dimensionality and ind-infinite dimensionality. That is, we could have an infinite-dimensional variety S that is the union $S = \cup_i S_i = \mathrm{colim}_i S_i$ of finite-dimensional varieties, or T that is the projective limit $T = \lim_j T_j$ of finite-dimensional varieties, e.g., a scheme of infinite type.

Any reasonable theory of D-modules will produce produce some kinds of de Rham homology and cohomology groups. We postulate as a basic principle that these groups should take values in discrete vector spaces, that is, we wish to avoid projective limits.

Then, in the ind-infinite dimensional case, the natural theory is the *cohomology* of S:

$$H_*(S) := \operatorname*{colim}_i H_*(S_i)$$

while in the pro-infinite dimensional case, the natural theory is the *cohomology* of T:

$$H^*(T) := \operatorname*{colim}_j H^*(T_j).$$

For varieties that are infinite-dimensional in both the ind and the pro directions, one requires a *semi-infinite* homology theory that is homology in the ind direction and cohomology in the pro direction.

Of course, such a theory requires some extra choices, as is immediately seen by considering the finite-dimensional case. For example, for a smooth variety, we have a choice of normalization for the cohomological shifts.

16.3. Theories of semi-infinite homology have appeared in many places in the literature. We do not pretend to survey the literature on the subject here, but note that in the case of the loop group, it is well-known that semi-infinite cohomology, in the sense above, may be defined using the semi-infinite cohomology of Lie algebras.

We provide such a theory in large generality below. In fact, in great generality, we develop two theories $D^!$ and D^* of derived categories of D-modules on indschemes of ind-infinite type. The theory $D^!$ is contravariant, and therefore carries a natural dualizing complex, and the theory D^* is covariant, and therefore is the place where cohomology is defined.

For *placid* indschemes, the two categories are identified after a choice of *dimension theory*, and therefore allows us to define the *renormalized* or *semi-infinite* cohomology of the scheme. The extra choice of dimension theory here precisely reflects the numerical choice of cohomological shifts discussed above.

16.4. The material in this section has been strongly influenced by [BD] §7, [Dri06] and [KV04]. We also thank Dennis Gaitsgory for many helpful discussions about this material; in particular, the idea of systematically distinguishing between $D^!$ and D^*, our very starting point, is due to him.

This section is lazy in certain notable respects. We work (essentially) in the setting of classical algebraic geometry throughout, in particular ignoring the relationship between D-modules and quasi-coherent sheaves.

16.5. Throughout, we impose the assumption that we are working with classical (i.e., non-derived) schemes. However, in some arguments we will explicitly move into the setting of derived algebraic geometry.

16.6. Due to the length of this section, let us describe in some detail the basic structure.

16.7. We give a review of the theory of Noetherian approximation in §16.11. This material will serve to bootstrap from the finite type setting to the infinite type setting.

Note that this idea is already essentially present in [KV04]; the authors of *loc. cit.* credit it to Drinfeld.

In §16.12-16.36 (the bulk of this section) we develop the theory of D-modules for quasi-compact quasi-separated schemes.

16.8. We begin in §16.12-16.19 with the basic theory of $D^!$-modules; functoriality properties and descent are the principal concerns. We then give the parallel theory of D^*-modules in §16.20-16.27. Recall from above that the crucial distinction between the two theories is that $D^!$ is contravariant and D^* is covariant.

We also note here that for a quasi-compact quasi-separated scheme S the DG category $D^!(S)$ admits a tensor product $\overset{!}{\otimes}$ and acts on $D^*(S)$ in a canonical way satisfying a version of the projection formula.

16.9. In §16.29 we will introduce the notion of *placidity*. One can understand this condition as saying that the singularities of a scheme are of finite type in a precise sense.

The key point of placid schemes is that they admit a "renormalized dualizing complex" that lies in $D^*(S)$: this is notable because, as we recall, D^* is covariant: its natural functoriality (with respect to infinite type morphisms) is through pushforwards. Moreover, the functor of action on the renormalized dualizing complex gives an equivalence $D^!(S) \simeq D^*(S)$. In particular, one obtains a covariant structure on $D^!$ and a contravariant structure on D^* is the placid setting. This material is developed in §16.30-16.36. For a morphism $f : S \to T$ of placid schemes, we let $f_{*,ren} : D^!(S) \to D^!(T)$ and $f^{!,ren} : D^!(T) \to D^!(S)$ denote the corresponding functors.

In general, these renormalized functors are very badly behaved, e.g., the pairs $(f^!, f_{*,ren})$ and $(f^{!,ren}, f_{*,dR})$ do not satisfy base-change.

In §16.37, we introduce a notion of placid morphism, which is something like a pro-smooth morphism. Proposition 16.38.1 (generalized to the indschematic setting by Proposition 16.59.1) says that for placid morphism, $f^!$ is left adjoint to $f_{*,ren}$, and similarly, $f^{!,ren}$ is left adjoint $f_{*,dR}$. Here the dimension shifts implicit in the infinite-dimensional setting work out to eliminate the usual cohomological shifts needed to make such statements in the finite-dimensional setting.

Moreover, Proposition 16.38.1 implies that there are good base-changed properties for placid morphisms.

16.10. In §16.41 we transition to the setting of indschemes. In §16.42-16.47 we define $D^!$ and D^*-modules for indschemes. We develop their basic functoriality properties and give descent theorems here as well. In §16.45-16.46, we recall the notion of *reasonable* indscheme from [BD] and examine how this condition interacts with the theories of D-modules.

Finally, in §16.49-16.57 we give a theory of placid indschemes with similar properties to the setting of placid schemes described above. It is here that dimension theories enter the story, and we discuss them in some detail in these sections as well.

16.11. **Noetherian approximation.** For the reader's convenience, we begin with a brief review of the theory of Noetherian approximation (alias: Noetherian descent). This theory is due to [Gro67] §8 and [TT90] Appendix C.

Let S be a quasi-compact quasi-separated base scheme. Let $\mathsf{Sch}^{f.p.}_{/S}$ denote the category of schemes finitely presentated (in particular: quasi-separated) over S. If S is Noetherian we will also use the notation $\mathsf{Sch}^{f.t.}_{/S}$ because in this case finite type is equivalent to finite presentation.

We say an S-scheme T is *almost affine* if for every $S' \to S$ of finite presentation every map $T \to S'$ factors as $T \to T' \to S'$ where $T \to T'$ is affine and $T' \to S'$ is finitely presented. Let $\mathsf{Sch}^{\mathrm{al.aff}}_{/S}$ denote the category of almost affine S-schemes.

Let $\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.p.}_{/S})$ denote the full subcategory of $\mathsf{Pro}(\mathsf{Sch}^{f.p.}_{/S})$ consisting of objects T that arise as filtered limits $T = \lim T_i$ of finitely presented S-schemes under affine structural morphisms $T_j \to T_i$. We recall that projective limits of such systems exist and that if each T_i is affine over S then T is as well. Clearly such limits commute with base-change.

Theorem 16.11.1. *(1) The right Kan extension:*

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}_{/S}^{f.p.}) \to \mathsf{Sch}_{/S}$$

of the embedding $\mathsf{Sch}_{/S}^{f.p.} \hookrightarrow \mathsf{Sch}_{/S}$ is defined and is fully-faithful. This right Kan extension maps into $\mathsf{Sch}_{/S}^{al.aff}$. If S is Noetherian and affine, then the essential image of this functor is all schemes over S that are quasi-compact and quasi-separated (in particular, quasi-compact quasi-separated k-schemes are almost affine).

(2) Suppose $T = \lim T_i$ is a filtered limit with each T_i finitely presented over S and $T_j \to T_i$ affine. Then if T' is a finitely presented T-scheme there exists an index i and a T_i-scheme T_i' of finite presentation such that $T' = T_i' \times_{T_i} T$ (as a T-scheme). If the map $T' \to T$ has any (finite) subset of the properties of being (e.g.) smooth, flat, proper, or surjective, then $T_i' \to T_i$ may be taken to have the same properties.

(3) Suppose $T = \lim_{i \in \mathfrak{I}^{op}} T_i$ as in (2)*. Then if $T \to S$ is an affine morphism, then there exists $i_0 \in \mathfrak{I}$ such that for every $i \in \mathfrak{I}_{i_0/}$, $T_i \to S$ is affine.*

(4) Suppose that $T = \lim T_i$ as in (2) *and $U \subseteq T$ is a quasi-compact open subscheme. Then for some index $i \in \mathfrak{I}$ and open $U_i \subseteq T_i$ we have $U = U_i \times_{T_i} T$ (as T-schemes).*

Remark 16.11.2. We note that (3) appears in [TT90] as Proposition C.6, where it is stated only in the case that S is affine. However, this immediately generalizes, since S is assumed quasi-compact and therefore admits a finite cover by affines.

We will also use the following result.

Proposition 16.11.3. *Suppose that $T = \lim_{i\in\mathcal{I}} T_i$ is a filtered limit of schemes under affine structure maps. Let $\alpha_i : T \to T_i$ denote the structure maps. Then passing to cotangent complexes, the canonical map:*

$$\operatorname*{colim}_{i\in\mathcal{I}} \alpha_i^*(\Omega^1_{T_i}) \to \Omega^1_T \in \mathsf{QCoh}(T)^{\leqslant 0}$$

is an equivalence.

Proof. Let DGSch denote the category of DG schemes. Note that filtered limits of derived schemes under affine structural maps exists as well, and satisfy the same properties as in the non-derived case: namely, if $T = \lim T_i$ in DGSch is a filtered limit under affine structural maps of affine S-schemes, then T is affine over S as well. In particular, we deduce that $\mathsf{Sch} \subseteq \mathsf{DGSch}$ is closed under such limits.

Now the result follows immediately from the description of the cotangent complex in terms of square-zero extensions in derived algebraic geometry.

$\square$

16.12. $D^!$**-modules.** Let Sch_{qcqs} denote the category of quasi-compact quasi-separated k-schemes. By Theorem 16.11.1, Sch_{qcqs} is a full subcategory of $\mathsf{Pro}(\mathsf{Sch}^{f.t.})$. We define the functor $D^! : \mathsf{Sch}_{qcqs}^{op} \to \mathsf{DGCat}_{cont}$ as the left Kan extension of the functor $D : \mathsf{Sch}^{f.t.,op} \to \mathsf{DGCat}_{cont}$ which attaches to a scheme S of finite type its DG category D-modules $D(S)$ and to a morphism $f : T \to S$ attaches the functor $D(S) \xrightarrow{f^!} D(T)$.

Remark 16.12.1. Suppose that $\mathcal{C}^0$ is an (essentially small) category and $\mathcal{C} \subseteq \mathsf{Ind}(\mathcal{C}^0)$ is a full subcategory containing $\mathcal{C}^0$. Suppose that we are given $F : \mathcal{C} \to \mathcal{D}$ a functor that is the left Kan extension of its restriction to $\mathcal{C}^0$. Then for any filtered colimit $X = \operatorname{colim}_i X_i \in \mathcal{C}$ in $\mathsf{Ind}(\mathcal{C}^0)$, we have $F(X) = \operatorname{colim} F(X_i)$. Indeed, by definition:

$$F(X) = \operatorname*{colim}_{X' \to X, X' \in \mathcal{C}^0} F(X').$$

But this also computes the left Kan extension from $\mathcal{C}^0$ to $\mathsf{Ind}(\mathcal{C}^0)$. Therefore, this claim reduces to the case $\mathcal{C} = \mathsf{Ind}(\mathcal{C}^0)$, where it is well-known.

Applying this in our setting, we see that for any realization $T = \lim_{i \in \mathcal{I}^{op}} T_i$ with $\mathcal{I}$ filtered, T_i finite type and $T_i \to T_j$ affine we have:

$$D^!(T) = \operatorname*{colim}_{i \in \mathcal{I}} D(T_i) \tag{16.12.1}$$

where the structure maps are !-pullback functors.

Example 16.12.2. If T is finite type then we canonically have $D^!(T) = D(T)$.

For any morphism $f : T \to S$ of quasi-compact quasi-separated schemes, we denote the induced pullback functor by $f^! : D(S) \to D(T)$. Note that there is no risk for confusion in this notation because in the finite type case the !-pullback functors identify under the canonical identification $D = D^!|_{\mathsf{Sch}^{f.t.,op}}$.

For T and S two quasi-compact schemes, we have a canonical equivalence:

$$D^!(T) \otimes D^!(S) \xrightarrow{\simeq} D^!(T \times S) \tag{16.12.2}$$

that immediately arises from the finite type case.

Remark 16.12.3. For S a quasi-compact quasi-separated scheme and $T = \lim T_i$ a filtered limit under affine morphisms of finitely presented S-schemes, we have:

$$D^!(T) = \operatorname*{colim}_{i \in \mathcal{I}} D^!(T_i)$$

generalizing (16.12.1). Indeed, it follows immediately from Noetherian descent that the limit $T = \lim T_i$ is preserved under the embedding in $\mathsf{Sch}_{qcqs} \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.})$ and therefore this follows general properties of Kan extensions, as in Remark 16.12.1.

16.13. For any $T \in \mathsf{Sch}_{qcqs}$ the category $D^!(T)$ carries a canonical symmetric monoidal structure $\overset{!}{\otimes}$ with unit $\omega_T := p_T^!(k)$ for $p_T : T \to \mathrm{Spec}(k)$ the structure map. For any $f : T \to S$ in Sch_{qcqs}, the functor $f^!$ is symmetric monoidal relative to these structures.

The symmetric monoidal structure $\overset{!}{\otimes}$ can be viewed as arising from the equalities $D^!(\prod_{i=1}^n T) = \otimes_{i=1}^n D^!(T)$ and the diagonal maps for T.

16.14. **Correspondences.** Next, we extend the functoriality of $D^!$.

Let $\mathsf{Sch}_{corr}^{f.t.}$ be the (1,1)-category of finite type schemes under correspondences. By [GR14], we have the functor $D : \mathsf{Sch}_{corr}^{f.t.} \to \mathsf{DGCat}_{cont}$ that attaches to a finite type scheme T its category $D(T)$ of D-modules and to a correspondence $T \xleftarrow{\alpha} H \xrightarrow{\beta} S$ (i.e., a map $T \to S$ in Sch_{corr}) attaches the functor $\beta_{*,dR}\alpha^!$.

Let $\mathsf{Sch}_{qcqs,corr;all,f.p.}$ denote the category of quasi-compact quasi-separated schemes under correspondences of the form:

$$
\begin{tikzcd}
& H \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
T & & S
\end{tikzcd}
$$

where $H \in \mathsf{Sch}_{qcqs}$, β is finitely presented and α is arbitrary. Note that $\mathsf{Sch}_{qcqs,corr;all,f.p}$ contains $\mathsf{Sch}_{corr}^{f.p.}$ as a full subcategory. It also contains Sch_{qcqs}^{op} as a non-full subcategory where morphisms are correspondences where the right arrow is an isomorphism.

We define the functor:

$$D^{!,enh} : \mathsf{Sch}_{qcqs,corr;all,f.p.} \to \mathsf{DGCat}_{cont}$$

by left Kan extension from $\mathsf{Sch}_{corr}^{f.t.}$.

Proposition 16.14.1. *The restriction of $D^{!,enh}$ to Sch_{qcqs}^{op} canonically identifies with the functor $D^! : \mathsf{Sch}_{qcqs}^{op} \to \mathsf{DGCat}_{cont}$.*

The proof will be given in §16.17.

16.15. We assume Proposition 16.14.1 until §16.16 so that we can discuss its consequences.

For $f : T \to S$ a map of quasi-compact quasi-separated schemes, the induced functor $D^{!,enh}(S) = D^!(S) \to D^{!,enh}(T) = D^!(T)$ coincides with $f^!$. If f is finitely presented we will denote the corresponding functor $D^!(T) \to D^!(S)$ by $f_{*,!-dR}$ (to avoid confusion with the functor $f_{*,dR} : D^*(T) \to D^*(S)$ defined in §16.20 below). We refer to the functor $f_{*,!-dR}$ as the "!-dR $*$-pushforward functor."

Note that the formalism of correspondences implies that we have base-change between $*$-pushforward and !-pullback for Cartesian squares.

Remark 16.15.1. Suppose that $f : T \to S$ is finitely presented. One can compute the functor $f_{*,!-dR}$ "algorithmically" as follows. Let f be obtained by base-change from $f' : T' \to S'$ a map of schemes of finite type via a map $S \to S'$. Write $S = \lim S_i$ where structure maps are affine and each S_i is a finite type S'-scheme. Then $T = \lim T_i$ for $T_i := S_i \times_{S'} T'$. Let $\alpha_i : S \to S_i$, $\beta_i : T \to T_i$ and $f_i : T_i \to S_i$ be the tautological maps.

Then for $\mathcal{F} \in D(T_i)$ we have $f_{*,!-dR}(\beta_i^!(\mathcal{F})) = \alpha_i^! f_{i,*,dR}(\mathcal{F})$, which completely determines the functor $f_{*,!-dR}$.

One readily deduces the following result from [GR14].

Proposition 16.15.2. *If $f : S \to T$ is a proper (in particular, finitely presented) morphism of quasi-compact quasi-separated schemes, then $f^!$ is canonically the right adjoint to $f_{*,!-dR}$. This identification is compatible with the correspondence structure: e.g., given a Cartesian diagram:*

$$\begin{tikzcd} S' \arrow[r, "\psi"] \arrow[d, "f'"] & S \arrow[d, "f"] \\ T' \arrow[r, "\varphi"] & T \end{tikzcd}$$

with f proper, the identification:

$$f_{*,dR}\varphi^! \xrightarrow{\simeq} \psi^! f'_{*,!-dR}$$

arising from the correspondence formalism is given by the adjunction morphism.

Similarly, we have the following.

Proposition 16.15.3. *If $f : S \to T$ is a smooth map of quasi-compact quasi-separated schemes, then $f^![-2 \cdot d_{S/T}]$ is left adjoint to $f_{*,!-dR}$. Here $d_{S/T}$ is the rank of $\Omega^1_{S/T}$ regarded as a locally constant function on S.*

Remark 16.15.4. By a *locally constant function* $T \to \mathbb{Z}$ on a scheme T, we mean a morphism of $T \to \mathbb{Z}$ with $\mathbb{Z}$ considered as the indscheme $\coprod_{n\in\mathbb{Z}} \mathrm{Spec}(k)$.

If T is quasi-compact quasi-separated and therefore a pro-finite type scheme $T = \lim T_i$ (under affine structure maps), then, by Noetherian approximation, any locally constant function on T arises by pullback from one on some T_i. In other words, if we define $\pi_0(T)$ as the profinite set $\lim_i \pi_0(T_i)$, then locally constant functions on T are equivalent to continuous functions on $\pi_0(T)$.

Remark 16.15.5. Recall that there is an automatic projection formula given the correspondence framework. Indeed, for $f : S \to T$ a finitely presented map of quasi-compact quasi-separated schemes, $\mathcal{F} \in D^!(T)$ and $\mathcal{G} \in D^!(S)$, we have a canonical isomorphisms:

$$f_{*,!-dR}\big(f^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}\big) \simeq \mathcal{F} \overset{!}{\otimes} f_{*,!-dR}(\mathcal{G})$$

coming base-change for $\mathcal{F} \boxtimes \mathcal{G} \in D^!(T \times S)$ and the Cartesian diagram:

$$\begin{tikzcd} S \arrow[r, "\Gamma_f"] \arrow[d, "f"] & T \times S \arrow[d, "\mathrm{id}_T \times f"] \\ T \arrow[r, "\Delta_T"] & T \times T \end{tikzcd}$$

where Γ_f is the graph of f and Δ_T is the diagonal.

By the finite type case, these isomorphisms are given by the adjunctions of Proposition 16.15.2 and 16.15.3 when f is proper or smooth.

16.16. In the proof of Proposition 16.14.1 we will need the following technical result.

Let T be a quasi-compact quasi-separated scheme. Consider the category $\mathcal{C}_T$ of correspondences:

$$\mathcal{C}_T := \{\ S \xleftarrow{\alpha} H \xrightarrow{\beta} T \ \mid \beta \text{ finitely presented, } S \in \mathsf{Sch}^{f.t.} \text{ and } H \in \mathsf{Sch}_{qcqs}\}.$$

Here, as usual, compositions are given by fiber products.

Note that $\mathcal{C}_T$ contains as a non-full subcategory $\mathsf{Sch}^{f.t.,op}_{T/}$ of maps $\curlyvee : T \to S$ with $S \in \mathsf{Sch}^{f.t.}$, where given such a map we attach the correspondence $S \xleftarrow{\curlyvee} T \xrightarrow{\mathrm{id}_T} T$.

Lemma 16.16.1. *The embedding* $\mathsf{Sch}^{f.t.}_{T/} \to \mathcal{C}_T$ *is cofinal.*

Proof. Fix a correspondence $(\ S \xleftarrow{\alpha} H \xrightarrow{\beta} T\ ) \in \mathcal{C}_T$. Translating Lurie's ∞-categorical Quillen Theorem A to this setting, we need to show the contractibility of the category $\mathcal{C}$ of commutative diagrams:

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "\beta"] & H' \arrow[d, "\beta'"] \arrow[dr, "\epsilon"] & \\ T \arrow[r, "\curlyvee"] & T' & S \end{tikzcd}$$

such that the square on the left is Cartesian, $H', T' \in \mathsf{Sch}^{f.t.}$ and $\epsilon \circ \delta = \alpha$. Here a morphism from one such diagram (denoted with subscripts "1") to another such diagram (denoted with subscripts "2") is given by maps $f : T'_1 \to T'_2$ and $g : H'_1 \to H'_2$ such that the following diagram commutes and all squares are Cartesian:

$$
\begin{tikzcd}
H \arrow[r, "\delta_1"'] \arrow[d, "\beta"] \arrow[rr, bend left, "\delta_2"] & H'_1 \arrow[r, "g"'] \arrow[d, "\beta'_1"] & H'_2 \arrow[d, "\beta'_2"] \arrow[dr, "\epsilon_2"] & \\
T \arrow[r, "\gamma_1"] \arrow[rr, bend right, "\gamma_2"'] & T'_1 \arrow[r, "f"] & T'_2 & S.
\end{tikzcd}
$$

First, we observe that the category $\mathcal{C}$ is non-empty. Indeed, because β is finitely presented we can find $T \to T' \in \mathsf{Sch}^{f.t.}$ and $\beta' : H' \to T'$ so that H is obtained from H' by base-change. Noting that H can be written as a limit under affine transition maps of H' obtained in this way and S is finite type, we see that $H \to S$ must factor though some H' obtained in this way.

To see that $\mathcal{C}$ is contractible, note that $\mathcal{C}$ admits non-empty finite limits (because Sch admits finite limits) and therefore $\mathcal{C}^{op}$ is filtered.

□

16.17. We now prove Proposition 16.14.1.

Proof of Proposition 16.14.1. We have an obvious natural transformation $D^! \to D^{!,enh}|_{\mathsf{Sch}^{op}_{qcqs}}$. It suffices to see that this natural transformation is an equivalence when evaluated on any fixed $T \in \mathsf{Sch}_{qcqs}$.

With the notation of §16.16, $D^{!,enh}$ is by definition the colimit over $(S \xleftarrow{\alpha} H \xrightarrow{\beta} T) \in \mathcal{C}_T$ of the category $D(S)$. By Lemma 16.16.1, this coincides with the colimit over diagrams where β is an isomorphism, as desired.

□

Remark 16.17.1. Neither Lemma 16.16.1 nor Proposition 16.14.1 is particular to schemes, but rather a general interaction between pro-objects in a category with finite limits and correspondences.

16.18. **Descent.** Next, we discuss descent for $D^!$.

For a map $f : S \to T$ of schemes and $[n] \in \mathbf{\Delta}$ let $\mathrm{Cech}^n(S/T)$ be defined as:

$$\mathrm{Cech}^n(S/T) := \underbrace{S \underset{T}{\times} \ldots \underset{T}{\times} S}_{n \text{ times}}.$$

Of course, $[n] \mapsto \mathrm{Cech}^n(S/T)$ forms a simplicial scheme in the usual way.

We use the terminology of Voevodsky's h-topology, developed in the infinite type setting in [Ryd10]. We simply recall that h-coverings are finitely presented[35] and include both the classes of fppf coverings and proper[36] coverings.

Proposition 16.18.1. *Let $f : S \to T$ be an h-covering of quasi-compact quasi-separated schemes. Then the canonical functor (induced by pullback):*

$$D^!(T) \to \lim_{[n]\in\mathbf{\Delta}} D^!(\mathrm{Cech}^n(S/T)) \tag{16.18.1}$$

is an equivalence.

Recall from [Ryd10] Theorem 8.4 that the h-topology of Sch_{qcqs} is generated by finitely presented Zariski coverings[37] and proper coverings. Therefore, it suffices to verify Lemmas 16.18.2 and 16.18.3 below.

Lemma 16.18.2. *$D^!$ satisfies proper descent, i.e., for every $f : T \to S$ a proper (in particular, finitely presented) surjective morphism of quasi-compact quasi-separated schemes the morphism* (16.18.1) *is an equivalence.*

[35]More honestly: it seems there is a bit of disagreement in the literature whether h-coverings are required to be finitely presented or merely finite type. We are using the convention that they are finitely presented.

[36]We include "finitely presented" in the definition of proper.

[37]We explicitly note that these are necessarily finitely presented because we work only with quasi-compact quasi-separated schemes. That is, any open embedding of quasi-compact quasi-separated schemes is necessarily of finite presentation: the only condition to check is that it is a quasi-compact morphism, and any morphism of quasi-compact schemes is itself quasi-compact.

Proof. We can find $f' : S' \to T'$ a proper covering between schemes of finite type and $T \to T'$ so that f is obtained by base-change. Let $T = \lim T_i$ where each T_i is a T'-scheme of finite type and structure maps are affine. Let $S_i := T_i \times_{T'} S'$.

We now decompose the map (16.18.1) as:

$$D^!(T) = \operatorname*{colim}_{i \in \mathcal{I}} D(T_i) \xrightarrow{\simeq} \operatorname*{colim}_{i \in \mathcal{I}} \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) \to$$

$$\lim_{[n] \in \mathbf{\Delta}} \operatorname*{colim}_{i \in \mathcal{I}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S/T)).$$

Here the isomorphism is by h-descent in the finite type setting.

Therefore, it suffices to see that the map:

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) \to \lim_{[n] \in \mathbf{\Delta}} \operatorname*{colim}_{i \in \mathcal{I}} D(\mathrm{Cech}^n(S_i/T_i))$$

is an isomorphism. It suffices to verify the Beck-Chevalley conditions in this case (c.f. [Lur12] Proposition 6.2.3.19). For each $i \in \mathcal{I}$ and each map $[n] \to [m]$ in $\mathcal{I}$, the functor:

$$D(\mathrm{Cech}^m(S_i/T_i)) \to D(\mathrm{Cech}^n(S_j/T_j))$$

admits a left adjoint given by the !-dR $*$-pushforward as in Proposition 16.15.2. By base change between upper-! and !-dR $*$-pushfoward (Proposition 16.14.1), the Beck-Chevalley conditions are satisfied since for every $j \to i$ in $\mathcal{I}$ and $[n] \to [m]$ in $\mathbf{\Delta}$ the diagram:

$$\begin{tikzcd}
\mathrm{Cech}^m(S_i/T_i) \arrow[r] \arrow[d] & \mathrm{Cech}^n(S_j/T_j) \arrow[d] \\
\mathrm{Cech}^n(S_i/T_i) \arrow[r] & \mathrm{Cech}^m(S_j/T_j)
\end{tikzcd}$$

is Cartesian.

$\square$

Lemma 16.18.3. *$D^! : \mathsf{Sch}^{op}_{qcqs} \to \mathsf{DGCat}_{cont}$ satisfies Zariski descent.*

Proof. It suffices to show for every S a quasi-compact quasi-separated scheme and $S = U \cup V$ a Zariski open covering of S by quasi-compact open subschemes that the canonical map:

$$D^!(S) \to D^!(U) \underset{D^!(U\cap V)}{\times} D^!(V)$$

is an equivalence.

Let $j_U : U \to S$, $j_V : V \to S$ and $j_{U\cap V} : U \cap V \to S$ be the corresponding (finitely presented) open embeddings.

Note that e.g. $j_{U,*,!-dR} : D^!(U) \to D^!(S)$ is fully-faithful. Indeed, by Proposition 16.15.3 we have an adjunction between $j_U^!$ and $j_{U,*,!-dR}$. The counit:

$$j_U^! j_{U,*,!-dR} \to \mathrm{id}_{D^!(U)}$$

is an equivalence by Remark 16.15.1 and the corresponding statement in the finite presentation setting.

Now we have a canonical map:

$$\mathrm{id}_{D^!(S)} \to \mathrm{Ker}\left(j_{U,*,!-dR}\, j_U^! \oplus j_{V,*,!-dR}\, j_V^! \to j_{U\cap V,*,!-dR}\, j_{U\cap V}^!\right)$$

and it suffices to see that this map is an equivalence. But this again follows by reduction to the finite presentation case via Remark 16.15.1.

□

16.19. **Equivariant setting.** Suppose that S is a quasi-compact quasi-separated base scheme and $\mathcal{G} \to S$ is a quasi-separated quasi-compact group scheme over S.

Suppose that P is a quasi-compact quasi-separated S-scheme with an action of $\mathcal{G}$. In this case, the semisimplicial bar complex:

$$\ldots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathcal{G} \underset{S}{\times} P \rightrightarrows P \tag{16.19.1}$$

induces the diagram:

$$D^!(P) \rightrightarrows D^!(\mathcal{G} \underset{S}{\times} P) \rightrightarrows D^!(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \rightrightarrows \dots .$$

and we define the *$\mathcal{G}$-equivariant derived category* $D^!(P)^{\mathcal{G}}$ of P to be the limit of this diagram.

Example 16.19.1. Suppose that $\mathcal{G}$ is constant, i.e., $\mathcal{G} = S \times \mathcal{G}_0$ for some quasi-compact quasi- separated group scheme $\mathcal{G}_0$ over Spec(k). Then, by (16.12.2), $D^!(\mathcal{G}_0)$ obtains a comonoidal structure in DGCat_{cont} in the usual way (e.g. the comulitplication is !-pullback along the multiplication for $\mathcal{G}_0$). As such, $D^!(\mathcal{G}_0)$ coacts on $D^!(P)$ and $D^!(P)^{\mathcal{G}}$ is the usual (strongly) $\mathcal{G}_0$-equivariant category, i.e., the limit of the diagram:

$$D^!(P) \rightrightarrows D^!(\mathcal{G}_0) \otimes D^!(P) \rightrightarrows D^!(\mathcal{G}_0) \otimes D^!(\mathcal{G}_0) \otimes D^!(P) \rightrightarrows \dots .$$

Let $\mathcal{P}_{\mathcal{G}} \to S$ be a $\mathcal{G}$-torsor, i.e., $\mathcal{G}$ acts on $\mathcal{P}_{\mathcal{G}}$ and after an appropriate fppf base-change $S' \to S$ we have a $\mathcal{G}$-equivariant identification:

$$\mathcal{P}_{\mathcal{G}} \underset{S}{\times} S' = \mathcal{G} \underset{S}{\times} S'.$$

We obtain a canonical functor:

$$\varphi : D^!(S) \to D^!(\mathcal{P}_{\mathcal{G}})^{\mathcal{G}}.$$

Proposition 16.19.2. *In the above setting the functor φ is an equivalence.*

Proof. By fppf descent (Proposition 16.18.1), we reduce to the case there $\mathcal{P}_{\mathcal{G}}$ is a trivial $\mathcal{G}$-bundle over T, i.e., $\mathcal{P}_{\mathcal{G}} = \mathcal{G} \times_S T$. Then the bar complex extends to a split simplicial object in the usual way from which we deduce the result.

□

Remark 16.19.3. If $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor, we will sometimes summarize the situation in writing $S = \mathcal{P}_{\mathcal{G}}/\mathcal{G}$.

16.20. **D^*-modules.** Next, we discuss the $*$-theory of D-modules.

We also let D denote the functor $\mathsf{Sch}^{f.t.} \to \mathsf{DGCat}_{cont}$ that attaches to any scheme its category of D-modules, and attaches to a morphism of schemes the corresponding de Rham pushforward functor.

We then define $D^* : \mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$ as the right Kan extension of this functor.

For any realization $T = \lim_{i \in \mathcal{I}^{op}} T_i$ as above we have $D^*(T) = \lim_{i \in \mathcal{I}^{op}} D(T_i)$ where the structure maps are !-pullback functors. If T is finite type, then we canonically have $D^*(T) = D(T)$.

For any morphism $f : T \to S$ of quasi-compact quasi-separated schemes, we denote the induced pushforward functor by $f_{*,dR} : D(T) \to D(S)$. As above, there is no risk for confusion here with the finite type case.

Remark 16.20.1. In the setting of Remark 16.12.3, similarly have:

$$D^*(T) = \lim_i D^*(T_i).$$

16.21. By the projection formula, for T quasi-compact quasi-separated there is a unique action $\overset{!}{\otimes}$ of $(D^!(T), \overset{!}{\otimes})$ on $D^*(T)$ such that for every $f : T \to S$ with S finite type and every $\mathcal{F} \in D(S) = D^!(S)$ and $\mathcal{G} \in D^*(T)$ we have:

$$f_{*,dR}\big(g^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}\big) = \mathcal{F} \overset{!}{\otimes} f_{*,dR}(\mathcal{G}). \tag{16.21.1}$$

Here on the left $\overset{!}{\otimes}$ denotes the action of $D^!(T)$ on $D^*(T)$ and on the right it denotes the usual tensor product of D-modules in $D(S) = D^*(S)$.

16.22. We now give a construction that encodes the projection formula in a more functorial way.

We claim that there is a canonical category we denote temporarily by $\mathcal{C}$ whose objects are pairs $\mathcal{A} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ and $\mathcal{M}$ a module for $\mathcal{A}$ in DGCat_{cont}, and where morphisms $(\mathcal{A}, \mathcal{M}) \to (\mathcal{B}, \mathcal{N})$ are pairs of a symmetric monoidal and continuous functor $\mathcal{A} \to \mathcal{B}$ plus $\mathcal{N} \to \mathcal{M}$ a continuous morphism of $\mathcal{A}$-module categories (where the $\mathcal{A}$-module category structure on $\mathcal{N}$ is induced by $\mathcal{A} \to \mathcal{B}$).[38]

One can compute filtered colimits in $\mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ as a colimit in the first variable and a limit in the second variable.

We claim that the of $D^!$ and D^* then upgrades to a functor $\mathsf{Sch}^{op}_{qcqs} \to \mathcal{C}$ sending S to $(D^!(S), D^*(S))$, upgrading the constructions of $D^!$ and D^*.

Indeed, first note that there is a functor $D : \mathsf{Sch}^{f.t.,op} \to \mathcal{C}$ sending S to $(D(S), D(S))$ equipped with upper-! functoriality in the first variable and lower-* functoriality in the second variable. Indeed, this follows from the formalism of correspondences from [GR14].

Then we obtain the functor $\mathsf{Sch}^{op}_{qcqs} \to \mathcal{C}$ as the left Kan extension of this functor.

16.23. Recall that for S a finite type scheme the category $D(S)$ is self-dual under Verdier duality and for a map $f : T \to S$ between finite type schemes the functor dual to $f^!$ is $f_{*,dR}$. Therefore, for S a quasi-compact quasi-separated scheme we obtain:

Proposition 16.23.1. *If $D^!(S)$ is a dualizable category, then its dual is canonically identified with $D^*(S)$.*

Note that in this case this is an identification of $(D^!(S), \overset{!}{\otimes})$-module categories. Moreover, the functor dual to $f^!$ continues to be $f_{*,dR}$.

16.24. **Constant sheaf.** For T quasi-compact quasi-separated, there is a canonical "constant sheaf" $k_T \in D^*(T)$ constructed as follows.

For any $S \in \mathsf{Sch}^{f.t.}$ and $\alpha : T \to S$, we define an object of $D(S) = D^*(S)$ that we denote formally as "$\alpha_{*,dR}(k_T)$" by the formula:

[38]A precise construction of this is given by combining the construction LMod, Remark 2.4.27 and Corollary 4.2.3.2 from [Lur12].

$$“\alpha_{*,dR}(k_T)” := \underset{\substack{T \xrightarrow{\beta} T' \xrightarrow{\Upsilon} S \\ T' \in \mathsf{Sch}^{f.t.}, \Upsilon \circ \beta = \alpha}}{\mathrm{colim}} \Upsilon_{*,dR}(k_{T'}).$$

For any triangle:

$$\begin{tikzcd} T \arrow[r, "\alpha'"] \arrow[d, "\alpha"'] & S' \\ S \arrow[ur, "f"'] & \end{tikzcd}$$

with S and $S' \in \mathsf{Sch}^{f.t.}$, we have a canonical isomorphism:

$$“\alpha'_{*,dR}(k_T)” \xrightarrow{\simeq} f_{*,dR}(“\alpha_{*,dR}(k_T)”)$$

and therefore we obtain the object $k_T \in D^*(T)$ (with each $\alpha_{*,dR}(k_T) = “\alpha_{*,dR}(k_T)”$) as desired.

The continuous functor $p_T^{*,dR} : \mathsf{Vect} \to D^*(T)$ sending k to k_T is readily seen to be the left adjoint to $p_{T,*,dR}$ (where $p_T : T \to \mathrm{Spec}(k)$ is the structure map).

16.25. **Correspondences.** Next, we extend the functoriality of D^* as in §16.14.

Let $\mathsf{Sch}_{qcqs,corr;f.p.,all}$ denote the category of quasi-compact quasi-separated schemes under correspondences of the form:

$$\begin{tikzcd} & H \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ T & & S \end{tikzcd}$$

where $H \in \mathsf{Sch}_{qcqs}$, α is finitely presented and β is arbitrary. Note that $\mathsf{Sch}_{qcqs,corr;f.p.,all}$ contains $\mathsf{Sch}^{f.t.}_{corr}$ as a full subcategory. It also contains Sch_{qcqs} as a non-full subcategory where morphisms are correspondences where the left arrow is an isomorphism.

We define the functor:

$$D^{*,enh} : \mathsf{Sch}_{qcqs,corr;f.p.,all} \to \mathsf{DGCat}_{cont}$$

by right Kan extension from $\mathsf{Sch}^{f.t.}_{corr}$.

Like Proposition 16.14.1, the following is immediate from Lemma 16.16.1.

Proposition 16.25.1. *The restriction of $D^{*,enh}$ to Sch_{qcqs} canonically identifies with the functor $D^* : \mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$.*

16.26. For $f : T \to S$ a map of quasi-compact quasi-separated schemes, the induced functor:

$$D^{*,enh}(T) = D^*(T) \to D^{*,enh}(S) = D^*(S)$$

coincides with $f_{*,dR}$. If f is finitely presented we will denote the corresponding functor $D^*(S) \to D^*(T)$ by $f^{¡}$ to avoid confusion with the functor $f^! : D^!(T) \to D^!(S)$. Note that the formalism of correspondences implies that we have base-change between $*$-pushforward and ¡-pullback for Cartesian squares.

Remark 16.26.1. Suppose that $f : T \to S$ is finitely presented. One can compute the functor $f^{¡}$ "algorithmically" as follows. In the notation of Remark 16.15.1, for $\mathcal{F} \in D(S)$ we have $\alpha_{i,*,dR} f^{¡}(\mathcal{F}) = f_i^! \beta_{i,*,dR}(\mathcal{F})$ by base-change, computing $f^{¡}(\mathcal{F})$ in $D(T) = \lim D(T_i)$ as promised.

One deduces from Remark 16.26.1 the following result.

Proposition 16.26.2. *If $f : S \to T$ is a finitely presented proper morphism of quasi-compact quasi-separated schemes, then $f^{¡}$ is canonically the right adjoint to $f_{*,dR}$.*

Similarly, we have:

Proposition 16.26.3. *If $f : S \to T$ is a smooth map of quasi-compact quasi-separated schemes, then $f^{¡}[-2d_{S/T}]$ is left adjoint to $f_{*,dR}$, with $d_{S/T}$ as in Proposition 16.15.3.*

16.27. **Descent.** Next, we discuss descent for D^*.

Proposition 16.27.1. *For $f : S \to T$ an h-covering of quasi-compact quasi-separated schemes the functor:*

$$D^*(T) \to \lim_{n \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T))$$

induced by the functors $f_n^!$ with $f_n : \mathrm{Cech}^n(S/T) \to T$ the canonical map is an equivalence.

Proof. Because f is finite presentation we can apply Noetherian approximation to find $f' : S' \to T'$ an h-covering between schemes of finite type and $T \to T'$ so that f is obtained by base-change. Let $T = \lim T_i$ where each T_i is a T'-scheme of finite type (and structure maps are affine) and let $S_i := T_i \times_{T'} S'$.

Then each $S_i \to T_i$ is an h-covering between finite type schemes. Note that $\mathrm{Cech}^n(S/T) = \lim \mathrm{Cech}^n(S_i/T_i)$.

Now we have:

$$D^*(T) = \lim_{i \in \mathcal{I}^{op}} D(T_i) \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n] \in \mathbf{\Delta}} \lim_{i \in \mathcal{I}^{op}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S/T)).$$

Here the indicated isomorphism is by usual h-descent for finite type schemes and Proposition 16.25.1.

□

Variant 16.27.2. One can similarly show that the functor:

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T)) \to D^*(T)$$

defined by de Rham pushforwards is an equivalence for $S \to T$ an h-covering. Indeed: it is easy to verify for Zariski coverings (the argument is basically the same as for Lemma 16.18.3), and for proper coverings, it follows automatically from Proposition 16.27.1.

This is the statement that should properly be thought of as dual to Proposition 16.18.1.

16.28. **Equivariant setting.** Suppose that we are in the setting of §16.19, i.e., $\mathcal{G}$ is a group scheme over S that acts on an S-scheme P.

In this case, (16.19.1) defines the *coequivariant derived category*:

$$D^*(P)_{\mathcal{G}} := \operatorname{colim}\Big(\; \ldots . \; \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} D^*(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} D^*(\mathcal{G} \underset{S}{\times} P) \rightrightarrows D^*(P) \; \Big) \tag{16.28.1}$$

with the colimit computed in DGCat_{cont}.

The analogue of Proposition 16.19.2 holds in this setting: if $P \to S$ is an $\mathcal{G}$-torsor, we obtain a functor:

$$D^*(P)_{\mathcal{G}} \to D^*(S)$$

that is an equivalence by essentially the same argument as in *loc. cit*, but using Variant 16.27.2 of Proposition 16.27.1.

16.29. **Placidity.** We now discuss an additional convenient hypothesis for quasi-compact quasi-separated schemes.

Definition 16.29.1. For $T \in \mathsf{Sch}$ we say an expression $T = \lim_{i \in \mathfrak{I}^{op}} T_i$ is a *placid presentation of* T if:

(1) The indexing category $\mathfrak{I}$ is filtered.
(2) Each T_i is finite type over k.
(3) For every $i \to j$ in $\mathfrak{I}$ the corresponding map $T_j \to T_i$ is an affine smooth covering.

We say that $T \in \mathsf{Sch}$ is *placid* if it admits a placid presentation.

Example 16.29.2. As is well known from the theory of group schemes, any affine group scheme is placid (we need the characteristic zero assumption on k here).

Example 16.29.3. Suppose that S is a finite type scheme and $\mathcal{G} \to S$ is a projective limit under smooth surjective affine maps of smooth S-group schemes. Suppose that $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor in the sense of §16.19. Then $\mathcal{P}_{\mathcal{G}}$ is placid.

Example 16.29.4. For a Cartesian square:

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & T_2 \arrow[d] \\ S_1 \arrow[r] & T_1 \end{tikzcd}$$

with T_1 finite type, S_1 and T_2 placid, the scheme S_2 is necessarily placid.

Indeed, for $S_1 = \lim_{i \in \mathcal{I}^{op}} S_{1,i}$ and $T_2 = \lim_{j \in \mathcal{J}^{op}} T_{2,j}$ placid presentations by T_1-schemes, we have:

$$S_2 = \lim_{(i,j) \in \mathcal{I}^{op} \times \mathcal{J}^{op}} S_{1,i} \underset{T_1}{\times} T_{2,j}.$$

Obviously all structure maps are smooth affine covers, so this is a placid presentation of S_2.

Remark 16.29.5. By Noetherian descent, if S is placid and $T \to S$ is finite presentation, then T is placid as well. Moreover, there always exist placid presentations $S = \lim_{i \in \mathcal{I}^{op}} S_i$, $T = \lim_{i \in \mathcal{I}^{op}} T_i$ and compatible morphisms $T_i \to S_i$ inducing $T \to S$, and such that, for every $i \to j \in \mathcal{I}$, the diagram:

$$\begin{tikzcd} T_j \arrow[r] \arrow[d] & S_j \arrow[d] \\ T_i \arrow[r] & S_i \end{tikzcd}$$

is Cartesian.

Remark 16.29.6. By [Gro67] Corollary 8.3.7, given a placid presentation $T = \lim_i T_i$, each structure morphism $T \to T_i$ is surjective on schematic points.

Remark 16.29.7. A placid scheme is tautologically quasi-compact and quasi-separated.

16.30. If T is a placid scheme with placid presentation $T = \lim_{i\in\mathcal{I}^{op}} T_i$ then we have:

$$D^*(T) = \operatorname*{colim}_{i\in\mathcal{I}} D(T_i) \tag{16.30.1}$$

where the structure functors are the $*$-pullback functors (defined because the maps $T_j \to T_i$ are smooth). For $i \in \mathcal{I}^{op}$ and $f_i : T \to T_i$ the corresponding structure map, we let $f_i^{*,dR}$ denote the functor $D^*(T_i) \to D^*(T)$ left adjoint to $f_{i,*,dR}$.

In particular, we see that $D^*(T)$ is compactly generated and therefore canonically dual to $D^!(T)$, which is also compactly generated. (Note that in the $D^!$-case, compact objects are !-pullbacks of compact objects from finite type schemes, where for D^* they are $*$-pullbacks).

Similarly, we obtain:

$$D^!(T) = \lim_{i\in\mathcal{I}^{op}} D(T_i) \tag{16.30.2}$$

where the structure functors are the right adjoints to the $f_i^!$ functors, i.e., shifted de Rham cohomology functors (again, these are adjoint by smoothness).

Remark 16.30.1. It follows from the identification of D^* as a colimit that for placid $T = \lim_{i\in\mathcal{I}^{op}} T_i$ as above and $\mathcal{F} \in D^*(T)$, the canonical map:

$$\operatorname*{colim}_{i\in\mathcal{I}} f_i^{*,dR} f_{i,*,dR}(\mathcal{F}) \to \mathcal{F} \tag{16.30.3}$$

is an equivalence.

16.31. Let T be a quasi-compact quasi-separated scheme.

Let $\mathsf{Pres}(T)$ denote the 1-category whose objects are placid presentations $(\mathcal{I}, \{T_i\}_{i\in\mathcal{I}})$ of T and where morphisms $(\mathcal{I}, \{T_i^1\}_{i\in\mathcal{I}}) \to (\mathcal{J}, \{T_j^2\}_{j\in\mathcal{J}})$ are given by a datum:

$$F : \mathcal{I} \to \mathcal{J} \text{ and } \{f_i : T_i^1 \to T^2_{F(i)}\}_{i \in \mathcal{I}} \text{ compatible morphisms of schemes under } T.$$

One easily shows that $\mathsf{Pres}(T)$ is filtered.

16.32. Fix two placid presentations $(\mathcal{I}, \{T_i^1\}_{i\in\mathcal{I}})$ and $(\mathcal{J}, \{T_j^2\}_{j\in\mathcal{J}})$ of a scheme T. We will make use of the following observation.

Lemma 16.32.1. *For every $j \in \mathcal{J}$ and every factorization $T \to T_i^1 \to T_j^2$ for $i \in \mathcal{I}$, the morphism $T_i^1 \to T_j^2$ is smooth.*

Proof. Suppose x is a geometric point of T. For each $i' \in \mathcal{I}$, let $x_{i'}$ denote the corresponding geometric point of $T^1_{i'}$.

Applying Proposition 16.11.3, we obtain:

$$\operatorname{Coker}\left(x_i^*(\Omega^1_{T_i^1/T_j^2}) \to x^*(\Omega^1_{T/T_j^2})\right) = \operatorname*{colim}_{i' \in \mathcal{I}_{i/}} \operatorname{Coker}\left(x_i^*(\Omega^1_{T_i^1/T_j^2}) \to x_{i'}^*(\Omega^1_{T^1_{i'}/T_j^2})\right) = \operatorname*{colim}_{i' \in \mathcal{I}_{i/}} x_{i'}^*(\Omega^1_{T^1_{i'}/T_i^1}).$$

Because the structure maps $T_j \to T_i$ are smooth the right hand side is a filtered limit of vector spaces concentrated in degree 0 and therefore is concentrated in degree 0 as well.

On cohomology we obtain a long exact sequence with segments:

$$\ldots \to H^{i-1}\Big(\operatorname*{colim}_{i' \in \mathcal{I}_{i/}} x_{i'}^*(\Omega^1_{T^1_{i'}/T^1_i})\Big) \to H^i\Big(x_i^*(\Omega^1_{T^1_i/T^2_j})\Big) \to H^i\Big(x^*(\Omega^1_{T/T^2_j})\Big) \to \ldots.$$

The left term is zero for $i \neq 1$ and the right term is zero for $i \neq 0$. But $x_i^*(\Omega^1_{T^1_i/T^2_j})$ is tautologically concentrated in degrees $\leqslant 0$, so it is concentrated in degree 0 as desired.

$\square$

16.33. **Dimensions.** We digress briefly to fix some terminology regarding dimensions.

Let T be a finite type scheme. We define the dimension function $\dim_T : T \to \mathbb{Z}^{\geqslant 0}$ to be the locally constant function that on a connected component is constant with value the Krull dimension of that connected component (i.e., the maximal dimension of an irreducible component of this connected component).

For $f : T \to S$ a map between finite type schemes, we let $\dim_{T/S} : T \to \mathbb{Z}$ be the locally constant function $\dim_T - f^*(\dim_S)$.

Example 16.33.1. If $f : T \to S$ is a smooth dominant morphism, then $\dim_{T/S}$ is the rank of the vector bundle $\Omega^1_{T/S}$.

Therefore, for a Cartesian diagram of finite type schemes:

$$\begin{tikzcd} T' \arrow[r, "\psi"] \arrow[d, "g"] & T \arrow[d, "f"] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

with φ and ψ both dominant smooth morphisms, $\dim_{T'/S'} = \psi^*(\dim_{T/S})$. In particular, this identity holds whenever φ is a smooth covering map.

Counterexample 16.33.2. We need not have $\dim_{T/S} = d_{S/T} := \operatorname{rank}(\Omega^1_{T/S})$ if $f : T \to S$ is smooth but not dominant.

For example, let $S = \mathbb{A}^2 \coprod_0 \mathbb{A}^1$ be a line and a plane glued along a point, and let $T = \mathbb{G}_m \times \mathbb{A}^1$ mapping to S via the composition:

$$\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \coprod_0 \mathbb{A}^1.$$

Then $d_{S/T}$ the constant function 1, while $\dim_{S/T}$ is the constant function $\dim_T - \dim_S = 2 - 2 = 0$.

Remark 16.33.3. By Remark 16.29.5, we see from Example 16.33.1 that $\dim_{T/S}$ can be defined as a locally constant function $T \to \mathbb{Z}$ for any finitely presented morphism $T \to S$ of placid schemes by Noetherian descent.

Given a pair of finitely presented morphisms $T \xrightarrow{f} S \to V$ of placid schemes, this construction satisfies the basic compatibility:

$$\dim_{T/V} = \dim_{T/S} - f^*(\dim_{S/V}). \tag{16.33.1}$$

16.34. **Renormalized dualizing sheaf.** Suppose that T is placid scheme. We will now define the *renormalized dualizing sheaf* $\omega_T^{ren} \in D^*(T)$.

Fix a placid presentation $T = \lim_{i \in \mathcal{I}^{op}} T_i$ of T. Because each structure map $\varphi_{ij} : T_j \to T_i$ is a smooth covering, we have canonical identifications:

$$\varphi_{ij}^{*,dR}(\omega_{T_i}[-2 \cdot \dim_{T_i}]) = \omega_{T_j}[-2 \cdot (\dim_{T_j})].$$

Therefore we have a uniquely defined sheaf ω_T^{ren} characterized by the fact that it is the $*$-pullback of $\omega_{T_i}[-2 \cdot \dim_{T_i}]$ from any T_i to T.

We claim that ω_T^{ren} canonically does not depend on the choice of placid presentation. Indeed, this follows from Lemma 16.32.1 and by filteredness of $\mathsf{Pres}(T)$.

Example 16.34.1. Let T be finite type. Then $\omega_T^{ren} \in D^*(T) = D(T)$ identifies with $\omega_T[-2 \cdot \dim_T]$.

Example 16.34.2. Suppose T admits a placid presentation $T = \lim T_i$ with each T_i smooth. Then $\omega_T^{ren} = k_T$.

16.35. Suppose that T is a placid scheme. We define the functor:

$$\eta_T : D^!(T) \to D^*(T)$$

by action on ω_T^{ren}.

Proposition 16.35.1. *The functor η_T is an equivalence.*

Proof. Choose $T = \lim_{i \in \mathcal{I}^{op}} T_i$ a placid presentation. We claim that the functor:

$$\eta_T : D^!(T) := \operatorname*{colim}_{i \in \mathcal{I}} D(T_i) \to D^*(T) \overset{(16.30.1)}{=} \operatorname*{colim}_{i \in \mathcal{I}} D(T_i).$$

is the colimit of the shifted identity functors $\mathrm{id}_{D(T_i)}[-2\cdot\dim_{T_i}]$. Indeed, the colimit of these functors is a morphism of $D^!(T)$-module categories and sends $\omega_T \in D^!(T)$ to $\omega_T^{ren} \in D^*(T)$.

Now the result obviously follows from this identification.

□

Example 16.35.2. If T is finite type then η_T is the composite equivalence $D^!(T) := D(T) =: D^*(T)$ shifted by $-2\dim_T$.

16.36. **Renormalized functors.** Let $f : T \to S$ a map of placid schemes.

We let $f_{*,ren} : D^!(T) \to D^!(S)$ denote the induced functor so that we have the commutative diagram:

$$\begin{tikzcd} D^!(T) \arrow[r, "f_{*,ren}"] \arrow[d, "\eta_T", "\simeq"'] & D^!(S) \arrow[d, "\eta_S", "\simeq"'] \\ D^*(T) \arrow[r, "f_{*,dR}"] & D^*(S). \end{tikzcd}$$

In the same way we obtain the functor $f^{!,ren} : D^*(S) \to D^*(T)$ fitting into a commutative diagram:

$$\begin{tikzcd} D^*(S) \arrow[r, "f^{!,ren}"] & D^*(T) \\ D^!(S) \arrow[r, "f^!"] \arrow[u, "\eta_S", "\simeq"'] & D^!(T) \arrow[u, "\eta_T", "\simeq"'] \end{tikzcd}$$

Note that we have a canonical isomorphism

$$f^{!,ren}(\omega_S^{ren}) = \omega_T^{ren} \tag{16.36.1}$$

because:

$$f^{!,ren}(\omega_S^{ren}) = f^!(\omega_S) \overset{!}{\otimes} \omega_T^{ren} = \omega_T \overset{!}{\otimes} \omega_T^{ren} = \omega_T^{ren}.$$

Example 16.36.1. Suppose $f : T \to S$ is a map between finite type schemes. We identify $D^!(S)$ and $D^!(T)$ with $D(S)$ and $D(T)$ in the canonical way.

Then the functor $f_{*,ren} : D(T) \to D(S)$ identifies with $f_{*,dR}[-2 \cdot \dim_{T/S}]$. In particular, if f is smooth and dominant, then $(f^!, f_{*,ren})$ form an adjoint pair of functors.

Note that in this setting the functor $f_{*,!-dR}$ coincides with the (non-renormalized) functor $f_{*,dR}$.

Warning 16.36.2. If $f : S \to T$ is a closed embedding of placid schemes, then $f_{*,ren}$ is not left adjoint to $f^!$ (c.f. Example 16.36.1). In fact, if f is a closed embedding of infinite codimension, then $f_{*,ren}$ does not preserve compact objects and therefore does not admit a continuous right adjoint at all.

Warning 16.36.3. Given a Cartesian diagram:

$$\begin{array}{ccc} T_1 & \xrightarrow{\psi} & S_1 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} \\ T_2 & \xrightarrow{g} & S_2 \end{array}$$

of finite type schemes, we find that:

$$f^! g_{*,ren} = f^! g_{*,dR}[-2 \cdot \dim_{T_2/S_2}] = \psi_{*,dR}\varphi^![-2 \cdot \dim_{T_2/S_2}]$$

while $\psi_{*,ren}\varphi^! = \psi_{*,dR}\varphi^![-2 \cdot \dim_{T_1/S_1}]$. Since dimensions do not always behave well under base-change, we see that base-change does not always hold between renormalized pushforward and upper-!.

Example 16.36.4. Suppose $f : T \to S$ is a map between finite type schemes. We identify $D^*(S)$ and $D^*(T)$ with $D(S)$ and $D(T)$ in the canonical way.

Then the functor $f^{!,ren} : D(S) \to D(T)$ identifies with $f^!(-)[-2 \dim_{T/S}]$. Note that if f is smooth and dominant, then $f^{!,ren}$ identifies canonically with $f^{*,dR}$.

The functor $f^{¡}$ coincides with the (non-renormalized) functor $f^!$.

Remark 16.36.5. We emphasize explicitly that the "renormalization" here has nothing to do with the renormalized de Rham cohomology functor from [DG12]. Rather, the terminology is taken from [Dri06] §6.8.

16.37. **Placid morphisms.** We will now further analyze the renormalized functors under certain very favorable circumstances.

We say a morphism $f : S \to T$ of placid schemes is *placid* if, for any placid presentations $S = \lim_{i \in \mathcal{I}^{op}} S_i$, $T = \lim_{j \in \mathcal{J}^{op}} T_j$, for every $j \in \mathcal{J}$ there exists $i \in \mathcal{I}$ with the morphism $S \to T \to T_j$ factoring as $S \to S_i \to T_j$ and with $S_i \to T_j$ a smooth covering.

Obviously, if this holds for one pair of placid presentations then it holds for any.

Example 16.37.1. By Noetherian descent and Remark 16.29.6, smooth morphisms that are surjective on geometric points are placid.

Example 16.37.2. Suppose that $S = \lim_{i \in \mathcal{I}^{op}} S_i$ and $T = \lim_{i \in \mathcal{I}^{op}} T_i$ are placid presentations, and suppose that we are given compatible smooth coverings $f_i : S_i \to T_i$ inducing $f : S \to T$ (by compatible, we do not assume that the relevant squares are Cartesian, only that they commute). Then f is a placid morphism.

Remark 16.37.3. For categorical arguments, it is convenient to use the following formulation of this definition.

Let $\mathsf{Sch}^{f.t.}_{sm-cov}$ denote the category of finite type schemes where we only allow smooth coverings as morphisms. Let:

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm-cov})$$

denote the full subcategory where we only allow objects obtained as projective limits under morphisms that are affine (in addition to being a priori smooth coverings).

Then the functor:

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \to \mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}) = \mathsf{Sch}_{qcqs}$$

is a (non-full) embedding of categories. Indeed, this is a general feature: (non-full) embeddings of $(1,1)$-categories induce embeddings on Ind or Pro categories, since filtered limits and colimits of injections in Set are still injections. Moreover, its essential image are placid schemes, and a morphism lies in this non-full subcategory if and only if it is placid.

Observe that $\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}_{sm\text{–}cov}) \to \mathsf{Sch}_{qcqs}$ commutes with filtered projective limits with affine structure maps, i.e., this functor is the right Kan extension of its restriction to $\mathsf{Sch}^{f.t.}_{sm\text{–}cov}$. Indeed, $\mathsf{Sch}_{qcqs} \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.})$ commutes with such filtered projective limits, and $\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}_{sm\text{–}cov}) \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm\text{–}cov})$ does too. Moreover, $\mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm\text{–}cov}) \to \mathsf{Pro}(\mathsf{Sch}^{f.t.})$ tautologically commutes with filtered limits, proving the claim.

Warning 16.37.4. Against the usual conventions for terminology in algebraic geometry, placid morphisms are not intended as a relative form of placidity.

Indeed, we can only speak about placid morphisms between between schemes already known to be placid. Moreover, for a placid scheme S, the structure map $S \to \mathrm{Spec}(k)$ may not be placid.

The terminology is rather taken by analogy with the definition of placid schemes, as in Remark 16.37.3.

Counterexample 16.37.5. It may be tempting to think of placid morphisms as being analogous to being a smooth covering morphisms, since this condition is equivalent for finite type schemes. The following example is meant to show the geometric limitations of this line of thought. We also note that this example models the geometry of Lemma 6.30.1.

Let $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^1 \times \mathbb{A}^{n-1}$ by:

$$\Big(\lambda, (x_1, \ldots, x_n)\Big) \mapsto \Big(\lambda, (x_1 - \lambda \cdot x_2, \ldots, x_{n-1} - \lambda \cdot x_n)\Big).$$

Each of these morphisms is a smooth covering. Moreover, these morphisms are compatible as n varies, and therefore induce a placid morphism (of infinite type):

$$\mathbb{A}^1 \times \mathbb{A}^\infty \to \mathbb{A}^1 \times \mathbb{A}^\infty$$

$$(\lambda, x_1, x_2, \ldots) \mapsto (\lambda, x_1 - \lambda \cdot x_2, x_2 - \lambda \cdot x_3, \ldots).$$

where we use the notation $\mathbb{A}^\infty = \lim_n \mathbb{A}^n$, the limit taken under structure maps $\mathbb{A}^m \to \mathbb{A}^n$ $(m \geqslant n)$ of projection onto the first n-coordinates.

Then for $0 \neq \lambda \in k$, the fiber of this map at $(\lambda, 0, 0, \ldots, 0)$ is a copy of $\mathbb{A}^1$, realized as the loci of points:

$$(\lambda, x_1, \lambda^{-1} \cdot x_1, \lambda^{-2} \cdot x_1, \ldots)$$

with $x_1 \in \mathbb{A}^1$ arbitrary.

However, the fiber at $(0, 0, 0, \ldots)$ is just the point scheme $\mathrm{Spec}(k)$, realized as the locus $(0, 0, 0, \ldots)$.

In particular, we see that fibers of placid morphisms can be finite type dimensional schemes that vary non-smoothly.

Lemma 16.37.6. *Given a Cartesian diagram:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

of placid schemes with g finite presentation and f a placid morphism, the morphism φ is placid as well.

Proof. Let $S_1 = \lim_i S_{1,i}$ and $T_1 = \lim_j T_{1,j}$ be placid presentations. We take a compatible placid presentation $T_2 = \lim_j T_{2,j}$ as in Remark 16.29.5.

Note that:

$$S_2 = \lim_j \lim_i S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}$$

where we really only take the limit under i such that the map $S_1 \to T_{1,j}$ factors (necessarily uniquely) through $S_{1,i}$.

For a pair of morphisms $(i_1 \to i_2)$ and $(j_1 \to j_2)$, we claim that the induced map:

$$S_{1,i_2} \underset{T_{1,j_2}}{\times} T_{2,j_2} \to S_{1,i_1} \underset{T_{1,j_1}}{\times} T_{2,j_1}$$

is an affine smooth covering. Indeed, we have $T_{2,j_2} = T_{1,j_2} \times_{T_{1,j_1}} T_{2,j_1}$ so that the left hand side of the above is $S_{1,i_2} \times_{T_{1,j_1}} T_{2,j_1}$. Because $S_{1,i_2} \to S_{1,i_1}$ is an affine smooth covering, we obtain the claim.

Therefore, the terms $S_{1,i} \times_{T_{1,j}} T_{2,j}$ define a placid presentation of S_2. But each map:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} \to T_{2,j}$$

is a smooth covering because each $S_{1,i} \to T_{1,j}$ is assumed to be, completing the proof.

□

The following results from the argument above.

Corollary 16.37.7. *Suppose that we have a Cartesian square:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

of placid schemes with g finite presentation and f a placid morphism. Then $\dim_{S_2/S_1} = \varphi^*(\dim_{T_2/T_1})$.

Proof. Let $S_1 = \lim_i S_{1,i}$, $T_1 = \lim_j T_{1,j}$ and $T_2 = \lim_j T_{2,j}$ be as in the proof of Lemma 16.37.6. As in *loc. cit.*, we have a placid presentation of S_2 with terms:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}.$$

Fixing and index j_0, as in *loc. cit.*, we have:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} = S_{1,i} \underset{T_{1,j_0}}{\times} T_{2,j_0}.$$

for every morphism $j_0 \to j$. Therefore, the morphisms $S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} \to S_{1,i}$ are obtained one from another by base-change, so that $\dim_{S_2/S_1}$ is defined as the pullback of the function:

$$\dim_{S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}/S_{1,i}}$$

for any choice of indices. But because our maps are smooth coverings, this function is the pullback of $\dim_{T_{2,j}/T_{1,j}}$, giving the result.

$\square$

16.38. For our purposes, the key feature of placid morphisms is given by the following proposition.

Proposition 16.38.1. *(1) For a placid morphism $f : S \to T$ of placid schemes, the left adjoint $f^{*,dR}$ to $f_{*,dR} : D^*(S) \to D^*(T)$ is defined.*

(2) For a placid morphism $f : S \to T$ of placid schemes, there is a canonical identification $f^{!,ren} \simeq f^{*,dR} : D^*(T) \to D^*(S)$.

More precisely, with Sch_{pl} denoting the category of placid schemes under placid morphisms, there is a canonical identification of functors:

$$(D^*, f^{*,dR}) \simeq (D^*, f^{!,ren}) : \mathsf{Sch}_{pl}^{op} \to \mathsf{DGCat}_{cont}$$

inducing the identity over the maximal subgroupoid of Sch_{pl}^{op}.

(3) *For a placid morphism $f : S \to T$ of placid schemes, the functor $f^! : D^!(T) \to D^!(S)$ admits a right adjoint, and this right adjoint is functorially identified with $f_{*,ren}$ in the sense above.*

(4) *For a Cartesian square of placid schemes:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd} \tag{16.38.1}$$

with f placid and g finitely presented, the canonical morphisms:

$$f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$$

$$f^! g_{*,ren} \to \psi_{*,ren} \varphi^!$$

arising from the adjunctions above are equivalences.

We begin with the following general remarks.

Let $\mathsf{DGCat}_{cont}^{ladj}$ denote the category of cocomplete DG categories under k-linear functors that admit continuous right adjoints. Let $\mathsf{DGCat}_{cont}^{radj}$ denote the category of cocomplete DG categories under k-linear functors that admit left adjoints.

We have an obvious equivalence $\mathsf{DGCat}_{cont}^{ladj} \simeq \mathsf{DGCat}_{cont}^{radj,op}$ given by passing to the adjoint functor.

One easily verifies:

Lemma 16.38.2. *The category $\mathsf{DGCat}_{cont}^{ladj}$ admits colimits, and the functor $\mathsf{DGCat}_{cont}^{ladj} \to \mathsf{DGCat}_{cont}$ preserves these colimits. Similarly, $\mathsf{DGCat}_{cont}^{radj}$ admits limits, and the functor $\mathsf{DGCat}_{cont}^{radj} \to \mathsf{DGCat}_{cont}$ commutes with limits.*

Proof. The content is that given a diagram $i \mapsto \mathcal{C}_i$ of cocomplete DG categories under structure functors admitting continuous right adjoints, a functor $\mathcal{C} := \operatorname{colim}_i \mathcal{C}_i \to \mathcal{D}$

admits a continuous right adjoint if and only if each $\mathcal{C}_i \to \mathcal{D}$ does. But this is obvious, since $\mathcal{C}$ is then also the limit of the $\mathcal{C}_i$ under the right adjoint functors.

□

Proof of Proposition 16.38.1. Recall from Remark 16.37.3 that Sch_{pl} is the full subcategory:

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm-cov}).$$

Moreover, because $\mathsf{Sch}_{pl} \to \mathsf{Sch}_{qcqs}$ is the right Kan extension of its restriction to $\mathsf{Sch}^{f.t.}_{sm-cov}$, we see that $D^*|_{\mathsf{Sch}_{pl}}$ is the right Kan extension of $D^*|_{\mathsf{Sch}^{f.t.}_{sm-cov}} = D|_{\mathsf{Sch}^{f.t.}_{sm-cov}}$.

Moreover, note that $D^*|_{\mathsf{Sch}^{f.t.}_{sm-cov}}$ factors through $\mathsf{DGCat}^{radj}_{cont}$ by smoothness.

As in Example 16.36.4, the corresponding functor:

$$D|_{\mathsf{Sch}^{f.t.}_{sm-cov}} \to \mathsf{DGCat}^{radj}_{cont} \simeq \mathsf{DGCat}^{ladj,op}_{cont}$$

identifies with $(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.}_{sm-cov}}$, i.e., the functor attaching to a scheme of finite type its category of D-modules, and to a smooth surjective morphism of schemes the corresponding renormalized pullback functor.[39]

By Lemma 16.38.2, the right Kan extension of this functor also factors through $\mathsf{DGCat}^{radj}_{cont}$, proving (1). Moreover, it follows that the corresponding functor to $\mathsf{Sch}^{op}_{pl} \to \mathsf{DGCat}^{ladj}$ encoding the left adjoints is the left Kan extension of $(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.,op}_{sm-cov}}$.

We have an equivalence:

$$(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.,op}} \simeq (D, f^!)|_{\mathsf{Sch}^{f.t.,op}}$$

computed termwise on a finite type scheme S as η_S^{-1}. Moreover, $(D^!, f^!)$ is the left Kan extension of the left hand side.

For a placid scheme S with placid presentation $S = \lim S_i$, we have:

[39] This identification is treated formally in the homotopical setting in [GR14].

$$\eta_S = \operatorname{colim}_i \eta_{S_i} : D^!(S) = \operatorname{colim}_i D^!(S_i) \to \operatorname{colim}_i D^*(S_i) = D(S)$$

the colimit on the right taken under renormalized pullback functors (equivalently: $*$-dR pullback). Indeed, this was already observed in the proof of Proposition 16.35.1.

Therefore, we see that $(D^!, f^{!,ren})$ is the left Kan extension of $(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.}_{sm-cov}}$, as desired. This completes the proof of (2).

Note that (3) is a formal consequence of (2). Therefore, it remains to show (4).

Suppose we are given a Cartesian square (16.38.1). It obviously suffices to show either of the base-change morphisms is an equivalence, so we treat the map $f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$.

First, suppose that T_1 and T_2 are finite type.

We take a placid presentation $S_1 = \lim_i S_{1,i}$. We can assume each $S_{1,i}$ is a T_1-scheme by Noetherian approximation.

Because $S_1 \to T_1$ is placid, each $S_{1,i} \to T_1$ is a smooth covering. Define $S_{2,i} = S_{1,i} \times_{T_1} T_2$.

We use the notation:

$$\begin{tikzcd} S_2 \arrow[r, "\beta_i"] \arrow[d, "\psi"] & S_{2,i} \arrow[r, "\varphi_i"] \arrow[d, "\psi_i"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "\alpha_i"] & S_{1,i} \arrow[r, "f_i"] & T_1. \end{tikzcd} \tag{16.38.2}$$

We now have:

$$f_{*,dR} f^{!,ren} g_{*,dR} = \operatorname{colim}_i f_{i,*,dR} f_i^{!,ren} g_{*,dR} = \operatorname{colim}_i f_{i,*,dR} \psi_{i,*,dR} \varphi_i^{!,ren} =$$
$$\operatorname{colim}_i g_{*,dR} \varphi_{i,*,dR} \varphi_i^{!,ren} = g_{*,dR} \varphi_{*,dR} \varphi^{!,ren} = f_{*,dR} \psi_{*,dR} \varphi^{!,ren}$$

Here the first and fourth equalities follows from filteredness of our index category and the adjunctions. The base-change in our second equality follows from the usual smooth base-change theorem in the finite type setting.

Applying the above argument to the left square of (16.38.2) and applying (finite-dimensional) smooth base-change to the right square, we see that the map:

$$\alpha_{i,*,dR} f^{!,ren} g_{*,dR} \to \alpha_{i,*,dR} \psi_{*,dR} \varphi^{!,ren}$$

is always an equivalence. But this suffices to see our base-change by definition of D^*.

We now treat the case of general g of finite presentation. Suppose that we have a diagram:

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[r, "\theta"] \arrow[d, "g"] & T_2' \arrow[d, "g'"] \\ S_1 \arrow[r, "f"] & T_1 \arrow[r, "\varepsilon"] & T_1' \end{tikzcd} \tag{16.38.3}$$

with both squares Cartesian, the schemes T_i' of finite type, and the maps θ and ε placid.

Then we have base-change maps:

$$f^{!,ren} \varepsilon^{!,ren} g'_{*,dR} \to f^{!,ren} g_{*,dR} \theta^{!,ren} \to \psi_{*,dR} \varphi^{!,ren} \theta^{!,ren}.$$

By our earlier analysis, the first map is an equivalence by considering the right square of (16.38.3), and the composite map is also an equivalence by considering the outer square of (16.38.3). Therefore, we see that the map:

$$f^{!,ren} g_{*,dR} \theta^{!,ren} \to f^{!,ren} g_{*,ren} \theta^{!,ren}$$

is an equivalence. Varying T_1' over some placid presentation of T_1, the corresponding functors $\theta^{!,ren}$ generate $D^*(T_2)$, so this suffices.

$\square$

16.39. As a consequence of Proposition 16.38.1, we show that some features from Examples 16.36.1 and 16.36.4 survive to greater generality.

Proposition 16.39.1. *For $f : T \to S$ a finitely presented morphism of placid schemes, we have canonical identifications:*

$$f^{¡}[-2 \cdot \dim_{T/S}] = f^{!,ren} : D^*(S) \to D^*(T)$$

$$f_{*,!-dR}[-2 \cdot \dim_{T/S}] = f_{*,ren} : D^!(T) \to D^!(S).$$

where $\dim_{T/S}$ *is defined as in §16.33.*

Proof. Let $S = \lim S_i$ be a placid presentation, and by Remark 16.29.5, we may assume we have a placid presentation $T = \lim T_i$ so that we have maps $f_i : T_i \to S_i$ with each $i \to j$ inducing a Cartesian diagram, and with f obtained by base-change from each of the f_i. Note that $\dim_{T/S}$ is then obtained by pullback from each $\dim_{T_i/S_i}$.

We use the notation:

$$\begin{tikzcd} T \arrow[r, "\psi_i"] \arrow[d, "f"] & T_i \arrow[d, "f_i"] \\ S \arrow[r, "\varphi_i"] & S_i. \end{tikzcd}$$

For the first part, note that by (16.30.3) and Example 16.36.4, we have:

$$f^{¡} = \operatorname*{colim}_i \psi_i^{*,dR} \psi_{i,*,dR} f^{¡} = \operatorname*{colim}_i \psi_i^{*,dR} f_i^{¡} \varphi_{i,*,dR} = \operatorname*{colim}_i \psi_i^{*,dR} f_i^{!,ren} \varphi_{i,*,dR}[2 \cdot \dim_{T/S}].$$

By Proposition 16.38.1, $\psi_i^{*,dR} = \psi_i^{!,ren}$. Therefore, we compute the above as:

$$\operatorname*{colim}_i \psi_i^{!,ren} f_i^{!,ren} \varphi_{i,*,dR}[2 \cdot \dim_{T/S}] = \operatorname*{colim}_i f^{!,ren} \varphi_i^{!,ren} \varphi_{i,*,dR}[2 \cdot \dim_{T/S}] = f^{!,ren}[2 \cdot \dim_{T/S}]$$

by again applying (16.30.3) and the identification $\varphi_i^{!,ren} = \varphi_i^{*,dR}$.

For the second part, note that we have functorial base change isomorphisms:

$$\varphi_i^! f_{i,*,ren} \simeq f_{*,ren} \psi_i^!$$

by Proposition 16.38.1. By Example 16.36.1, $f_{i,*,!-dR}[-2 \cdot \dim_{T/S}] = f_{i,*,ren}$. Moreover, these cohomological shifts are compatible with varying i, so we obtain the result by definition of $f_{*,!-dR}$.

$\square$

Corollary 16.39.2. *Suppose we are given a Cartesian square:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

with S_1 and T_2 placid schemes, f and g placid morphisms, and T_1 finite type. Then the canonical morphisms:

$$f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$$

$$f^! g_{*,ren} \to \psi_{*,ren} \varphi^!$$

are equivalences.

Proof. Note that we have already seen in Example 16.29.4 that S_2 is actually a placid scheme.

It tautologically suffices to prove that the first base-change morphism is an equivalence.

We form the diagram:

$$\begin{tikzcd} S_2 \arrow[r, "i"] \arrow[d, "\psi"] & S_1 \times T_2 \arrow[r, "p_2"] \arrow[d, "\mathrm{id}_{S_1} \times g"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "\Delta_f"] & S_1 \times T_1 \arrow[r, "p_2"] & T_1. \end{tikzcd}$$

Here Δ_f is the graph of f. Note that each of these squares is Cartesian. In particular, i is a finitely presented morphism. We are reduced to proving the base-change result for each of these squares separately.

For the right square, the result is essentially obvious: it follows from the compatibility of push-forward with products of schemes.

For the left square, note that the base-change result holds with the upper-¡ functor in place of the renormalized upper-! functor by the correspondence formalism. Therefore, the result follows from Proposition 16.39.1.

□

16.40. **Holonomic D-modules.** Let S be a scheme of finite type. Let $D_{coh,hol}(S)$ denote the full subcategory of $D_{coh}(S)$ (the compact objects in $D(S)$) composed of those coherent complexes with holonomic cohomologies, defined in the usual way. Let $D_{hol}(S) \subseteq D(S)$ denote the full subcategory:

$$D_{hol}(S) := \mathsf{Ind}(D_{coh,hol}(S)) \subseteq D(S).$$

We refer to objects of $D_{hol}(S)$ simply as holonomic objects.[40]

For $f : S \to T$ a map of finite type schemes, the usual theory of D-modules implies that the functors $f_{*,dR}$ and $f^!$ preserve the subcategories of holonomic objects.

For S a quasi-compact quasi-separated scheme, we obtain the categories:

$$D^!_{hol}(S) \text{ and } D^*_{hol}(S)$$

defined by a Kan extension, as in the case of $D^!$ and D^*. We have obvious functors $D^!_{hol}(S) \to D^!(S)$ and $D^*_{hol}(S) \to D^*(S)$, the latter being fully-faithful. For S placid, we can express $D^*_{hol}(S)$ as a limit as for $D^*(S)$, and therefore we see that $D^*_{hol}(S) \to D^*(S)$ is fully-faithful in this case as well. We refer to subobjects of $D^*(S)$ lying in $D^*_{hol}(S)$ as holonomic objects, and similarly for $D^!$ when S is placid.

[40]We note that, of course, this condition completely ruins all the nice finiteness conditions that "usual" (coherent) holonomic complexes satisfy, e.g., finiteness of de Rham cohomology. This is necessary for obvious reasons in the infinite-dimensional setting.

We have upper-! and lower-* functors for $D^!_{hol}(S)$ and $D^*_{hol}(S)$ respectively, compatible with the forgetful functors.

Proposition 16.40.1. *For $f : S \to T$ a morphism of quasi-compact quasi-separated schemes, the morphism $f_{*,dR} : D^*_{hol}(S) \to D^*_{hol}(T)$ admits a left adjoint $f^{*,dR}$.*

If T is placid and f is finitely presented, then the morphism $f^! : D^!_{hol}(T) \to D^!_{hol}(S)$ admits a left adjoint $f_!$.

Moreover, in each of the above settings, these left adjoints are well-behaved with respect to maps to non-holonomic objects as well, i.e., the partially-defined left adjoints to $f_{*,dR} : D^*(S) \to D^*(T)$ and $f^! : D^!(T) \to D^!(S)$ are defined on holonomic objects, and these left adjoints preserve the holonomic subcategories (and therefore are computed by the above functors). Of course, we are assuming f finitely presented and T placid when discussing $f_!$.

We begin with the following lemma.

Lemma 16.40.2. *Let $\mathcal{I}$ be an indexing category with $\mathcal{I}^{op}$ filtered. Let $(i \mapsto \mathcal{C}_i)$ and $(i \mapsto \mathcal{D}_i)$ are two $\mathcal{I}$-shaped diagrams of cocomplete categories under continuous functors, with structure functors:*

$$\psi_\alpha : \mathcal{C}_i \to \mathcal{C}_j \qquad \psi_i : \mathcal{C} := \lim_{j \in \mathcal{I}} \mathcal{C}_j \to \mathcal{C}_i$$

$$\varphi_\alpha : \mathcal{D}_i \to \mathcal{D}_j \qquad \varphi_i : \mathcal{D} := \lim_{j \in \mathcal{I}} \mathcal{D}_j \to \mathcal{D}_i$$

for $\alpha : i \to j$ in $\mathcal{I}$ and for $i \in \mathcal{I}$.

Suppose $G_i : \mathcal{C}_i \to \mathcal{D}_i$ are compatible functors with induced functor:

$$G : \mathcal{C} \to \mathcal{D}.$$

If each functor G_i admits a left adjoint F_i, then G admits a left adjoint $F : \mathcal{D} \to \mathcal{C}$ such that, for every $j \in \mathcal{I}$, we have:

$$\psi_j F = \operatorname*{colim}_{(\alpha: i \to j) \in (\mathcal{I}_{/j})^{op}} \psi_\alpha F_i \varphi_i.$$

Proof. For $j \in \mathcal{I}$ fixed, note that for any diagram:

$$i' \xrightarrow{\beta} i \xrightarrow{\alpha} j$$

we have the natural map:

$$\varphi_\beta \to \varphi_\beta G_{i'} F_{i'} \to G_i \psi_\beta F_{i'}.$$

By adjunction, this gives rise to a map:

$$F_i \varphi_\beta \to \psi_\beta F_{i'}.$$

Composing on the left with ψ_α and on the right with $\varphi_{i'}$, we obtain the map:

$$\psi_\alpha F_i \varphi_\beta \varphi_{i'} = \psi_\alpha F_i \varphi_i \to \psi_{\alpha \circ \beta} F_{i'} \varphi_{i'} = \psi_\alpha \psi_\beta F_{i'} \varphi_{i'}.$$

Expressing this in the obvious homotopy-compatible way, we obtain a diagram of functors:

$$(\alpha : i \to j) \in (\mathcal{I}_{/j})^{op} \mapsto \psi_\alpha F_i \varphi_i.$$

Define the functor:

$$\text{“}\psi_j F\text{”} := \operatorname*{colim}_{(\alpha: i \to j) \in (\mathcal{I}_{/j})^{op}} \psi_\alpha F_i \varphi_i.$$

As j varies, we see by filteredness that these functors are homotopy compatible, and therefore we obtain a functor $F : \mathcal{D} \to \mathcal{C}$ with the property that we have functorial identifications:

$$\psi_j F = \text{“}\psi_j F\text{”}$$

with “$\psi_j F$” as above.

For every $j \in \mathcal{I}$, we have the map:

$$\psi_j FG = \underset{(\alpha:i\to j)\in(\mathcal{I}_{/j})^{op}}{\operatorname{colim}} \psi_\alpha F_i \varphi_i G = \psi_\alpha F_i G_i \psi_i \to \psi_\alpha \psi_i = \psi_j.$$

As $j \in \mathcal{I}$ varies, these maps are homotopy compatible and therefore we obtain the counit map:

$$FG \to \mathrm{id}_{\mathcal{C}}.$$

Similarly, for every $j \in \mathcal{I}$, we have the map:

$$\varphi_j = \underset{(\alpha:i\to j)\in(\mathcal{I}_{/j})^{op}}{\operatorname{colim}} \varphi_j = \underset{(\alpha:i\to j)\in(\mathcal{I}_{/j})^{op}}{\operatorname{colim}} \varphi_\alpha \varphi_i \to \underset{(\alpha:i\to j)\in(\mathcal{I}_{/j})^{op}}{\operatorname{colim}} \varphi_\alpha G_i F_i \varphi_i =$$

$$\underset{(\alpha:i\to j)\in(\mathcal{I}_{/j})^{op}}{\operatorname{colim}} G_j \psi_\alpha F_i \varphi_i = G_j \psi_j F = \varphi_j GF.$$

As j varies, these maps are homotopy compatible and therefore give the unit map:

$$\mathrm{id}_{\mathcal{D}} \to GF.$$

One readily checks that the unit and counit maps constructed above actually define an adjunction.

□

Proof of Proposition 16.40.1. For any map $f : S \to T$, it is easy to see that we can arrange to have $S = \lim_{i\in\mathcal{J}^{op}} S_i$, $T = \lim_{i\in\mathcal{J}^{op}}$ filtered systems of finite type schemes under affine maps and with compatible maps $f_i : S_i \to T_i$ inducing f in the limit (note that we do not assume any diagrams are Cartesian). Therefore, the existence of the left adjoint $f^{*,dR}$ follows immediately from Lemma 16.40.2.

Let us see that these objects map in the obvious way to non-holonomic objects. For $\alpha : i \to j$, let $\varphi_i : S \to S_i$, $\varphi_\alpha : S_j \to S_i$, $\psi_i : T \to T_i$, $\psi_\alpha : T_j \to T_i$ denote the structure maps.

Note that e.g. $D^*_{hol}(T) \to D^*(T)$ is continuous. Therefore, for $\mathcal{F} \in D^*_{hol}(T)$ and $\mathcal{G} \in D^*(S)$, we have:

$$\mathrm{Hom}_{D^*(T)}(f^{*,dR}(\mathcal{F}), \mathcal{G}) = \lim_i \lim_{\alpha:i\to j} \mathrm{Hom}_{D(T_i)}(\psi_{\alpha,*,dR} f_j^{*,dR} \varphi_{j,*,dR}(\mathcal{F}), \psi_{i,*,dR}(\mathcal{G})) =$$

$$\lim_i \mathrm{Hom}_{D(T_i)}(f_i^{*,dR} \varphi_{i,*,dR}(\mathcal{F}), \psi_{i,*,dR}(\mathcal{G})) = \mathrm{Hom}_{D(T_i)}(\varphi_{i,*,dR}(\mathcal{F}), f_{i,*,dR}\psi_{i,*,dR}(\mathcal{G})) =$$

$$\mathrm{Hom}_{D(T_i)}(\varphi_{i,*,dR}(\mathcal{F}), \varphi_{i,*,dR} f_{*,dR}(\mathcal{G})) = \mathrm{Hom}_{D^*(T)}(\mathcal{F}, f_{*,dR}(\mathcal{G}))$$

For f finite presentation, we can take placid presentations $S = \lim S_i$ and $T = \lim T_i$ as in Remark 16.29.5: by base-change, the upper-! functors are compatible with the shifted lower-* functors expressing D^* as a limit (using placidity), so Lemma 16.40.2 again applies. The same argument as above treats maps to non-holonomic objects.

□

We also have the following observation.

Proposition 16.40.3. *If S is placid, then η_S identifies the subcategories $D^!_{hol}(S)$ and $D^*_{hol}(S)$.*

Proof. Suppose $\mathcal{F} \in D^!(S)$. We will show that $\mathcal{F} \in D^!_{hol}(S)$ if and only if $\eta_S(\mathcal{F}) \in D^*_{hol}(\mathcal{F})$.

Let $S = \lim_i S_i$ be a placid presentation of S and let $\alpha_i : S \to S_i$ denote the structure maps.

By definition, $\eta_S(\mathcal{F})$ is in $D^*_{hol}(\mathcal{F})$ if and only if $\alpha_{i,*,ren}(\mathcal{F}) \in D_{hol}(S_i)$ for every i. By (16.30.3) and Proposition 16.38.1, we have:

$$\mathcal{F} = \operatorname*{colim}_i \alpha_i^! \alpha_{i,*,ren}(\mathcal{F})$$

giving the result.

To see that for $\mathcal{F} = D^!_{hol}(S)$ we have $\alpha_{i,*,ren}(\mathcal{F}) \in D_{hol}(S_i)$, note that $D^!_{hol}(S)$ is tautologically generated under colimits by objects $\alpha^!_j(\mathcal{F}_j)$, for $\mathcal{F}_j \in D_{hol}(S_j)$. By filteredness of our indexing category, we can compute $\alpha_{i,*,ren}\alpha^!_j(\mathcal{F}_j)$ as a colimit of objects obtained by pushing and pulling along correspondences $S_i \leftarrow S_k \rightarrow S_j$ (coming from correspondences $i \rightarrow k \leftarrow j$ in the indexing category).

□

Corollary 16.40.4. *For $f : S \rightarrow T$ a morphism of placid schemes, the functors $f_{*,ren}$ and $f^{!,ren}$ preserve holonomic objects in $D^!$ and D^* respectively.*

16.41. **Indschemes.** We now move to the setting of indschemes.

We say that $T \in \mathsf{PreStk}$ is a (classical) indscheme if $T = \mathrm{colim}_{i \in \mathcal{I}} T_i$ in PreStk where $\mathcal{I}$ is filtered, $T_i \in \mathsf{Sch}_{qcqs} \subseteq \mathsf{PreStk}$ and each structure map $T_i \rightarrow T_j$ is a closed embedding (recall that in this case $T \in \mathrm{Stk} \subseteq \mathsf{PreStk}$).

16.42. We define the functor $D^! : \mathsf{IndSch}^{op} \rightarrow \mathsf{DGCat}_{cont}$ as the right Kan extension of the functor $D^! : \mathsf{Sch}^{op}_{qcqs} \rightarrow \mathsf{DGCat}_{cont}$. Therefore, for $T = \mathrm{colim}\, T_i$ we have $D^!(T) = \lim D^!(T_i)$ where the structure functors are !-pullback functors.

For $f : T \rightarrow S$ a map of indschemes, we let $f^! : D^!(S) \rightarrow D^!(T)$ denote the corresponding pullback functor.

The functor $D^!$ lifts to a functor $D^! : \mathsf{Sch}^{op}_{qcqs} \rightarrow \mathrm{ComAlg}(\mathsf{DGCat}_{cont})$, i.e., each $D^!(T)$ has a symmetric monoidal structure $\overset{!}{\otimes}$ and every map $f : T \rightarrow S$ induces a symmetric monoidal functor $f^! : D^!(T) \rightarrow D^!(S)$. The unit of the symmetric monoidal structure is $\omega_T := p^!_T(k) \in D^!(T)$ for $T \rightarrow \mathrm{Spec}(k)$.

16.43. Similarly, we define the functor $D^* : \mathsf{IndSch} \rightarrow \mathsf{DGCat}_{cont}$ as the left Kan extension of the functor $D^* : \mathsf{Sch}_{qcqs} \rightarrow \mathsf{DGCat}_{cont}$. For $T = \mathrm{colim}\, T_i$, we have $D^*(T) = \mathrm{colim}\, D^*(T_i)$ where the structure functors are $*$-pushforward functors.

For $f : T \rightarrow S$ a map of indschemes, we let $f_{*,dR} : D^*(T) \rightarrow D^*(S)$ denote the corresponding pushforward functor.

For every indscheme T and quasi-compact quasi-separated closed subscheme $T' \subseteq T$, we have the symmetric monoidal functor $D^!(T) \to D^!(T')$, so that $D^*(T')$ is a module category for $D^!(T)$. By the projection formula (16.21.1), for every $T' \subseteq T'' \subseteq T$ with T' and T'' quasi-compact quasi-separated closed subschemes, the $*$-pushforward $D^*(T') \to D^*(T'')$ is a morphism of $D^!(T)$-module categories. Passing to the colimit, we obtain that $D^*(T)$ is a module category for $D^!(T)$ canonically.

We again have a projection formula, i.e., for $f : T \to S$ a map of indschemes the functor $f_{*,dR} : D^*(T) \to D^*(S)$ is a morphism of $D^!(S)$-module categories.

If $D^!(T)$ is dualizable and $D^*(T')$ is dualizable for every $T' \subseteq T$ a quasi-compact quasi-separated closed subscheme, then $D^!(T)$ is canonically dual to $D^*(T)$. This identification is compatible with $D^!(T)$-module category structures.

Notation 16.43.1. If T is an indscheme of ind-finite type then $D^!(T)$ and $D^*(T)$ are canonically identified. Indeed, the former is the colimit under left adjoints and the latter is the limit under right adjoints.

As in the finite type case, we denote this category simply by $D(T)$, as there is no risk for confusion.

16.44. **Correspondences.** We say a morphism $f : T \to S$ of indschemes is *finitely presented* if f is schematic and its base-change by any scheme is a finitely presented morphism.

Exactly parallel to Propositions 16.14.1 and 16.25.1 one shows that $D^!$ and D^* upgrade (via Kan extensions) to functors $D^{!,enh}$ and $D^{*,enh}$ from the categories of indschemes under correspondences where the "right" (resp. "left") map is finitely presented.

For $f : S \to T$ finitely presented we have the corresponding functors $f_{*,!-dR} : D^!(S) \to D^!(T)$ and $f^{\text{¡}} : D^*(T) \to D^*(S)$. The analogue of Proposition 16.15.2 holds as well.

Remark 16.44.1. We emphasize that by *schematic*, we mean schematic in the sense of classical (i.e., non-derived) algebraic geometry, which is a more forgiving notion than that of derived algebraic geometry. This is relevant, say, for considering the embedding of 0 inside of the indscheme associated with an infinite-dimensional k-vector space, which is a classically schematic embedding but not a DG schematic embedding.

16.45. **Reasonable indschemes.** The following definition is taken from [BD] §7.

Definition 16.45.1. A subscheme $S \subseteq T$ is a *reasonable subscheme* of T if S is a quasi-compact quasi-separated closed subscheme such that, for every closed subscheme S' of T containing S, the closed embedding $S \hookrightarrow S'$ is finitely presented.

T is a *reasonable* indscheme if T is the colimit of its reasonable subschemes.

Example 16.45.2. Every quasi-compact quasi-separated scheme is reasonable when regarded as an indscheme.

Example 16.45.3. Every indscheme of ind-finite type is reasonable.

Example 16.45.4. For an ind-pro finite set T, considered as an indscheme in the obvious way, a subset $S \subseteq T$ is reasonable if and only if it is compact and open in the ind-pro topology.

Terminology 16.45.5. Because of Example 16.45.4, we sometimes refer to reasonable subschemes as compact open subschemes. We especially use this terminology in the group setting, where we speak of compact open subgroups, meaning group subschemes that are reasonable as subschemes.

Lemma 16.45.6. *Suppose T is a reasonable indscheme and $f : S \to T$ a finitely presented morphism of indschemes. Then S is a reasonable indscheme, and for every reasonable subscheme $T' \subseteq T$, $f^{-1}(T') \subseteq S$ is a reasonable subscheme.*

Proof. Fix a reasonable subscheme $T' \subseteq T$. It suffices to show that $f^{-1}(T') \subseteq S$ is a reasonable subscheme.

First, suppose that $T' \subseteq T'' \subseteq T$ is a reasonable subscheme of T. We will show that $f^{-1}(T') \hookrightarrow f^{-1}(T'')$ is a finitely presented closed embedding.

Note that $f^{-1}(T') \to T'$ is finitely presented because f is, and similarly for T''. Moreover, $f^{-1}(T') \to T''$ is finitely presented, since it factors as $f^{-1}(T') \to T' \to T''$ with the latter morphism being finitely presented because T' is reasonable.

Therefore, since $f^{-1}(T') \to f^{-1}(T'')$ sits in the diagram:

$$f^{-1}(T') \to f^{-1}(T'') \to T''$$

with the composite morphism and the right morphism finitely presented, the morphism $f^{-1}(T') \to f^{-1}(T'')$ is finitely presented as well (the relevant "two out of three" principle appears in [Gro67] Proposition 1.6.2).

To see that this suffices: suppose that $f^{-1}(T') \subseteq S' \subseteq T$ is closed subscheme. We can take T'' as above we $S' \to T$ factoring through T''. Therefore, we have:

$$f^{-1}(T') \subseteq S' \subseteq f^{-1}(T'').$$

That $f^{-1}(T') \to f^{-1}(T'')$ is finite presentation means that the ideal sheaf of $f^{-1}(T')$ is finitely generated over the structure sheaf of $f^{-1}(T'')$. Therefore, we see that it is finitely generated over the structure sheaf of S' as well, so that our closed embedding $f^{-1}(T') \subseteq S'$ is itself finitely presented.

□

16.46. The key feature of reasonable indschemes is the following. Suppose $T = \operatorname{colim}_{i \in \mathcal{I}} T_i$ as in the definition.

Then every $\alpha : T_i \to T_j$ is a finitely presented closed embedding and therefore $\alpha^! : D^!(T_j) \to D^!(T_i)$ admits the left adjoint $\alpha_{*,!-dR}$ and $\alpha_{*,dR} : D^*(T_i) \to D^*(T_j)$ admits the right adjoint $\alpha^{¡}$.

Therefore, we have:

$$D^!(T) = \operatorname*{colim}_{i \in \mathcal{I}} D^!(T_i)$$
$$D^*(T) = \lim_{i \in \mathcal{I}^{op}} D^*(T_i) \tag{16.46.1}$$

where on the left we use functors $\alpha_{*,!-dR}$ and on the right we use functors $\alpha^{¡}$.

We deduce that for T and S reasonable indschemes we have canonical equivalences:

$$D^!(T \times S) = D^!(T) \otimes D^!(S). \tag{16.46.2}$$

16.47. **Descent.** We say a morphism $f : T \to S$ of indschemes is an h-covering if its base-change by any affine scheme is an h-covering.

Proposition 16.47.1. *Let $f : S \to T$ be an h-covering of indschemes. Then the canonical functor:*

$$D^!(T) \to \lim_{[n] \in \mathbf{\Delta}} D^!(\mathrm{Cech}^n(S/T))$$

given by !*-pullback is an equivalence.*

Proof. This is obvious from Proposition 16.18.1: it just amounts to commuting limits with limits.

□

Similarly, we have the following result under more restrictive hypotheses.

Proposition 16.47.2. *Let $f : S \to T$ be an h-covering of reasonable indschemes. Then the canonical functor:*

$$D^*(T) \to \lim_{[n] \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T))$$

given by ¡-pullback is an equivalence.

Proof. As above, this follows from Proposition 16.27.1 by commuting limits with limits, using the presentation (16.46.1) of D^*.

$\square$

16.48. **Equivariant setting.** We now render the material of §16.19 and §16.28 to the indscheme setting.

Suppose that S is an indscheme and $\mathcal{G} \to S$ is a group indscheme over S.

Suppose P is an indscheme with a morphism $P \to S$ and an action of $\mathcal{G}$. We define the equivariant derived category $D^!(P)^{\mathcal{G}}$ as the limit of the diagram formed using (16.19.1):

$$D^!(P)^{\mathcal{G}} := \lim \left(D^!(P) \rightrightarrows D^!(\mathcal{G} \underset{S}{\times} P) \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \ D^!(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \ \ldots . \right)$$

Similarly, we define the coequivariant derived category by (16.28.1).

Now suppose that $\mathcal{P}_{\mathcal{G}} \to S$ is an indscheme with a $\mathcal{G}$-action as above and that $\mathcal{P}_{\mathcal{G}}$ is a $\mathcal{G}$-torsor in the sense that, for every closed subscheme S' of S, the fiber product $\mathcal{P}_{\mathcal{G}} \times_S S'$ is a $\mathcal{G} \times_S S'$-torsor in the sense of §16.19: after an fppf base-change in S', $\mathcal{P}_{\mathcal{G}} \times_S S' \to S'$ is $\mathcal{G}$-equivariantly isomorphic to $\mathcal{G}$.

Proposition 16.48.1. *The pullback functor:*

$$D^!(S) \to D^!(\mathcal{P}_{\mathcal{G}})^{\mathcal{G}}$$

is an equivalence.

The pushforward functor:

$$D^*(\mathcal{P}_{\mathcal{G}})_{\mathcal{G}} \to D^*(S)$$

is an equivalence if S is reasonable, and $\mathcal{G}$ is a union $\mathcal{G} = \cup \mathcal{G}_i$ where the $\mathcal{G}_i$ are closed group indschemes in $\mathcal{G}$ with the property that $\mathcal{G}_i \times_S S' \to \mathcal{G} \times_S S'$ is a reasonable subscheme for every reasonable subscheme $S' \subseteq S$.

Proof. For the first functor, we commute limits with limits to dévissage to the case where S is a quasi-compact quasi-separated scheme. Then the result follows as in Proposition 16.19.2: by Proposition 16.47.1 we reduce to the case of a trivial $\mathcal{G}$-bundle where it follows by using split simplicial objects.

The second functor is analyzed similarly: commuting colimits with colimits, we reduce to the case where S is a quasi-compact quasi-separated scheme.

Note that $\mathcal{P}_{\mathcal{G}}$ must be induced as a torsor from some $\mathcal{G}_i$-torsor for some i_0. Therefore, $\mathcal{P}_{\mathcal{G}}$ is reasonable: it is a union of the induced $\mathcal{G}_i$-torsors for $i \to i_0$, and these are obviously reasonable subschemes. Therefore, we can apply Proposition 16.47.2 to again reduce to the case of a trivial torsor.

$\square$

Remark 16.48.2. When our indschemes are reasonable, Example 16.19.1 translates verbatim to the present setting by using (16.46.2).

Remark 16.48.3. We will sometimes use the notational convention of Remark 16.19.3 in the above setting as well.

16.49. **Placidity.** We now give an indscheme analogue of the notion of placidity.

Definition 16.49.1. We say that $T \in \mathsf{IndSch}$ is a *placid indscheme* if T is reasonable and every reasonable subscheme of T is placid.

Remark 16.49.2. By Remark 16.29.5, we see that T is placid if and only if we can write $T = \operatorname{colim}_{i \in \mathcal{I}} T_i$ as in the definition of indscheme so that each T_i is placid and a reasonable subscheme of T.

Remark 16.49.3. By (16.46.1) and §16.30, for T placid the categories $D^!(T)$ and $D^*(T)$ are compactly generated and canonically dual.

The following is the indscheme analogue of Example 16.29.3.

Example 16.49.4. Suppose that S is a placid indscheme and $\mathcal{G} \to S$ is a group indscheme over S. Suppose moreover that for every closed subscheme S' of S the fiber product $\mathcal{G} \times_S S' \to S'$ is a group scheme that can be written as a projective limit under smooth maps of group schemes $\mathcal{G}_i$ smooth and affine over S'. Then $\mathcal{G}$ is a placid indscheme.

More generally, if $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor over S in the sense of §16.48 then $\mathcal{P}_{\mathcal{G}}$ is a placid indscheme. Indeed, we reduce to showing that if S as above is actually a placid scheme, then $\mathcal{P}_{\mathcal{G}} \to S$ is a placid morphism. But $\mathcal{P}_{\mathcal{G}}$ is the projective limit of the induced $\mathcal{G}_i$-torsors, giving the result.

16.50. **Fiber products.** We digress somewhat to give the following technical result that we will need in the body of the text.

Proposition 16.50.1. *Let $S_1 \to S_2$ and $T \to S_2$ be morphisms of indschemes.*

(1) If S_1 and S_2 are finite type schemes, then the canonical morphisms:

$$D^!(T) \underset{D(S_2)}{\otimes} D(S_1) \to D^!(T \underset{S_2}{\times} S_1)$$

$$D^*(T) \underset{D(S_2)}{\otimes} D(S_1) \to D^*(T \underset{S_2}{\times} S_1)$$

of ! and ¡-pullback respectively are equivalences.

(2) If S_1 is a placid indscheme and S_2 is a finite type scheme and T is an arbitrary indscheme, then:

$$D^!(T) \underset{D(S_2)}{\otimes} D^!(S_1) \to D^!(T \underset{S_2}{\times} S_1)$$

is an equivalence.

We will deduce Proposition 16.50.1 from the following two lemmas from the finite-dimensional setting.

Lemma 16.50.2. *Let $S_1 \to S_2$ and $T \to S_2$ morphisms of finite type schemes, the canonical morphism:*

$$D(T) \underset{D(S_2)}{\otimes} D(S_1) \to D(T \underset{S_2}{\times} S_1)$$

is an equivalence.

This is well-known, though we do not know a reference in the literature for this fact. However, it follows from Corollary 19.9.3 and Theorem 19.8.1.

Lemma 16.50.3. *For $f : S \to T$ a morphism of finite type schemes, $D(S)$ is dualizable as a $D(T)$-module category.*

This result follows immediately from Theorem 19.18.1, but we give a more direct proof below.

Proof. We will show that $D(S)$ is self-dual as a $D(T)$-module category.

Let Δ_f denote the diagonal embedding $S \to S \times_T S$.

We have the evaluation:

$$D(S) \underset{D(T)}{\otimes} D(S) \simeq D(S \underset{T}{\times} S) \xrightarrow{\Delta_f^!} D(S) \xrightarrow{f_{*,dR}} D(T)$$

and coevaluation:

$$D(T) \xrightarrow{f^!} D(S) \xrightarrow{\Delta_{f,*,dR}} D(S \underset{T}{\times} S) \simeq D(S) \underset{D(T)}{\otimes} D(S).$$

One readily checks by base-change that these define a duality datum as required.

$\square$

Proof of Proposition 16.50.1. For (1): the category $D(S_1)$ is dualizable as a $D(S_2)$-module category. Therefore, tensoring over $D(S_2)$ with $D(S_1)$ commutes with limits of categories. Applying the definition of $D^!$, the result then immediately follows from the finite type case.

Similarly, to prove (2) it suffices to show that $D^!(S_1)$ is dualizable as a $D(S_2)$-module category. This follows from the finite type case combined with Proposition 19.12.4 (3) and (16.46.1).

$\square$

16.51. **Dimension theories.** Let T be a placid indscheme. We use the notation of §16.33 here.

Definition 16.51.1. A *dimension theory* $\tau = \tau^T$ on T is a rule that assigns to every reasonable subscheme S of T a locally constant function:

$$\tau_S : S \to \mathbb{Z}$$

such that for any pair of reasonable subschemes $S' \subseteq S \subseteq T$ we have:

$$\tau_{S'} = \tau_S|_{S'} + \dim_{S'/S}. \tag{16.51.1}$$

Example 16.51.2. By Remark 16.33.3, every placid scheme T carries a canonical dimension theory normalized by the condition that $\dim_T$ be identically zero.

Example 16.51.3. Let T be an indscheme of ind-finite type. Then a reasonable subscheme of T is just a closed finite type subscheme S, and the rule $\tau_S := \dim_S$ is a dimension theory on T.

Remark 16.51.4. If $T = \cup_i S_i$ is written as a union of reasonable subschemes, it suffices to define the τ_{S_i} satisfying the compatibility (16.51.1). Indeed, this again follows from Remark 16.33.3.

Example 16.51.5. By Remark 16.51.4, the product $T_1 \times T_2$ of indschemes T_i equipped with dimension theories τ^{T_i} inherits a canonical dimension theory $\tau^{T_1 \times T_2}$ such that, for every pair $S_i \subseteq T_i$, $i = 1, 2$ of reasonable subschemes, we have:

$$\tau_{S_1 \times S_2}^{T_1 \times T_2} = p_1^*(\tau_{S_1}^{T_1}) + p_2^*(\tau_{S_2}^{T_2})$$

with p_i^* denoting the restriction of a function along the projection.

Remark 16.51.6. Dimension theories are étale local.

Remark 16.51.7. For T a group indscheme, the choice of dimension theory may be seen as analogous to the choice of a Haar measure in the p-adic setting.

Remark 16.51.8. See [Dri06] for relevant material on dimension theories. In particular, questions of existence (and non-existence) are treated in some detail.

16.52. We now give something of a classification of the set of dimension theories.

Definition 16.52.1. A *locally constant function* $T \to \mathbb{Z}$ on an indscheme T is a morphism of indschemes $T \to \mathbb{Z} = \coprod_{n \in \mathbb{Z}} \operatorname{Spec}(k)$.

Remark 16.52.2. For $T = \operatorname{colim} T_i$, a locally constant function on T is equivalent to a compatible system of locally constant functions on the T_i. As in Remark 16.15.4, we can make sense of $\pi_0(T)$ as an ind-profinite set, and a locally constant function on T is equivalent to a continuous function $\pi_0(T) \to \mathbb{Z}$, with π_0 equipped with its natural topology as an ind-profinite set.

Clearly locally constant functions form an abelian group under addition. Moreover, they obviously act on the set of dimension theories on T: given $d : T \to \mathbb{Z}$ and τ a dimension theory on T, we obtain a new dimension theory $d + \tau$ with $(d + \tau)_S = d|_S + \tau_S$ for every reasonable subscheme S of T.

Proposition 16.52.3. *Suppose that S is a placid indscheme that admits a dimension theory. Then the set of dimension theories for S is a torsor for the set of locally constant functions $S \to \mathbb{Z}$, i.e., the above action of locally constant functions on dimension theories is a simply transitive action.*

Proof. The difference between two dimension theories obviously defines a locally constant function on S.

$\square$

16.53. We will repeatedly use the following construction of dimension theories.

Definition 16.53.1. A morphism $f : T \to S$ of placid indschemes is *healthy* if there exists a reasonable subscheme $S' \subseteq S$ such that:

(1) The inverse image of any closed subscheme $S' \subseteq S'' \subseteq S$ is a reasonable subscheme of T.

(2) For every closed subscheme $S' \subseteq S'' \subseteq S$, we have:

$$\dim_{T'/T''} = f'^{,*}(\dim_{S'/S''})$$

with $f' : T' \to S'$ the fiber product of f along S' and T'' the fiber along S''.

We say a subscheme $S' \subseteq S$ is f-healthy if it is reasonable and satisfies the above conditions (so f is healthy if and only if there exists an f-healthy subscheme of S).

Example 16.53.2. Every morphism $f : T \to S$ of placid schemes is healthy: S itself is f-healthy.

Counterexample 16.53.3. For $n \geqslant 0$, let S_n be the union of a line, a plane, up to an affine n-space all glued together along 0. Let $S = \operatorname{colim} S_n$. Let T_n be the union of n (ordered) lines glued along 0, mapping to S_n by embedding the rth irreducible component into $\mathbb{A}^r$ as a line into a vector space. Let $T = \operatorname{colim}_n T_n$. Then the resulting map $T \to S$ is not healthy.

Example 16.53.4. In §16.58, we will give a definition of placid morphism of placid indschemes such that every placid morphism is healthy.

Remark 16.53.5. Any reasonable subscheme containing an f-healthy subscheme is itself f-healthy.

In particular, we see that given two choices S'_1, S'_2 of f-healthy subschemes of S, there is always a third S'_3 containing both.

Our key use of this definition is the following construction.

Construction 16.53.6. For $f : T \to S$ a healthy morphism of placid indschemes, any dimension theory τ^S on S induces a unique dimension theory τ^T on T such that for any f-healthy reasonable subscheme $S' \subseteq S$, we have $\tau^T_{T'} = f'^{,*}(\tau^S_{S'})$ for $f' : T' \to S'$ the base-change of f along $S' \hookrightarrow S$.

Indeed, that this construction can be performed follows immediately from Remarks 16.51.4 and 16.53.5.

Remark 16.53.7. Healthy morphisms are obviously preserved under compositions, and Construction 16.53.6 is obviously compatible with compositions.

16.54. As §16.53 generalizes Example 16.51.2, we now generalize Example 16.51.3.

We say a morphism $f : T \to S$ of reasonable indschemes is *ind-finitely presented* if $T = \operatorname{colim} T_i$ with each $T_i \to T$ a reasonable subscheme such that $T_i \to S$ factors through a reasonable subscheme S_i of S with $T_i \to S_i$ finite presentation.

We claim under this hypothesis that T inherits a canonical dimension theory τ^T from a dimension theory τ^S of S.

Indeed, for $T' \subseteq T$ a reasonable subscheme, the morphism $T' \to S$ factors through some reasonable subscheme $S' \subseteq S$, and $f' : T' \to S'$ is finite presentation by assumption. We take:

$$\tau^T_{T'} := \dim_{T'/S'} + f'^{,*}(\tau^S_{S'}).$$

To simultaneously show that τ^T is well-defined and actually defines a dimension theory, take $T' \xhookrightarrow{i_1} T'' \subseteq T$ reasonable subschemes mapping via f' and f'' to reasonable subschemes $S' \xhookrightarrow{i_2} S'' \subseteq S$ respectively, and compute:

$$\tau^T_{T'} - i_1^*(\tau^T_{T''}) := \dim_{T'/S'} - i_1^*(\dim_{T''/S''}) + f'^{,*}(\tau^S_{S'}) - f'^{,*} i_2^*(\tau^S_{S''}) =$$

$$= -f'^{,*}(\dim_{S'/S''}) + \dim_{T'/T''} + f'^{,*}(\dim_{S'/S''}) = \dim_{T'/T''}$$

as desired, where we have used the expansions:

$$\dim_{T'/S'} = \dim_{T'/S''} - f'^{,*}(\dim_{S'/S''})$$

$$i_1^*(\dim_{T''/S''}) = \dim_{T'/S''} - \dim_{T'/T''}$$

of (16.33.1).

Example 16.54.1. If T is a reasonable subscheme of a placid indscheme S, then the embedding $T \hookrightarrow S$ satisfies the hypotheses of this section. If τ^S is a dimension theory on S, the induced dimension theory τ^T on T constructed above is the "obvious" one, which to a reasonable subscheme $T' \subseteq T$ assigns the function $\tau^T_{T'} := \tau^S_{T'}$.

Warning 16.54.2. If $f : T \to S$ is a finitely presented morphism of placid schemes, the pullback constructed above of the dimension theory τ^S given in Example 16.51.2 is *not* (generally) the dimension theory on T constructed in Example 16.51.2: they differ by $\dim_{T/S}$.

16.55. **Renormalization.** Let T be a placid indscheme and let τ be a dimension theory on T. We will define the "τ-renormalized dualizing sheaf" $\omega^\tau_T \in D^*(T)$ below.

Let $i : S \hookrightarrow T$ be a reasonable subscheme. We formally define:

$$\text{“}i^{\textbf{!}}(\omega^\tau_T)\text{”} := \omega^{ren}_S[2\tau_S] \in D^*(S).$$

Suppose that for S as above $\iota : S' \to S$ is a reasonable subscheme (equivalently: of S or of T, or equivalently ι is a finitely presented closed embedding). Then we have canonical isomorphisms:

$$\iota^{\textbf{!}}(\text{“}i^{\textbf{!}}(\omega^\tau_T)\text{”}) = \iota^{\textbf{!}}(\omega^{ren}_S)[2\tau_S] = \iota^{!,ren}(\omega^{ren}_S)[2\cdot(\tau_S + \dim_{S'/S})] = (\omega^{ren}_{S'})[2\cdot(\tau_S + \dim_{S'/S})] =: \text{“}(i \circ \iota)^{\textbf{!}}(\omega^\tau_T)\text{”}$$

where the second equality is Proposition 16.39.1 and the third equality is (16.36.1).

These identifications are readily made homotopy compatible and therefore define ω_T^τ in $D^*(T)$ so that $\iota^{\natural}(\omega_T^\tau) =$ "$\iota^{\natural}(\omega_T^\tau)$" for all $\iota : S \hookrightarrow T$ as above.

16.56. Let T and τ be as in §16.55.

Let $\eta_T^\tau : D^!(T) \to D^*(T)$ denote the functor of action on ω_T^τ. We immediately deduce from Proposition 16.35.1 that η_T^τ is an equivalence.

16.57. Let $f : T \to S$ a morphism of placid indschemes equipped with dimension theories τ^T and τ^S.

Then as in §16.36 we obtain functors $f_{*,\tau} : D^!(T) \to D^!(S)$ and $f^{!,\tau} : D^!(S) \to D^!(T)$ so that we have the commuting diagram:

$$\begin{array}{ccc} D^!(T) & \xrightarrow{f_{*,\tau}} & D^!(S) \\ \simeq \downarrow \eta_T^{\tau^T} & & \simeq \downarrow \eta_S^{\tau^S} \\ D^*(T) & \xrightarrow{f_{*,dR}} & D^*(S) \end{array} \qquad \begin{array}{ccc} D^*(S) & \xrightarrow{f^{!,\tau}} & D^*(T) \\ \simeq \downarrow \eta_S^{\tau^S} & & \simeq \downarrow \eta_T^{\tau^T} \\ D^!(S) & \xrightarrow{f^!} & D^!(T). \end{array}$$

Example 16.57.1. If $f : T \to S$ is a map of placid schemes, each equipped with their canonical dimension theories (see Example 16.51.2), then the functors constructed above are the renormalized functors of §16.36.

Notation 16.57.2. In light of Example 16.57.1, when the relative dimension theory τ is implicit we denote the functors $f_{\tau,ren}$ and $f^{!,ren}$ above simply by $f_{*,ren}$ and $f^{!,ren}$.

Fixing a map $f : T \to S$ of placid indschemes, we obtain a pullback map for locally constant functions and therefore an induced diagonal action of locally constant functions on S on the set of pairs (τ^T, τ^S) of dimension theories for T and S:

$$\Big(d : S \to \mathbb{Z}, (\tau^T, \tau^S)\Big) \mapsto (\tau^T + d \circ f, \tau^S + d).$$

Definition 16.57.3. A *relative dimension theory* for T and S is an equivalence class of pairs (τ^T, τ^S) of dimension theories for T and for S modulo the above action of locally constant functions on S.

Clearly the functors $f^{!,ren}$ and $f_{*,ren}$ only depend on the relative dimension theory defined by the pair (τ^T, τ^S).

Example 16.57.4. Let $f : T \to S$ be an ind-finitely presented morphism of placid indschemes with S equipped with dimension theory. By §16.54, we obtain a dimension theory on T and therefore a relative dimension theory for f.

As in Examples 16.36.1 and 16.36.4, the functors $f_{*,ren}$ and $f^{!,ren}$ canonically identify with $f_{*,!-dR}$ and $f^{¡}$ respectively.[41]

16.58. Next, we extend the notion of placid morphism from §16.37 to the indscheme framework.

Definition 16.58.1. A morphism $f : T \to S$ of placid indschemes is *placid* if there exists a reasonable subscheme $S' \subseteq S$ such that:

(1) The inverse image of any closed subscheme $S' \subseteq S'' \subseteq S$ is a reasonable subscheme of T.
(2) For every closed subscheme $S' \subseteq S'' \subseteq S$, the morphism $T'' := S'' \times_S T \to S''$ is placid.

Remark 16.58.2. By Corollary 16.37.7, we immediately see that any placid morphism is healthy.

Example 16.58.3. If f is smooth and surjective on geometric points (in particular schematic and finitely presented), then f is placid.

[41]Unlike Example 16.36.1, there are no cohomological shifts in this formula. There is no real discrepancy because of Warning 16.54.2.

Example 16.58.4. Suppose that S is a placid indscheme and $\mathcal{G} \to S$ is a group indscheme satisfying the hypotheses of Example 16.49.4. Suppose $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor on S. Then $\mathcal{P}_{\mathcal{G}} \to S$ is placid. In particular, this morphism is healthy. Indeed, this follows by Example 16.49.4.

16.59. We have the following indschematic version of Proposition 16.38.1.

Proposition 16.59.1. *Let $f : T \to S$ be placid and suppose that S is equipped with a dimension theory. By Construction 16.53.6, this choice induces a dimension theory on T.*

(1) The functors:

$$f_{*,dR} : D^*(T) \to D^*(S)$$

$$f_{*,ren} : D^!(T) \to D^!(S)$$

admit left adjoints. Moreover, these left adjoints are canonically identified with $f^{!,ren}$ and $f^!$ respectively.

(2) Suppose that we are given a Cartesian diagram:

$$\begin{tikzcd} T' \arrow[r, "\varphi"] \arrow[d, "\psi"] & S' \arrow[d, "g"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

of placid indschemes with f placid and g finitely presented. Then φ is also placid, and the natural transformations:

$$f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$$

$$f^! g_{*,ren} \to \psi_{*,ren} \varphi^!$$

are equivalences. Here we have equipped S' and T' with the dimension theories of §16.54 using the finitely presented maps g and ψ.

Proof. It suffices to prove each of these statements in the $D^!$-setting.

Then (1) then follows immediately Proposition 16.38.1 (say, by applying a simplified version of Lemma 16.40.2). So it remains to show (2).

Let S_0 be a reasonable subscheme of S satisfying the hypotheses of the definition of placid morphism for f. Then combining Lemmas 16.37.6 and 16.45.6., we find that its pullback to S' satisfies the same conditions for φ. In particular, we see that φ is placid.

We form the commutative cube:

$$\begin{tikzcd}
T_0' \arrow[rr, "\varphi_0"] \arrow[dr, "\iota'"] \arrow[dd, "\psi_0"'] & & S_0' \arrow[dr, "i'"] \arrow[dd, "g_0"] & \\
& T' \arrow[rr] \arrow[dd] & & S' \arrow[dd] \\
T_0 \arrow[rr, "f_0"] \arrow[dr, "\iota"] & & S_0 \arrow[dr, "i"] & \\
& T \arrow[rr] & & S
\end{tikzcd}$$

where all faces are taken to be Cartesian squares. We equip these new schemes with the dimension theories obtained using Example 16.54.1.

Note that the dimension theories on the back square are not (necessarily) the canonical ones on placid schemes from Example 16.51.2.

Still, the relative dimension theories of T_0/S_0 and T_0'/S_0' are the same, so renormalized functors for these dimension theories coincide with those of §16.36.

Moreover, the dimension theories for S_0'/S_0 differs from the "canonical" one by $\dim_{S_0'/S_0}$, and similarly for T_0'/T_0. Note that this error term $\dim_{S_0'/S_0}$ pulls back to T_0' as $\dim_{T_0'/T_0}$ by Corollary 16.37.7.

We will use the notation e.g. $g_{0,*,ren}$ here for the renormalized functor corresponding to our given dimension theory, therefore differing by cohomological shifts from the so-named functor in §16.36.

In this notation, we see from the above discussion that we can apply Proposition 16.38.1 to deduce:

$$f_0^! g_{0,*,ren} \xrightarrow{\simeq} \psi_{0,*,ren}\varphi_0^!.$$

Because $D^!(S')$ is generated under colimits by D-modules of the form $i'_{*,!-dR}(\mathcal{F}) = i'_{*,ren}(\mathcal{F})$ as we increase S_0, it suffices to show that the natural transformation:

$$f^! g_{*,ren} i'_{*,ren} \to \psi_{*,ren}\varphi^! i'_{*,ren}$$

is an equivalence.

Similarly, since T is a union of the schemes T_0 as S_0 varies, it suffices to show that the natural transformation:

$$\iota^! f^! g_{*,ren} i'_{*,ren} \to \iota^! \psi_{*,ren}\varphi^! i'_{*,ren}$$

is an equivalence.

Now we compute:

$$\iota^! f^! g_{*,ren} i'_{*,ren} = f_0^! i^! i_{*,ren} g_{0,*,ren} = f_0^! g_{0,*,ren} \xrightarrow{\simeq} \psi_{0,*,ren}\varphi_0^! = \iota^! \iota_{*,ren}\psi_{0,*,ren}\varphi_0^! =$$
$$\iota^! \psi_{*,ren}\iota'_{*,ren}\varphi_0^! i'^{,!} i'_{*,ren} = \iota^! \psi_{*,ren}\iota'_{*,ren}\iota'^{,!}\varphi^! i'_{*,ren} = \iota^! \psi_{*,ren}\varphi^! i'_{*,ren}$$

as desired.

$\square$

16.60. **Holonomic D-modules.** For T an indscheme, we define $D^!_{hol}(S)$ and $D^*_{hol}(S)$ by Kan extension, as in the definition of $D^!$ and D^*.

We have canonical forgetful functors:

$$D^!_{hol}(S) \to D^!(S) \text{ and } D^*_{hol}(S) \to D^*(S)$$

and compatible upper-! and lower-* functoriality, respectively. For S reasonable (resp. placid), $D^*_{hol}(S) \to D^*(S)$ (resp. $D^!_{hol}(S) \to D^!_{hol}(S)$) is fully-faithful.

Definition 16.60.1. A morphism $f : S \to T$ of reasonable indschemes is a *reasonable morphism* if there exists cofinal system $T = \cup T_i$ of reasonable subschemes such that $f^{-1}(T_i)$ is a reasonable subscheme in S (in particular: f is schematic).

Proposition 16.60.2. *If $f : S \to T$ is a reasonable morphism of reasonable indschemes, then the partially-defined left adjoint $f^{*,dR}$ to $f_{*,dR}$ is defined on holonomic objects in $D^*(T)$.*

Similarly, if f is a morphism of ind-finite presentation of placid indschemes, then the partially-defined left adjoint $f_!$ to $f^! : D^!(T) \to D^!(S)$ is defined on holonomic objects.

Proof. Follows from the combination of Proposition 16.40.1 and Lemma 16.40.2 by the same argument as in Proposition 16.40.1.

□

We have the following counterparts to Proposition 16.40.3 and its Corollary 16.40.4, proved by the same arguments.

Proposition 16.60.3. *For S a placid indscheme with a dimension theory τ, η^τ_S identifies $D^!_{hol}(S)$ with $D^*_{hol}(S)$.*

Corollary 16.60.4. *For S and T placid indschemes with a dimension theories and $f : S \to T$ a morphism, $f_{*,ren}$ and $f^{!,ren}$ preserve holonomic objects in $D^!$ and D^* respectively.*

17. Iwahori vs. semi-infinite Borel

17.1. Define $\mathsf{Whit}^{\frac{\infty}{2}}$ as the $N(K)T(O)$-coinvariants of the Whittaker invariants of $D^!(G(K))$, these notions being introduced in §16 and §6: we emphasize that we work over a single point here.

The purpose of §17-18 is to show that this category coincides with the category $D(\mathrm{Fl}^{\mathrm{aff}}_G)^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ considered in [AB09].

There are two comparisons to be made: in the present section, we treat the $N(K)T(O)$ side, and in §18, we treat the Whittaker side.

17.2. The main result of this section is the following.

Theorem 17.2.1. *Let* $\mathcal{C}$ *be a category acted on by* $G(K)$.[42] *Then the functor:*

$$\mathcal{C}_I \xrightarrow{\mathrm{Nm}} \mathcal{C}_{B(O)} \to \mathcal{C}_{N(K)T(O)}$$

is an equivalence. Here Nm *is the* norm *map, which by definition corresponds to* Oblv *under the equivalences* $\mathcal{C}_I \simeq \mathcal{C}^I$ *and* $\mathcal{C}_{B(O)} \simeq \mathcal{C}^{B(O)}$.

Remark 17.2.2. Note that this result is borrowed from the theory of reductive p-adic groups: c.f. [Cas80] Proposition 2.4.

Corollary 17.2.3. *For* $\mathcal{C}$ *as above, the functor* $\mathcal{C}^{N(K)T(O)} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{B(O)} \xrightarrow{\mathrm{Av}_*} \mathcal{C}^I$ *is an equivalence.*

Proof that Theorem 17.2.1 implies Corollary 17.2.3. We have:

$$\mathrm{Hom}_{D^*(G(K))\text{–}\mathsf{mod}}(D^*(G(K))_{N(K)T(O)}, \mathcal{C}) \simeq \mathcal{C}^{N(K)T(O)}$$

and similarly for Iwahori invariants. Therefore, we deduce the result from Theorem 17.2.1 applied to the regular representation.

□

17.3. For every $\check{\lambda} \in \check{\Lambda}$, we use the notation:

$$I^{\check{\lambda}} := \mathrm{Ad}_{-\check{\lambda}(t)}(I) \subseteq G(K)$$

$$B(O)^{\check{\lambda}} := \mathrm{Ad}_{-\check{\lambda}(t)}(B(O)) \subseteq G(K)$$

where $t \in K$ is a uniformizer.

[42] I.e., a $D^!(G(K))$-comodule category in DGCat_{cont}, or equivalently, a $D^*(G(K))$-module category.

Remark 17.3.1. The normalization with $-\check{\lambda}(t)$ is so we can work with $\check{\lambda} \in \check{\Lambda}^+$ instead of $-\check{\Lambda}^+$.

17.4. The key fact we will use is the following one.

Lemma 17.4.1. *For $\mathcal{C}$ acted on by $G(K)$ and $\check{\lambda}, \check{\eta}$ coweights, the functor:*

$$\mathrm{Av}_*^{I^{\check{\mu}}} : \mathcal{C}^{I^{\check{\lambda}}} \to \mathcal{C}^{I^{\check{\mu}}}$$

(properly defined by forgetting to $I^{\check{\lambda}} \cap I^{\check{\mu}}$ and then averaging) is an equivalence.

Proof. Up to translations, this follows from the invertibility of Mirkovic-Wakimoto sheaves in the Iwahori-Hecke algebra (see [AB09] Lemma 8).

□

Remark 17.4.2. We denote the inverse functor by $\mathrm{Av}_!^{I^{\check{\lambda}}}$, since it is evidently given by (forgetting down to $I^{\check{\lambda}} \cap I^{\check{\mu}}$ and then) applying such a !-averaging.

17.5. Before preceding, we record a technical general lemma we will need. The reader may prefer to skip this section and refer back to it as necessary.

Suppose that $\mathcal{I}$ is a filtered category, and suppose we are given diagrams:

$$i \mapsto \mathcal{C}_i \in \mathsf{DGCat}_{cont}$$

$$i \mapsto \mathcal{D}_i \in \mathsf{DGCat}_{cont}.$$

Let $\mathcal{C}$ (resp. $\mathcal{D}$) denote the colimit category in DGCat_{cont}. For $\alpha : i \to j \in \mathcal{I}$, let ψ_α (resp. φ_α) denote the structure functor $\mathcal{C}_i \to \mathcal{C}_j$ (resp. $\mathcal{D}_i \to \mathcal{D}_j$). We let $\psi_i : \mathcal{C}_i \to \mathcal{C}$ and $\varphi_i : \mathcal{D}_i \to \mathcal{D}$ denote the structure functors.

Suppose we are given compatible functors $F_i : \mathcal{C}_i \to \mathcal{D}_i$, and suppose that each functor F_i admits a continuous right adjoint G_i. We do not assume that the functors G_i are compatible with the structure maps (though they are automatically *lax* compatible).

Let F denote the induced functor $F : \mathcal{C} \to sD$.

Construction 17.5.1. For every i, define the continuous functor "$G\varphi_i$" $: \mathcal{D}_i \to \mathcal{C}$ by the formula:[43]

$$\text{“}G\varphi_i\text{”} := \operatorname*{colim}_{\alpha:i\to j} \psi_j G_j \varphi_\alpha.$$

For $\beta : k \to i$, observe that we have:

$$\text{“}G\varphi_i\text{”} \circ \varphi_\beta = \operatorname*{colim}_{\alpha:i\to j} \psi_j G_j \varphi_\alpha \varphi_\beta = \operatorname*{colim}_{\Upsilon:k\to j} \psi_j G_j \varphi_\Upsilon = \text{“}G\varphi_k\text{”}$$

where we use filteredness to deduce the second equality. There, we have a functor $G : \mathcal{D} \to \mathcal{C}$ characterized by the identities $G\varphi_i \simeq$ "$G\varphi_i$."

Lemma 17.5.2. *The functor G is the right adjoint to the functor F.*

Proof. We construct the unit and counit of the adjunction explicitly.

Let i be a fixed index. We have:

$$FG\varphi_i = \operatorname*{colim}_{\alpha:i\to j} F\psi_j G_j \varphi_\alpha = \operatorname*{colim}_{\alpha:i\to j} \varphi_j F_j G_j \varphi_\alpha \to \operatorname*{colim}_{\alpha:i\to j} \varphi_j \varphi_\alpha = \varphi_i.$$

These functors are compatible as we vary i, and therefore define a natural transformation:

$$FG \to \mathrm{id}_{\mathcal{D}}\,.$$

Fixing i again, we similarly obtain:

$$\psi_i = \operatorname*{colim}_{\alpha:i\to j} \psi_j \psi_\alpha \to \operatorname*{colim}_{\alpha:i\to j} \psi_j G_j F_j \psi_\alpha = \operatorname*{colim}_{\alpha:i\to j} \psi_j G_j \varphi_\alpha F_i = G\varphi_i F_i = GF\psi_i$$

and then by passing to the limit, we obtain the natural transformation:

$$\mathrm{id}_{\mathcal{C}} \to GF.$$

[43] Note that for maps $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ of indices, we have the map $\psi_j G_j \varphi_\alpha = \psi_k \psi_\beta G_j \varphi_\alpha \to \psi_k G_k \varphi_{\beta\alpha} = \psi_k G_k \varphi_\beta \circ \varphi_\alpha$ given by the base-change map $\psi_\beta G_j \to G_k \varphi_\beta$, meaning that the arrows go in the correct direction in our colimit diagram.

One easily finds that these natural transformations define the counit and unit of an adjunction.

$\square$

Corollary 17.5.3. *Suppose that $\mathcal{I}$ is a filtered as above and $i \mapsto \mathcal{D}_i \in \mathsf{DGCat}_{cont}$ is a diagram with structure maps denoted by φ as above.*

Suppose i_0 is a fixed index in $\mathcal{I}$ and we are given $X_{i_0} \in \mathcal{D}_{i_0}$ such that, for every $\alpha : i_0 \to j$, the functor $\mathcal{D}_{i_0} \to \mathcal{D}_j$ sends X_{i_0} to a compact object $\varphi_\alpha(X_{i_0})$ in $\mathcal{D}_j$.

Then $\varphi_{i_0}(X)$ is compact in $\mathcal{D} = \operatorname{colim}_i \mathcal{D}_i$. Moreover, for every $\alpha : i_0 \to j$, the resulting continuous functor:

$$\mathcal{D}_j \to \mathcal{D} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\varphi_{i_0}(X),-)} \mathsf{Vect}$$

is computed explicitly by the formula:

$$Y \mapsto \operatorname*{colim}_{\beta:j\to k} \mathrm{Hom}_{\mathcal{D}_k}(\varphi_{\beta\alpha}(X_{i_0}), \varphi_\beta(Y)).$$

Proof. First, replacing $\mathcal{I}$ by $\mathcal{I}_{i_0/}$ by filteredness, we may assume i_0 is initial in $\mathcal{I}$. Then for any $j \in \mathcal{I}$, let $X_j \in \mathcal{D}_j$ obtained from functoriality from X_{i_0} using the structure functor $\mathcal{D}_{i_0} \to \mathcal{D}_j$. Let $X \in \mathcal{D}$ denote the object $\varphi_{i_0}(X_{i_0})$.

Then we apply Lemma 17.5.2 with $\mathcal{C}_j = \mathsf{Vect}$ for every j, with the compatible functors $\mathsf{Vect} \to \mathcal{D}_j$ given by $k \mapsto \varphi_{\alpha_j}(X_{i_0})$. Note that the corresponding functor $\mathsf{Vect} \to \mathcal{D}$ sends the trivial vector space k to X.

The lemma applies because each of these functors admits the continuous right adjoint $\mathrm{Hom}_{\mathcal{D}_j}(X_j, -)$ (or rather: we should take the Vect-enriched Hom here).

Then Lemma 17.5.2 ensures that the functor $\mathsf{Vect} \to \mathcal{D}$, $k \mapsto X$, admits a continuous right adjoint $\mathrm{Hom}_{\mathcal{D}}(X, -)$, and therefore X is compact. Then the explicit formula for the right adjoint given in Lemma 17.5.2 translates to the stated formula for $\mathrm{Hom}_{\mathcal{D}}(X, -)$.

$\square$

17.6. We now give the proof of Theorem 17.2.1.

Proof of Theorem 17.2.1. For every $\check{\lambda} \in \check{\Lambda}$, let $\mathfrak{p}^{\check{\lambda}}$ denote the projection functor $\mathcal{C}^{B(O)^{\check{\lambda}}} \simeq \mathcal{C}_{B(O)^{\check{\lambda}}} \to \mathcal{C}_{N(K)T(O)}$. For $\check{\lambda} = 0$, we use the notation $\mathfrak{p}$ instead.

Step 1. First, we show that $\mathcal{C}^I \to \mathcal{C}_{N(K)T(O)}$ generates the target under colimits.

Certainly $\mathcal{C}_{N(K)T(O)}$ is generated under colimits by the image of the functor $\mathfrak{p}$.

Note that:

$$\operatorname*{colim}_{\check{\lambda} \in \check{\Lambda}^+} \delta_{I^{\check{\lambda}} \cap I} \simeq \delta_{B(O)}.$$

Therefore, for $X \in \mathcal{C}^{B(O)}$, we have:

$$X \simeq \operatorname*{colim}_{\check{\lambda} \in \check{\Lambda}^+} \operatorname{Av}_*^{I^{\check{\lambda}} \cap I}(X)$$

and therefore $\mathcal{C}_{N(K)T(O)}$ is generated under colimits by the images of the functors $\mathcal{C}^{I^{\check{\lambda}} \cap I} \hookrightarrow \mathcal{C}^{B(O)} \xrightarrow{\mathfrak{p}} \mathcal{C}_{N(K)T(O)}$ as $\check{\lambda}$ ranges over $\check{\Lambda}^+$.

Now observe that for any $X \in \mathcal{C}^{B(O)}$, we have:

$$\mathfrak{p}(\operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)) \xrightarrow{\simeq} \mathfrak{p}(X)$$

by definition of the coinvariants. For $X \in \mathcal{C}^{I^{\check{\lambda}} \cap I}$, we then see that $\operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)$ is $I^{\check{\lambda}}$-equivariant, so that, by Lemma 17.4.1, we have:

$$\operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X) \xrightarrow{\simeq} \operatorname{Av}_*^{B(O)^{\check{\lambda}}} \operatorname{Av}_!^I \operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)$$

and therefore:

$$\mathfrak{p}(\operatorname{Av}_!^I \operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)) = \mathfrak{p}^{\check{\lambda}}(\operatorname{Av}_*^{B(O)^{\check{\lambda}}} \operatorname{Av}_!^I \operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)) \simeq \mathfrak{p}^{\check{\lambda}}(\operatorname{Av}_*^{B(O)^{\check{\lambda}}}(X)) = \mathfrak{p}(X).$$

Therefore, since the former term is $\mathfrak{p}$ applied to an Iwahori-equivariant object, we obtain the claim.

Step 2. Next, suppose that $X \in \mathcal{C}^I$ is compact.

From Lemma 17.4.1, we find that $\mathrm{Av}_*^{I^{\check{\lambda}}}(X)$ is compact in $\mathcal{C}^{I^{\check{\lambda}}}$ and therefore compact in $\mathcal{C}^{B(O)^{\check{\lambda}}}$. For $\check{\lambda} \in \check{\Lambda}^+$, we have $\mathrm{Av}_*^{I^{\check{\lambda}}}(X) = \mathrm{Av}_*^{B(O)^{\check{\lambda}}}(X)$, so, we conclude that $\mathrm{Av}_*^{B(O)^{\check{\lambda}}}(X)$ is compact for every $\check{\lambda} \in \check{\Lambda}^+$.

Now observe that for any $Y \in \mathcal{C}^I$, the map:

$$\mathrm{Hom}_{\mathcal{C}^I}(X, Y) \to \mathrm{Hom}_{\mathcal{C}^{B(O)^{\check{\lambda}}}}(\mathrm{Av}_*^{B(O)^{\check{\lambda}}}(X), \mathrm{Av}_*^{B(O)^{\check{\lambda}}}(Y))$$

is an isomorphism, since we can compute these averages as $\mathrm{Av}_*^{I^{\check{\lambda}}}$.

Therefore, Corollary 17.5.3 implies that:

$$\mathrm{Hom}_{\mathcal{C}^I}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_{N(K)T(O)}}(\mathfrak{p}(X), \mathfrak{p}(Y))$$

is an equivalence for every Y.

Step 3. Combining Steps 1 and 2, we obtain that our functor is an equivalence whenever $\mathcal{C}^I$ is compactly generated.

In particular, this applies to $\mathcal{C} = D^*(G(K))$, since $D^*(G(K))_I \simeq D(\mathrm{Fl}_G^{\mathrm{aff}})$ is compactly generated.

To treat the case of general $\mathcal{C}$, we use the same method as Corollary 17.2.3:

$$\mathcal{C}_I \simeq \mathcal{C} \underset{D^*(G(K))}{\otimes} D^*(G(K))_I \simeq \mathcal{C} \underset{D^*(G(K))}{\otimes} D^*(G(K))_{N(K)T(O)} \simeq \mathcal{C}_{N(K)T(O)}.$$

$\square$

18. Comparison of baby and big Whittaker categories

18.1. To complete the task set in §17.1, this section will compare the *baby Whittaker* category $D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ (see §1.8) considered in [AB09] to $\mathsf{Whit}(D(\mathrm{Fl}_G^{\mathrm{aff}}))$, which by Theorem 17.2.1 is equivalent to $\mathsf{Whit}(D^!(\mathfrak{Fl}^{\frac{\infty}{2}}))$, the main category considered in this thesis.

Our main result is Theorem 18.3.1, showing that these two categories are equivalent.

18.2. **Shifted Whittaker objects.** For convenience, we take Whittaker objects with respect to a character of non-zero conductor.

For $\mathcal{C}$ a category acted on by $G(K)$, we use the notation Whit' to denote the *shifted Whittaker category* of objects equivariant with respect to the character sheaf on $N^-(K)$ corresponding to the character $\psi_{N^-(K)} : \mathfrak{n}^-(K) \to k$ of its Lie algebra defined by:

$$\psi'_{N^-(K)}(x) = \psi_{N^-(K)}(t^{-1}x)$$

where we recall that $\psi_{N^-(K)}$ was defined in (1.20.1).

We use the notation ψ' for the corresponding character sheaf on $N^-(K)$.

Remark 18.2.1. We have an obvious equivalence $\mathsf{Whit}(\mathcal{C}) \simeq \mathsf{Whit}'(\mathcal{C})$, so this change does not make much difference. It is just for convenience in comparing Whittaker and baby Whittaker categories.

Remark 18.2.2. The convenience of the shifted Whittaker character is that

$$\psi_{\overset{o}{I}^-}|_{\mathfrak{n}^-(O)} = \psi'_{N^-(K)}|_{\mathfrak{n}^-(O)}.$$

Here we recall that $\psi_{\overset{o}{I}^-}$ was defined in §1.7.

18.3. We have a functor $\mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}})) \to D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ given by forgetting the Whittaker condition and then $*$-averaging against $\overset{o}{I}^-, \psi_{\overset{o}{I}^-}$. We denote this functor by $\mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$.

It is easy to see that this functor admits a left adjoint, since every object in the right hand side is (ind-)holonomic and because $\overset{o}{I}{}^- \subseteq N^-(K)$ is a compact open subgroup: one applies Proposition 16.60.2. We denote this left adjoint by $\mathrm{Av}_!^{\mathsf{Whit}'}$.

Theorem 18.3.1. *The adjoint functors:*

$$D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}{}^-,\psi_{\overset{o}{I}{}^-}} \underset{\mathrm{Av}_*^{\overset{o}{I}{}^-,\psi_{\overset{o}{I}{}^-}}}{\overset{\mathrm{Av}_!^{\mathsf{Whit}'}}{\rightleftarrows}} \mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}}))$$

are mutually inverse equivalences.

18.4. Let $1_{\mathrm{Fl}_G^{\mathrm{aff}}}$ denote the canonical point of $\mathrm{Fl}_{G^{\mathrm{aff}}}$.

18.5. **Relevant orbits.** We begin by analyzing which orbits admit baby and shifted Whittaker sheaves on $\mathrm{Fl}_G^{\mathrm{aff}}$.

Let $W^{\mathrm{aff,ext}}$ denote the *extended affine Weyl group* $W \ltimes \check{\Lambda}$. Let W^{aff} be the *non-extended affine Weyl group* given as the semidirect product of W and the $\mathbb{Z}$-span of the coroots.

Remark 18.5.1. After a choice of Borel in G, one knows that W^{aff} picks up a canonical structure of Coxeter group, i.e., the corresponding simple reflections are determined. We use the Borel B^- in making these conventions. This choice reflects the fact that we are using $\overset{o}{I}{}^-$ and $N^-(K)$ for our characters. (But we continue to reference positive and dominant co/weights for G using B to define positivity).

We alert the reader that the same convention is implicitly used in [AB09].

Remark 18.5.2. Recall that the length function on W^{aff} extends in a canonical way to one on $W^{\mathrm{aff,ext}}$. (This is recalled explicitly in the proof of Proposition 18.5.9).

Notation 18.5.3. In the affine Weyl group, we use the notation $w\check{\lambda}$ to denote the product of the elements w and $\check{\lambda}$. This should not be confused with $w(\check{\lambda})$, the result of letting the Weyl group act on $\check{\Lambda}$.

The map $W^{\mathrm{aff,ext}} \to \mathrm{Fl}_G^{\mathrm{aff}}$ given by $\check{\lambda}w \mapsto \check{\lambda}(t)w1_{\mathrm{Fl}_G^{\mathrm{aff}}}$ (we choose representatives in G for elements of the Weyl group) gives a set of points indexing both the $\overset{o}{I}{}^-$ orbits and the $N^-(K)$ orbits on $\mathrm{Fl}_G^{\mathrm{aff}}$.

Remark 18.5.4. The closure relations among the former are given by the Bruhat ordering on the extended affine Weyl group, while closure relations among the latter are given by the *semi-infinite Bruhat ordering*, c.f. [FFKM99] §5. However, we will not explicitly need either of these facts in what follows.

For $g \in G(K)$ with $\overline{g}$ the induced point $g \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$ in $\mathrm{Fl}_G^{\mathrm{aff}}$, note that the orbit $N^-(K)\overline{g}$ supports a shifted Whittaker sheaf[44] if and only if:

$$\mathfrak{n}^-(K) \cap \mathrm{Ad}_g(\mathrm{Lie}(I)) \subseteq \mathrm{Ker}(\psi'_{N^-(K)}) \tag{18.5.1}$$

and similarly, the orbit supports a baby Whittaker sheaf if and only if:

$$\mathrm{Lie}(\overset{o}{I}{}^-) \cap \mathrm{Ad}_g \mathrm{Lie}(I) \subseteq \mathrm{Ker}(\psi_{\overset{o}{I}{}^-}). \tag{18.5.2}$$

For our explicit orbit representatives, we easily find:

Proposition 18.5.5. *For $\check{\lambda}w \in W^{\mathrm{aff,ext}}$, the corresponding $N^-(K)$-orbit (resp. $\overset{o}{I}{}^-$-orbit) supports a Whittaker sheaf if and only if:*

$$\begin{cases} (\check{\lambda}, \alpha_i) \leqslant 0 & \text{if } w^{-1}(\alpha_i) > 0 \\ (\check{\lambda}, \alpha_i) < 0 & \text{if } w^{-1}(\alpha_i) < 0 \end{cases} \tag{18.5.3}$$

for every $i \in \mathcal{I}_G$.

Definition 18.5.6. We say that $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ (or the corresponding $N^-(K)$ or $\overset{o}{I}{}^-$ orbit) is *relevant* if (18.5.3) is satisfied.

[44]I.e. Whit' of the corresponding orbit is non-zero.

Remark 18.5.7. As we will see in the proof of Proposition 18.5.9, the inequalities (18.5.3) force the generalization where we allow general positive roots α in place of the simple roots α_i.

Remark 18.5.8. If $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ is relevant, then $B(O) \cdot \check{\lambda}w = \check{\lambda}w \in \mathrm{Fl}_G^{\mathrm{aff}}$. It follows that:

$$\overset{o}{I}{}^{-} \cdot \check{\lambda}w \subseteq N^{-}(K) \cdot \check{\lambda}w.$$

To compare with [AB09], we include the following computation, well-known and implicit in *loc. cit.*, but for which we are not sure of a good reference and therefore include for the reader's convenience. The reader may safely skip this material.

Proposition 18.5.9. $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ *is relevant if and only if* $\check{\lambda}w$ *is the unique element of minimal length in* $W \cdot \check{\mu}$ *for some* $\check{\mu} \in \check{\Lambda}$.

Proof. The existence of a unique minimal length element in this coset follows from the fact that W is a parabolic subgroup (in the sense of Coxeter groups) in the affine Weyl group W^{aff}.

Recall that we can compute the length of an element $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ by the formula:[45]

$$\ell(\check{\lambda}w) = \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(\check{\lambda}, \alpha)| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(\check{\lambda}, \alpha) + 1|.$$

For $\check{\lambda} = w(\check{\mu})$, so that $\check{\lambda}w = w\check{\mu}$, we find:

$$\ell(w\check{\mu}) = \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(w(\check{\mu}), \alpha)| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(w(\check{\mu}), \alpha) + 1| = \\ \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(\check{\mu}, w^{-1}(\alpha))| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(\check{\mu}, w^{-1}(\alpha)) + 1|. \tag{18.5.4}$$

[45]This formula relies on the convention of Remark 18.5.1. One usually finds this formula written relative to the positive Borel, in which case the formula would have last term $|(\check{\lambda}, \alpha) - 1|$, but switching α with $-\alpha$ everywhere, we obviously recover the formula in its given form.

Let $w_{\check{\mu}}$ be the minimal length element of W such that $w_{\check{\mu}}(-\check{\mu})$ lies in the dominant chamber: the uniqueness of a minimal length such element is again guaranteed by the fact that the appropriate stabilizer group is a parabolic subgroup of W.

We claim that $w_{\check{\mu}}$ is characterized in W by the identities:

$$\begin{cases} (w_{\check{\mu}}(\check{\mu}), \alpha) \leqslant 0 & \text{for } \alpha > 0 \text{ with } w_{\check{\mu}}^{-1}(\alpha) > 0 \\ (w_{\check{\mu}}(\check{\mu}), \alpha) < 0 & \text{for } \alpha > 0 \text{ with } w_{\check{\mu}}^{-1}(\alpha) < 0. \end{cases} \tag{18.5.5}$$

Indeed, we have $(w_{\check{\mu}}(\check{\mu}), \alpha) \leqslant 0$ for all $\alpha > 0$ by dominance of $-w_{\check{\mu}}(\check{\mu})$. Then recall that for $\alpha > 0$, $w^{-1}(\alpha) < 0$ is equivalent to $\ell(s_\alpha w) < \ell(w)$.[46] Therefore, if we had $w_{\check{\mu}}^{-1}(\alpha) < 0$ and $(w_{\check{\mu}}(\check{\mu}), \alpha) = 0$, this would force:

$$\ell(s_\alpha w_{\check{\mu}}) < \ell(w_{\check{\mu}})$$

$$s_\alpha w_{\check{\mu}}(\check{\mu}) = w_{\check{\mu}}(\check{\mu}) - (w_{\check{\mu}}(\check{\mu}), \alpha)\alpha = w_{\check{\mu}}(\check{\mu})$$

contradicting the minimality of $w_{\check{\mu}}$.

We see from this argument that it is enough to verify (18.5.5) in the case that α is a simple root.

Next, we claim that $w_{\check{\mu}}$ minimizes (18.5.4).

Indeed, let $w \in W$ other than $w_{\check{\mu}}$. Since we noted that $w_{\check{\mu}}$ is characterized by the identities (18.5.5) for α a simple root, we see that $w \neq w_{\check{\mu}}$ implies that either there exists a simple root α_i with $w^{-1}(\alpha_i) > 0$ and $w(\check{\mu}, \alpha_i) > 0$, or else there exists α_i with $w^{-1}(\alpha_i) < 0$ and $(w(\check{\mu}), \alpha_i) \geqslant 0$.

In the former case, using the fact that s_i permutes the non-α_i positive roots, one finds:

[46] This fact is certainly standard for α a simple root, but perhaps warrants a proof for general $\alpha > 0$ since e.g. it does not appear in [Hum90] Chapter 1. We prove the claim by induction on $\ell(w)$, the case $\ell(w) = 0$ being obvious. Choose $i \in \mathcal{I}_G$ with $w(\alpha_i) < 0$; let s_i denote the corresponding simple reflection. If $w(\alpha_i) \neq -\alpha$, then $\ell(ws_i) < \ell(w)$ and $ws_i(\alpha) < 0$, so by induction, $\ell(s_\alpha w s_i) < \ell(ws_i) = \ell(w) - 1$, but $\ell(s_\alpha w s_i) \geqslant \ell(s_\alpha w) - 1$, giving the claim in this case. Otherwise, $ws_i(\alpha_i) = \alpha$, so $(ws_i)^{-1} s_\alpha w s_i = s_i$, so $s_\alpha w = ws_i$, but $w(\alpha_i) < 0$ implies that $\ell(s_\alpha w) = \ell(ws_i) < \ell(w)$.

$$\ell(s_i w\check{\mu}) - \ell(w\check{\mu}) = |(s_i w(\check{\mu}), \alpha_i) + 1| - |(w(\check{\mu}), \alpha_i)| = |-(w(\check{\mu}), \alpha_i) + 1| - |(w(\check{\mu}), \alpha_i)| = -1$$

and in the latter case, one similarly finds:

$$\ell(s_i w\check{\mu}) - \ell(w\check{\mu}) = |(s_i w(\check{\mu}), \alpha_i)| - |(w(\check{\mu}), \alpha_i) + 1| = |(w(\check{\mu}), \alpha_i)| - |(w(\check{\mu}), \alpha_i) + 1| = -1.$$

Either way, $\ell(s_i w\check{\mu}) < \ell(w\check{\mu})$, meaning that $w\check{\mu}$ was not of minimal length.

Finally, one immediately sees that in terms of $\check{\lambda} = w_{\check{\mu}}(\check{\mu})$, (18.5.5) exactly translates into (18.5.3), as desired (appealing to the fact that it is enough to verify (18.5.5) for simple roots.)

□

18.6. **Minimal orbits.** We introduce two parallel pictures for $\overset{o}{I}{}^-$ and $N^-(K)$ orbits on $\mathrm{Fl}_G^{\mathrm{aff}}$.

We define the *minimal* $N^-(K)$-orbit (resp. $\overset{o}{I}{}^-$) orbit to be the orbit through $1_{\mathrm{Fl}_G^{\mathrm{aff}}}$.

We define $j_!^{\mathrm{min},\mathsf{Whit}'} \in \mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}}))$ and $j_!^{\mathrm{min,baby}} \in D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}{}^-, \psi_{\overset{o}{I}{}^-}}$ be the !-extensions of the relevant character sheaves supported on these orbits.[47]

18.7. **Cleanness.** The main point in proving Theorem 18.3.1 are the following two *cleanness* results.

Remark 18.7.1. Suppose that $j : U \hookrightarrow Z$ is a locally closed embedding of schemes of finite type. Recall that $\mathcal{F} \in D(Z)$ is said to be *cleanly extended* from U if the maps $j_! j^!(\mathcal{F}) \to \mathcal{F} \to j_{*,dR} j^{*,dR}(\mathcal{F})$ are isomorphisms. This definition extends to the setting of ind-schemes of ind-finite type in the obvious way.

Proposition 18.7.2. *The object* $j_!^{\mathrm{min,baby}}$ *is cleanly extended from the orbit* $\overset{o}{I}{}^- \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$.

[47]To see that $j_!^{\mathrm{min},\mathsf{Whit}'}$ actually lies in the shifted Whittaker subcategory, exhaust $N^-(K)$ by compact open subgroups and exploit placidity of these subgroups.

Proposition 18.7.3. *The object $j_!^{\min,\mathsf{Whit}'}$ is cleanly extended from the orbit $N^-(K)\cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$.*

Each of these results follows easily from the closure relations noted above, but we give complete proofs below.

Proof of Proposition 18.7.2. We have:

$$\overset{o}{I}{}^- \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}} = N^- \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}} \overset{\text{open}}{\subseteq} G/B \overset{\text{closed}}{\subseteq} \mathrm{Fl}_G^{\mathrm{aff}}.$$

On N^-, our sheaf is a non-degenerate character sheaf, and this obviously extends cleanly to G/B.

□

Proof of Proposition 18.7.3. We use the techniques of §7 freely here.

Let $Z \subseteq \mathrm{Fl}_G^{\mathrm{aff}}$ be the pullback of $\overline{\mathrm{Gr}}_{N^-} \subseteq \mathrm{Gr}_G$. Then Z is ind-closed in $\mathrm{Fl}_G^{\mathrm{aff}}$ and contains the orbit $N^-(K)\cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$ as an ind-open subscheme.

Clearly the only $N^-(K)$-orbits in Z pass through points $\check{\lambda}w$ with $\check{\lambda} \in \check{\Lambda}^{pos}$.

We claim that the only such $\check{\lambda}w$ supporting a Whittaker sheaf is $\check{\lambda} = 0$, $w = 1$. Indeed, as in the proof of Proposition 18.5.9, the inequalities (18.5.3) force the same inequalities for a general positive root, not merely a simple root. Then we see $\check{\lambda} \in \check{\Lambda}^{pos}$ forces:

$$0 \leqslant (\check{\lambda},\rho)(\check{\lambda}, \frac{1}{2}\sum_{\alpha>0}\alpha) = \frac{1}{2}\sum_{\alpha>0}(\check{\lambda},\alpha) \leqslant 0$$

so we must have equality, forcing $\check{\lambda} = 0$, and then we further see from (18.5.3) that we must have $w = 1$ as well.

This now gives the cleanness result.

□

Corollary 18.7.4. *The unit and counit maps:*

$$j_!^{\mathrm{min,baby}} \to \mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}} \mathrm{Av}_!^{\mathsf{Whit}'}(j_!^{\mathrm{min,baby}})$$

$$\mathrm{Av}_!^{\mathsf{Whit}'} \mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\mathrm{min},\mathsf{Whit}'}) \to j_!^{\mathrm{min},\mathsf{Whit}'}$$

are isomorphisms.

Proof. By Remark 18.5.8, we obtain that:

$$\mathrm{Av}_!^{\mathsf{Whit}'}(j_!^{\mathrm{min,baby}}) \simeq j_!^{\mathrm{min},\mathsf{Whit}'}.$$

Note that Remark 18.5.8 implies that the only relevant $\overset{o}{I}^-$-orbit intersecting $N^-(K) \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$ is $\overset{o}{I}^- \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$.

Therefore, applying cleanness of the $j_!^{\mathrm{min},\mathsf{Whit}'}$, we obtain that $\mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\mathrm{min},\mathsf{Whit}'})$ is the $*$-extension of our character sheaf from $\overset{o}{I}^- \cdot 1_{\mathrm{Fl}_G^{\mathrm{aff}}}$. Moreover, applying cleanness of the latter, we obtain:

$$\mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\mathrm{min},\mathsf{Whit}'}) \simeq j_!^{\mathrm{min},\mathsf{Whit}'}$$

as desired.

□

18.8. **Compatibility with the affine Hecke algebra.** Both categories $D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ and $\mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}}))$ are acted on by the geometric affine Hecke algebra $H_{\mathrm{aff}} := D(\mathrm{Fl}_G^{\mathrm{aff}})^I$ by the convolution action of H_{aff} on $D(\mathrm{Fl}_G^{\mathrm{aff}})$.

Moreover, the functor $\mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ is given by a convolution, and therefore commutes with H_{aff}-actions.

One can further see that $\mathrm{Av}_!^{\mathsf{Whit}'}$ commutes with the H_{aff}-actions by exploiting the ind-properness of $\mathrm{Fl}_G^{\mathrm{aff}}$. Alternatively: we don't actually need this fact; we will only need that $\mathrm{Av}_!^{\mathsf{Whit}'}$ commutes with convolution with Mirkovic-Wakimoto sheaves, and this follows formally from their invertibility and the fact that $\mathrm{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ commutes with such convolutions.

18.9. We now prove Theorem 18.3.1.

Proof of Theorem 18.3.1. The category $D(\mathrm{Fl}_G^{\mathrm{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ is compactly generated by objects !-extended from relevant orbits, and similarly for $\mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}}))$. For $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ relevant, let $j_!^{\check{\lambda}w,\mathrm{baby}}$ and $j_!^{\check{\lambda}w,\mathsf{Whit}'}$ denote the corresponding objects.

As in [AB09] Lemma 4, the object $j_!^{\check{\lambda}w,\mathrm{baby}}$ is obtained from $j_!^{\mathrm{min,baby}}$ by convolving with an appropriate invertible object of H_{aff}.

Therefore, by Corollary 18.7.4 and §18.8, the unit map of the adjunction applied to $j_!^{\check{\lambda}w,\mathrm{baby}}$ is an equivalence.

Moreover, we claim that:

$$\mathrm{Av}_!^{\mathsf{Whit}'}(j_!^{\check{\lambda}w,\mathrm{baby}}) \xrightarrow{\simeq} j_!^{\check{\lambda}w,\mathsf{Whit}'}.$$

Indeed, this is immediate from Remark 18.5.8. Therefore, $j_!^{\check{\lambda}w,\mathsf{Whit}'}$ is similarly obtained from $j_!^{\mathrm{min,baby}}$ by convolving with the appropriate invertible object of H_{aff}. Therefore, as for the baby Whittaker category, we see that the counit for $j_!^{\check{\lambda}w,\mathsf{Whit}'}$ is an equivalence.

By compact generation, we now obtain the result.

□

19. Sheaves of categories

19.1. The purpose of this section is to recall the rudiments of the theory of sheaves of categories on prestacks, and the theory of 1-affineness from [Gai12b].

Incidentally, we prove Theorem 19.18.1 on the relationship between local and global duality for de Rham prestacks; this result is not needed elsewhere in the text.

19.2. **Linear categories.** We begin with a quick review of the theory of sheaves of categories from [Gai12b] and [Lur11b].

Recall that DGCat_{cont} denotes the category of cocomplete DG categories under continuous functors, and that DGCat_{cont} is equipped with a symmetric monoidal structure $\otimes$ with unit Vect, and whose tensor product commutes with colimits in each variable.

Let A be a commutative algebra. An *A-linear category* is an A–mod-module category in DGCat_{cont}. A functor of A-linear categories is A-linear if it is a continuous functor of A–mod-module categories. When A is connective, we denote the category of A-linear categories under A-linear functors by $\mathsf{ShvCat}_{/\mathrm{Spec}(A)}$.

Remark 19.2.1. Note that $\mathsf{ShvCat}_{/\mathrm{Spec}(A)}$ is a symmetric monoidal category with tensor product $(\mathsf{C}, \mathsf{D}) \mapsto \mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} \mathsf{D}$. This symmetric monoidal structure has unit A–mod.

For $A \to B$ a map of commutative rings, we have the symmetric monoidal functor:

$$A\text{–}\mathsf{mod} \to B\text{–}\mathsf{mod} \tag{19.2.1}$$

sending $M \to M \otimes_A B$ and therefore we obtain the adjoint functors:

$$(A\text{–}\mathsf{mod})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) \underset{}{\overset{\mathsf{C} \mapsto \mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}{\rightleftarrows}} (B\text{–}\mathsf{mod})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) \tag{19.2.2}$$

where the right adjoint is restriction along (19.2.1). Each of these functors commutes with arbitrary colimits.

Remark 19.2.2. According to [Gai12a], rigidity of A–mod implies that B–mod is dualizable as an A–mod-module category. Therefore, the left adjoint in (19.2.2) commutes with limits as well.

Lemma 19.2.3. *For a morphism $A \to B$ of commutative algebras and for an A-linear category C, the tautological functor:*

$$\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to \mathsf{C}$$

is conservative and admits an A-linear left adjoint.

Notation 19.2.4. In the setting of Lemma 19.2.3, we denote this left adjoint by:

$$X \mapsto X \underset{A}{\otimes} B.$$

Proof of Lemma 19.2.3. The existence of a left adjoint follows from the existence of the adjoint A-linear functors:

$$A\text{–}\mathsf{mod} \rightleftarrows B\text{–}\mathsf{mod}$$

It suffices to see that this left adjoint generates the category $\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}$ under colimits. Because B generates $B\text{–}\mathsf{mod}$ under colimits and shifts, it suffices to see that the essential image of the (non-exact) functor:

$$\mathsf{C} \times B\text{–}\mathsf{mod} \to \mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}$$

generates under colimits. But this is immediate from the universal property of the tensor product of categories.

□

19.3. **Sheaves of categories.** We consider $\mathsf{ShvCat}_{/\operatorname{Spec}(-)}$ as a functor $\mathsf{AffSch}^{op} \to \mathsf{DGCat}_{cont}$ via the left adjoint functor in (19.2.1). We let $\mathsf{ShvCat}_{/-} : \mathsf{PreStk}^{op} \to \mathsf{DGCat}_{cont}$ denote the right Kan extension of this functor.

For any prestack $\mathcal{Y}$, $\mathsf{ShvCat}_{/\mathcal{Y}}$ is a symmetric monoidal category with tensor product computed "locally" using Remark 19.2.1. We denote the tensor product by:

$$(\mathsf{C}, \mathsf{D}) \mapsto \mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D}.$$

For a prestack $\mathcal{Y}$ we refer to objects of $\mathsf{ShvCat}_{/\mathcal{Y}}$ as *sheaves of categories* on $\mathcal{Y}$. For a sheaf of categories C on $\mathcal{Y}$ we let

$$\Gamma(\mathcal{Y}, \mathsf{C}) \in \mathsf{DGCat}_{cont}$$

denote the global sections of the category. We let $\mathsf{QCoh}_{\mathcal{Y}} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ denote the canonical object with global sections $\mathsf{QCoh}(\mathcal{Y})$. For $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ the category $\Gamma(\mathcal{Y}, \mathsf{C})$ is canonically a $\mathsf{QCoh}(\mathcal{Y})$-module caetgory.

For $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ and $f : \mathcal{Y}' \to \mathcal{Y}$ we use both notations $\mathsf{C}_{\mathcal{Y}'}$ and $f^*(\mathsf{C})$ for the pullback of C to $\mathcal{Y}'$. Note that if f is an affine (schematic) morphism then the functor $f^* : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Y}'}$ admits a continuous right adjoint f_* computed "locally" using (19.2.2).

Remark 19.3.1. By Remark 19.2.2, limits in $\mathsf{ShvCat}_{/\mathcal{Y}}$ are computed locally, i.e., pullbacks of sheaves of categories commute with limits.

19.4. **Fully-faithful functors.** For $\mathcal{Y}$ a prestack, we say that a morphism $\mathsf{D} \to \mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ is *locally fully-faithful*, or simply *fully-faithful*,[48] if, for every affine scheme S with a morphism $f : S \to \mathcal{Y}$, the induced functor:

$$\Gamma(S, \mathsf{D}) \to \Gamma(S, \mathsf{C})$$

is fully-faithful.

Example 19.4.1. If $\mathsf{D} \to \mathsf{C}$ admits a right (resp. left) adjoint in the 2-category $\mathsf{ShvCat}_{/\mathcal{Y}}$ with unit (resp. counit) an equivalence, then this functor is locally fully-faithful.

Terminology 19.4.2. We sometimes simply summarize the situation in saying that D is a full subcategory of C, and write $\mathsf{D} \subseteq \mathsf{C}$.

The following result helps to identify locally fully-faithful functors.

[48]This terminology is justified by Proposition 19.4.3.

Proposition 19.4.3. *For $\mathcal{Y} = \mathrm{Spec}(A)$, a functor $F : \mathsf{D} \to \mathsf{C}$ of A-linear categories is locally fully-faithful if and only if it is fully-faithful as a mere functor.*

Proof. It suffices to show that for every morphism $A \to B$ of commutative algebras, the induced functor:

$$F_B : \mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to \mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}$$

is fully-faithful.

Let $\mathrm{Oblv}^B_{\mathsf{D}}$ denote the forgetful functor:

$$\mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to \mathsf{D}$$

and similarly for C.

By Lemma 19.2.3, it suffices to show that, for $X \in \mathsf{D}$ and $Y \in \mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}$, the morphism:

$$\mathrm{Hom}_{\mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}(X \underset{A}{\otimes} B, Y) \to \mathrm{Hom}_{\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}(F_B(X \underset{A}{\otimes} B), F_B(Y)) \tag{19.4.1}$$

is an equivalence.

Note that both operations Oblv^B_{-} and $- \otimes_A B$ commute with A-linear functors. Moreover, under the identifications:

$$\mathrm{Hom}_{\mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}(X \underset{A}{\otimes} B, Y) = \mathrm{Hom}_{\mathsf{D}}(X, \mathrm{Oblv}^B_{\mathsf{D}}(Y))$$

and:

$$\mathrm{Hom}_{\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}(F_B(X \underset{A}{\otimes} B), F_B(Y)) = \mathrm{Hom}_{\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}}(F(X) \underset{A}{\otimes} B, F_B(Y)) =$$
$$\mathrm{Hom}_{\mathsf{C}}\left(F(X), \mathrm{Oblv}^B_{\mathsf{C}}(F_B(Y))\right) = \mathrm{Hom}_{\mathsf{C}}(F(X), F \circ \mathrm{Oblv}^B_{\mathsf{D}}(Y))$$

the morphism (19.4.1) is given by the canonical map:

$$\mathrm{Hom}_{\mathsf{D}}(X, \mathrm{Oblv}^B_{\mathsf{D}}(Y)) \to \mathrm{Hom}_{\mathsf{C}}(F(X), F \circ \mathrm{Oblv}^B_{\mathsf{D}}(Y))$$

so that the result follows from the hypothesis that F is fully-faithful.

□

We also note the following basic stability.

Proposition 19.4.4. *Given an* $\mathcal{I}$*-shaped diagram of fully-faithful functors* $\mathsf{C}_i \to \mathsf{D}_i \in \mathsf{ShvCat}_{/\mathcal{Y}}$*, the induced functor:*

$$\lim_{i \in \mathcal{I}} \mathsf{C}_i \to \lim_{i \in \mathcal{I}} \mathsf{D}_i$$

is fully-faithful as well.

This follows immediately from the corresponding statement for DG categories.

Corollary 19.4.5. *Given a system of subcategories* $i \mapsto \mathsf{C}_i \subseteq \mathsf{C}$ *indexed by a contractible category* $\mathcal{I}$ *(i.e., the groupoid obtained by inverting all arrows is contractible), the induced functor* $\lim_{i \in \mathcal{I}} \mathsf{C}_i \to \mathsf{C}$ *is fully-faithful as well.*

Proof. Apply Proposition 19.4.4 to the functors:

$$\mathsf{C}_i \hookrightarrow \mathsf{C}$$

and note that contractibility of $\mathcal{I}$ implies that $\lim_{\in \mathcal{I}} \mathsf{C} \xrightarrow{\simeq} \mathsf{C}$.

□

19.5. Let $F : \mathsf{C} \to \mathsf{D}$ be a morphism of A-linear categories. We define $F(\mathsf{C})$ as the subcategory of D generated under colimits by objects $F(X)$, $X \in \mathsf{C}$. Note that $F(\mathsf{C})$ is an A-linear subcategory since $A\text{–}\mathsf{mod}$ is generated under colimits by A.

Lemma 19.5.1. *$A \to B$ be a morphism of commutative algebras and let $F : \mathsf{C} \to \mathsf{D}$ be an A-linear morphism of A-linear categories. Let F^B denote the induced functor:*

$$F^B : \mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to \mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}.$$

Then the canonical functor:

$$F(\mathsf{C}) \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to F^B(\mathsf{C} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}) \tag{19.5.1}$$

is an equivalence.

Proof. The morphism $F(\mathsf{C}) \to \mathsf{D}$ is fully-faithful, so by Proposition 19.4.3 the morphism:

$$F(\mathsf{C}) \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod} \to \mathsf{D} \underset{A\text{–}\mathsf{mod}}{\otimes} B\text{–}\mathsf{mod}$$

is as well. Therefore, it remains to show essential surjectivity of (19.5.1).

By Lemma 19.2.3, $\mathsf{C} \otimes_{A\text{–}\mathsf{mod}} B\text{–}\mathsf{mod}$ is generated under colimits by objects induced from C, giving the result.

□

By the lemma, for $F : \mathsf{C} \to \mathsf{D}$ a morphism of sheaves of categories on $\mathcal{Y} \in \mathsf{PreStk}$, we can make sense of $F(\mathsf{C})$ so that its formation commutes with base-change. Note that $F(\mathsf{C}) \to \mathsf{D}$ is locally fully-faithful.

19.6. **Localizations.** Let A be a fixed commutative algebra. Let C be a A-linear category, and let $\mathsf{D} \subseteq \mathsf{C}$ be a subcategory closed under colimits. As above, since A–mod is generated under colimits by A, D is an $(A\text{–}\mathsf{mod})$-submodule category.

In this case, we can form the quotient category C/D, that is computed as a pushout:

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{C} \arrow[d] \\ 0 \arrow[r] & \mathsf{C}/\mathsf{D} \end{tikzcd} \tag{19.6.1}$$

in the category of A-linear categories.

Lemma 19.6.1. *Given $B \to A$ a map of commutative algebras, the induced restriction functor:*

$$\{A\text{-linear categories}\} \to \{B\text{-linear categories}\}$$

commutes with formation of quotients.

Proof. Indeed, this functor commutes with arbitrary colimits, since it is the a restriction functor for modules in DGCat_{cont} from A–mod to B–mod (c.f. [Lur12] 4.2.3.5).

□

Remark 19.6.2. Applying the lemma for $B = k$, we obtain an explicit description of the quotient in the category of A-linear categories: it is the usual quotient of DG categories, which may be computed by applying the usual localization procedure from [Lur09] §5.5.4.

More generally, one can form quotients for locally fully-faithful functors of sheaves of categories on an arbitrary prestack, defined also as a pushout. This operation tautologically commutes with pullback of sheaves of categories, and then can be computed "locally" using Lemma 19.6.1.

19.7. For $\mathcal{Y}$ a prestack and F a morphism $F : \mathsf{C} \to \mathsf{D} \in \mathsf{ShvCat}_{/\mathcal{Y}}$, the *kernel* $\mathrm{Ker}(F)$ of F is by definition the fiber product $\mathsf{C} \times_{\mathsf{D}} 0$. By Remark 19.3.1, formation of kernels commutes with base-change.

Note that the natural morphism $\mathrm{Ker}(F) \to \mathsf{C}$ is always locally fully-faithful. Indeed, this tautologically reduces to the case where $\mathcal{Y}$ is an affine scheme, where it is obvious.

Definition 19.7.1. A morphism $F : \mathsf{C} \to \mathsf{D} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ is a *localization functor* in $\mathsf{ShvCat}_{/\mathcal{Y}}$ if the natural morphism:

$$\mathsf{C}/\mathrm{Ker}(F) \to \mathsf{D}$$

is an equivalence.

We have the following equivalence between subcategories and localization functors.

Proposition 19.7.2. *Let* C *be a sheaf of categories on a prestack* $\mathcal{Y}$*, and let* $\mathsf{C}^0 \subseteq \mathsf{C}$ *be a full subcategory.*

(1) *The kernel of the functor* $\mathsf{C} \to \mathsf{C}/\mathsf{C}^0$ *is* C^0*.*
(2) *The functor* $\mathsf{C} \to \mathsf{C}/\mathsf{C}^0$ *is a localization functor.*

Proof. The first statement immediately reduces to the affine case, where it is well-known, and the second statement follows tautologically from the first.

□

Proposition 19.7.3. *Suppose that* $\mathsf{C} = \mathrm{colim}_{i \in \mathcal{I}} \mathsf{C}_i \in \mathsf{ShvCat}_{/\mathcal{Y}}$*, and suppose that* $\mathcal{I}$ *is filtered and each structure map* $\mathsf{C}_i \to \mathsf{C}_j$ *is a localization functor.*

Then for every $i_0 \in \mathcal{I}$*, the functor* $\mathsf{C}_{i_0} \to \mathsf{C}$ *is a localization functor.*

We first need the following lemma, which is obvious in the affine case and therefore in general.

Lemma 19.7.4. *Let* $F : \mathsf{D} \to \mathsf{C}$ *be a (not necessarily fully-faithful) morphism of sheaves of categories and let* C/D *denote the corresponding pushout. Then* $\mathsf{C}/\mathsf{D} = \mathsf{C}/F(\mathsf{D})$*. In particular,* $\mathsf{C} \to \mathsf{C}/\mathsf{D}$ *is a localization functor.*

Proof of Proposition 19.7.3. We can assume i_0 is initial in $\mathcal{I}$ by filteredness. A functor $\mathsf{C} \to \mathsf{D}$ is equivalent to compatible functors $\mathsf{C}_i \to \mathsf{D}$, which in turn are equivalent to functors $\mathsf{C}_{i_0} \to \mathsf{D}$ mapping $\mathrm{Ker}(\mathsf{C}_{i_0} \to \mathsf{C}_i)$ to 0. But this is obviously equivalent to giving a functor $\mathsf{C}_{i_0} \to \mathsf{D}$ mapping $\mathrm{colim}_i \mathrm{Ker}(\mathsf{C}_{i_0} \to \mathsf{C}_i)$ to 0, so the result follows from Lemma 19.7.4.

□

19.8. **1-affineness.** We follow [Gai12b] in saying a prestack $\mathcal{Y}$ is *1-affine* if the morphism:

$$\Gamma : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$$

is an equivalence.

The following useful results are proved in [Gai12b].

Theorem 19.8.1. *(1) Any quasi-compact quasi-separated scheme is 1-affine.*

(2) If T is a quasi-compact quasi-separated scheme, S is a closed subscheme with quasi-compact complement, and $T_S^\wedge$ is the (indscheme) formal completion, then $T_S^\wedge$ is 1-affine.

(3) For S an almost finite type scheme, S_{dR} is 1-affine.

We also need a relative version: we say that a morphism $f : \mathcal{Y} \to \mathcal{Z}$ of prestacks is *1-affine* if for every affine scheme S and map $S \to \mathcal{Z}$, the prestack $\mathcal{Y} \underset{\mathcal{Z}}{\times} S$ is 1-affine.

We immediately deduce from Theorem 19.8.1 the following:

Proposition 19.8.2. *Any quasi-compact quasi-separated morphism is 1-affine.*

Remark 19.8.3. It is not tautological that a 1-affine prestack $\mathcal{Y}$ has 1-affine structure map $\mathcal{Y} \to \mathrm{Spec}(k)$. However, we will prove this in Corollary 19.10.5 below.

19.9. **Pushforwards.** Next, we discuss the pushforward construction for sheaves of categories.

Proposition 19.9.1. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of prestacks.*

(1) The functor:

$$f^* : \mathsf{ShvCat}_{/\mathcal{Z}} \to \mathsf{ShvCat}_{/\mathcal{Y}}$$

admits a right adjoint f_* compatible with arbitrary base-change.

(2) If f is 1-affine, then $f_* : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Z}}$ commutes with arbitrary colimits and satisfies the projection formula in the sense that it is a morphism of $\mathsf{ShvCat}_{/\mathcal{Z}}$-module categories.

(3) If f is quasi-compact quasi-separated, then for every $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Z}}$ the unit map:

$$\mathsf{C} \to f_* f^*(\mathsf{C})$$

admits a right adjoint in the 2-category $\mathsf{ShvCat}_{/\mathcal{Z}}$. This right adjoint commutes with base-change in the natural sense.

Corollary 19.9.2. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi-compact quasi-separated schematic morphism of prestacks. Then for every $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Z}}$ the morphism:*

$$f_{\mathsf{C}}^* : \Gamma(\mathcal{Z}, \mathsf{C}) \to \Gamma(\mathcal{Y}, f^*(\mathsf{C})) = \Gamma(\mathcal{Z}, f_* f^*(\mathsf{C}))$$

admits a continuous right adjoint $f_{\mathsf{C},*}$.

This right adjoint commutes with base-change in the sense that for every Cartesian diagram:

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[r, "f_1"] \arrow[d, "\varphi"] & \mathcal{Z}_1 \arrow[d, "\psi"] \\ \mathcal{Y}_2 \arrow[r, "f_2"] & \mathcal{Z}_2 \end{tikzcd}$$

with f_2 quasi-compact quasi-separated and schematic and every $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Z}}$ the natural morphism:

$$\psi_{\mathsf{C}}^* \circ f_{2,\mathsf{C},*} \to f_{1,\mathsf{C}_{\mathcal{Y}_2},*} \circ \varphi_{\mathsf{C}_{\mathcal{Y}_2}}^*$$

is an equivalence.

Proof of Proposition 19.9.1. We begin with (1).

Suppose first that $\mathcal{Z} = S$ is an affine scheme. Now the functor:

$$\begin{aligned} \mathsf{QCoh}(S)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) &\to \mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) \\ \mathcal{M} &\mapsto \mathcal{M} \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(\mathcal{Y}) \end{aligned} \tag{19.9.1}$$

obviously admits a right adjoint given by restriction along $\mathsf{QCoh}(S) \to \mathsf{QCoh}(\mathcal{Y})$. This functor commutes with colimits by [Lur12] 4.2.3.5 and tautologically satisfies the projection formula.

We then see that the right adjoint $f_* : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{QCoh}(S)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$ is computed as the composition:

$$\mathsf{ShvCat}_{/\mathcal{Y}} \xrightarrow{\Gamma(\mathcal{Y},-)} \mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) \xrightarrow{\text{restriction}} \mathsf{QCoh}(S)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}) = \mathsf{ShvCat}_{/S}.$$

We now verify the base-change property of this functor. Suppose first that we are given a Cartesian diagram:

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "\varphi"] \arrow[d, "f'"] & \mathcal{Y} \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd} \tag{19.9.2}$$

with S' and S affine schemes. Then for $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$, we compute:

$$\Gamma(S', g^* f_*(\mathsf{C})) = \Gamma(\mathcal{Y}, \mathsf{C}) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S') = \Big(\lim_{\substack{\alpha: T \to \mathcal{Y} \\ T \in \mathsf{AffSch}}} \Gamma(T, \alpha^*(\mathsf{C})) \Big) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S') =$$
$$\lim_{\substack{\alpha: T \to \mathcal{Y} \\ T \in \mathsf{AffSch}}} \Big(\Gamma(T, \alpha^*(\mathsf{C})) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S') \Big) = \Gamma(\underset{\substack{\alpha: T \to \mathcal{Y} \\ T \in \mathsf{AffSch}}}{\operatorname{colim}} T \underset{S}{\times} S', p_1^* \alpha^*(\mathsf{C})) = \Gamma(\mathcal{Y}', \varphi^*(\mathsf{C})).$$

This verifies base-change for the Cartesian diagram (19.9.2), when S' is assumed affine; the case when S' is allowed to be an arbitrary prestack immediately reduces to this one.

We obtain the existence of a right adjoint in (1) compatible with base-change by an immediate reduction to the case when $\mathcal{Z}$ is affine.

The claims of (2) follow from the observations we have already made about (19.9.1).

Using the same dévissage we obtain (3), using that in the quasi-compact quasi-separated case with affine target S we have the continuous right adjoint $f_* : \mathsf{QCoh}(\mathcal{Y}) \to \mathsf{QCoh}(S)$ satisfying the projection formula.

□

Corollary 19.9.3. *Suppose that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are 1-affine prestacks and we are given a diagram:*

$$\begin{tikzcd} \mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{Y} \arrow[r, "\varphi"] \arrow[d, "\psi"] & \mathcal{Y} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Z} \end{tikzcd}$$

with f 1-affine. Then the natural functor:

$$\mathsf{QCoh}(\mathcal{X}) \underset{\mathsf{QCoh}(\mathcal{Z})}{\otimes} \mathsf{QCoh}(\mathcal{Y}) \to \mathsf{QCoh}(\mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{Y})$$

is an equivalence.

Proof. By Proposition 19.9.1, we have:

$$g^* f_*(\mathsf{QCoh}_{\mathcal{X}}) \xrightarrow{\simeq} \varphi_*(\mathsf{QCoh}_{\mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{Y}}) \in \mathsf{ShvCat}_{/\mathcal{Y}}. \tag{19.9.3}$$

Applying global sections on $\mathcal{Y}$, the left hand side of (19.9.3) becomes:

$$\mathsf{QCoh}(\mathcal{X}) \underset{\mathsf{QCoh}(\mathcal{Z})}{\otimes} \mathsf{QCoh}(\mathcal{Y})$$

by our assumptions of 1-affinity, and the right hand side obviously becomes $\mathsf{QCoh}(\mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{Y})$.

□

19.10. We will prove the following technical result.

Proposition 19.10.1. *The composition of 1-affine morphisms is 1-affine.*

We will prove the following more precise form of Proposition 19.10.1.

Lemma 19.10.2. *If $f : \mathcal{Y} \to \mathcal{Z}$ is a 1-affine morphism of prestacks with $\mathcal{Z}$ a 1-affine prestack, then $\mathcal{Y}$ is 1-affine.*

Proof of Proposition 19.10.1 given Lemma 19.10.2. Suppose $\mathcal{Y} \to \mathcal{Z} \to \mathcal{S}$ are 1-affine morphisms. To show that the composition is 1-affine, we reduce to showing that in the case when $\mathcal{S}$ is an affine scheme, $\mathcal{Y}$ is 1-affine. But in this case, $\mathcal{Z}$ is a 1-affine prestack, so the result follows from Lemma 19.10.2.

□

We need the following result first.

Lemma 19.10.3. *For $f : \mathcal{Y} \to \mathcal{Z}$ a 1-affine morphism, the pushforward $f_* : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Z}}$ is conservative.*

Proof. Suppose that C and D are two sheaves of categories on $\mathcal{Y}$ and $\varphi : \mathsf{C} \to \mathsf{D}$ is a map such that $f_*(\varphi)$ is an equivalence. We will show that φ is an equivalence.

Let S be an affine scheme with a map $g : S \to \mathcal{Y}$. It suffices to show that for every such datum, $g^*(\varphi)$ is an equivalence.

We form the commutative diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\mathrm{id}_S"'] & \mathcal{Y} \underset{\mathcal{Z}}{\times} S \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "f"] \\ & S \arrow[r, "f \circ g"] & \mathcal{Z}. \end{tikzcd}$$

Note that pushforward along $\mathcal{Y} \underset{\mathcal{Z}}{\times} S \to S$ is conservative because:

$$\Gamma(\mathcal{Y} \underset{\mathcal{Z}}{\times} S, -)$$

is conservative by 1-affineness of f. But now base-change and this conservativity imply that the pullback of φ to $\mathcal{Y} \underset{\mathcal{Z}}{\times} S$ is an equivalence, giving the result after further restriction to S.

□

Proof of Lemma 19.10.2. Because f_* commutes with arbitrary colimits by Proposition 19.9.1 and is conservative by Lemma 19.10.3, Barr-Beck implies that we have:

$$f_* f^*\text{–}\mathsf{mod}(\mathsf{ShvCat}_{/\mathcal{Z}}) \simeq \mathsf{ShvCat}_{/\mathcal{Y}}.$$

Therefore, we deduce:

$$\mathsf{ShvCat}_{/\mathcal{Y}} = f_*(\mathsf{QCoh}_{/\mathcal{Y}})\text{–}\mathsf{mod}(\mathsf{ShvCat}_{/\mathcal{Z}}) \overset{\Gamma}{\simeq} \mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod}(\mathsf{QCoh}(\mathcal{Z})\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})) = \mathsf{QCoh}(\mathcal{Y})\text{–}\mathsf{mod}$$

as desired.

□

Corollary 19.10.4. *For any pair of 1-affine prestacks $\mathcal{Y}$ and $\mathcal{Z}$, the product $\mathcal{Y} \times \mathcal{Z}$ is 1-affine.*

Proof. It suffices to show that the projection $\mathcal{Y} \times \mathcal{Z} \to \mathcal{Y}$ is 1-affine. By the definition, we reduce showing that in the case where S is an affine scheme, $S \times \mathcal{Z}$ is a 1-affine prestack.

Note that the morphism $S \times \mathcal{Z} \to \mathcal{Z}$ is affine and therefore 1-affine, so the result follows from Lemma 19.10.2.

□

Corollary 19.10.5. *A prestack $\mathcal{Y}$ is 1-affine if and only if the structure map $\mathcal{Y} \to \operatorname{Spec}(k)$ is 1-affine.*

Proposition 19.10.6. *Given a commutative diagram of prestacks:*

$$\begin{tikzcd} \mathcal{W} \arrow[rr, "g"] \arrow[dr, "f\circ g"'] & & \mathcal{Y} \arrow[dl, "f"] \\ & \mathcal{Z} & \end{tikzcd}$$

with f and $f \circ g$ 1-affine, and such that the diagonal $\Delta_f : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ is 1-affine, the morphism g is 1-affine.

Proof. Applying base-change by any map to $\mathcal{Z}$ from an affine scheme, we reduce to showing in the case $\mathcal{Z} = S \in \mathsf{AffSch}$ that $g : \mathcal{W} \to \mathcal{Y}$ is 1-affine.

The graph morphism $\mathcal{W} \to \mathcal{W} \times_S \mathcal{Y}$ is obtained by base-change along $\mathcal{W} \to \mathcal{Y}$ from the diagonal $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$, and therefore by assumption is 1-affine. But the morphism g factors as:

$$\mathcal{W} \to \mathcal{W} \underset{S}{\times} \mathcal{Y} \to \mathcal{Y}$$

and the second morphism is 1-affine since it is obtained by base-change from $\mathcal{W} \to S$.

□

19.11. **Correspondences.** Let $\mathsf{PreStk}_{corr;all,1\text{-}aff}$ denote the category of prestacks under correspondences of the form:

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathcal{Y} & & \mathcal{Y}' \end{tikzcd} \tag{19.11.1}$$

where β is a 1-affine morphism.

We consider $\mathsf{PreStk}_{corr;all,1\text{-}aff}$ as a symmetric monoidal category using the Cartesian monoidal structure on PreStk.

From the Gaitsgory-Rozenblyum theory [GR14] of correspondences, we obtain the following result from Proposition 19.9.1 (1)

Corollary 19.11.1. *There is a canonical lax symmetric monoidal functor:*

$$\mathsf{ShvCat}^{enh}_{/-} : \mathsf{PreStk}_{corr;all,\text{1-aff}} \to \mathsf{Cat}$$

sending a prestack $\mathcal{Y}$ *to* $\mathsf{ShvCat}_{/\mathcal{Y}}$ *and sending:*

$$\left(\begin{tikzcd} & \mathcal{H} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathcal{Y} & & \mathcal{Y}' \end{tikzcd}\right) \mapsto (\beta_* \alpha^* : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Y}'}).$$

The lax symmetric monoidal structure is given by exterior products.

19.12. **Dualizability for sheaves of categories.** Let $\mathcal{Y}$ be a fixed prestack. As in §19.3, $\mathsf{ShvCat}_{/\mathcal{Y}}$ is a symmetric monoidal category with unit $\mathsf{QCoh}_{\mathcal{Y}}$.

We will say that a sheaf of categories C on $\mathcal{Y}$ is *dualizable* if it is dualizable as an object of the symmetric monoidal category $\mathsf{ShvCat}_{/\mathcal{Y}}$. For C dualizable, we let $\mathsf{C}^\vee \in \mathsf{ShvCat}_{/\mathcal{Y}}$ denote its dual.

Proposition 19.12.1. *The sheaf of categories* $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ *is dualizable if and only if for every* $f : S \to \mathcal{Y}$ *a map from an affine scheme* S*, the category* $\Gamma(S, \mathsf{C}_S)$ *is dualizable as a DG category.*

Proof. Let S be an affine scheme. By [Gai12a] a sheaf of categories:

$$\mathsf{D} \in \mathsf{ShvCat}_{/S} = \mathsf{QCoh}(S)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$$

is dualizable if and only if $\Gamma(S, \mathsf{D})$ is dualizable as an object of DGCat_{cont}.

Restriction functors for sheaves of categories are symmetric monoidal and therefore preserve dualizability and canonically commute with passage to the dual. Therefore, we see that dualizability for $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ can be tested after pullback to any affine scheme, and now the result follows from the above.

□

Lemma 19.12.2. *For any dualizable* $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ *the functor:*

$$\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} - : \mathsf{ShvCat}_{/\mathcal{Y}} \to \mathsf{ShvCat}_{/\mathcal{Y}}$$

commutes with limits.

Proof. Combining Remark 19.3.1 with Proposition 19.12.1, we immediately reduce to the affine case, which is contained in [Gai12a].

□

Construction 19.12.3. Let $i \mapsto \mathsf{C}_i$ be an $\mathcal{I}$-shaped diagram of dualizable sheaves of categories on $\mathcal{Y}$ with each C_i is dualizable. Let $\mathsf{C} := \operatorname{colim}_{i\in\mathcal{I}} \mathsf{C}_i$ and let $\overline{\mathsf{C}} := \lim_{i\in\mathcal{I}^{op}} \mathsf{C}_i^\vee$, where the limit is taken over the duals to the structure functors.

Then there is a canonical pairing:

$$\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \overline{\mathsf{C}} \to \mathsf{QCoh}_{\mathcal{Y}} \tag{19.12.1}$$

constructed as:

$$\Big(\lim_{i\in\mathcal{I}^{op}} \mathsf{C}_i^\vee\Big) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \Big(\operatorname*{colim}_{j\in\mathcal{I}} \mathsf{C}_j\Big) = \operatorname*{colim}_{j\in\mathcal{I}} \Big(\big(\lim_{i\in\mathcal{I}^{op}} \mathsf{C}_i^\vee\big) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_j\Big) \to \operatorname*{colim}_{j\in\mathcal{I}} \Big(\mathsf{C}_j^\vee \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_j\Big) \to \mathsf{QCoh}_{\mathcal{Y}}.$$

Here the latter map is defined by compatible family of evaluation maps for each C_i.

The following result is taken from [Gai12a].

Proposition 19.12.4. *Let* $i \mapsto \mathsf{C}_i$, C *and* $\overline{\mathsf{C}}$ *be as in Construction 19.12.3.*

(1) If C *is dualizable, then* (19.12.1) *realizes* $\overline{\mathsf{C}}$ *as the dual of* C.

(2) C *is dualizable if and only if, for every* $\mathsf{D} \in \mathsf{ShvCat}_{/\mathcal{Y}}$, *the tautological map:*

$$\mathsf{D} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \overline{\mathsf{C}} \to \lim_{i\in\mathcal{I}^{op}} \Big(\mathsf{D} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_i^\vee\Big) \tag{19.12.2}$$

is an equivalence.

(3) If each functor $\mathsf{C}_i \to \mathsf{C}_j$ admits a right adjoint in $\mathsf{ShvCat}_{/\mathcal{Y}}$, then C is dualizable.

Proof. Suppose first that C is dualizable.

For every $i \in \mathcal{I}$ the coevaluation for C_i gives the canonical map:

$$\mathsf{QCoh}_{\mathcal{Y}} \to \mathsf{C}_i^\vee \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_i \to \mathsf{C}_i^\vee \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}.$$

These maps are compatible as i varies, and therefore we obtain the map:

$$\mathsf{QCoh}_{\mathcal{Y}} \to \lim_{i \in \mathcal{I}^{op}} \Big(\mathsf{C}_i^\vee \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C} \Big). \tag{19.12.3}$$

Because C is dualizable, Lemma 19.12.2 gives:

$$\big(\lim_{i \in \mathcal{I}^{op}} \mathsf{C}_i^\vee\big) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C} \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \Big(\mathsf{C}_i^\vee \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C} \Big) \tag{19.12.4}$$

so (19.12.3) gives a coevaluation map, which one easily sees defines a duality datum alongside the evaluation pairing above. This completes the proof of (1).

For (2), suppose first that C is dualizable. By (1), we have $\overline{\mathsf{C}} = \mathsf{C}^\vee$. Therefore, we see that for any $\mathsf{D}_1, \mathsf{D}_2 \in \mathsf{ShvCat}_{/\mathcal{Y}}$, we have:

$$\mathrm{Hom}(\mathsf{D}_2, \mathsf{D}_1 \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \overline{\mathsf{C}}) = \mathrm{Hom}(\mathsf{D}_2 \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}, \mathsf{D}_1) = \mathrm{Hom}\Big(\underset{i \in \mathcal{I}}{\mathrm{colim}}(\mathsf{D}_2 \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_i), \mathsf{D}_1\Big) =$$

$$\lim_{i \in \mathcal{I}^{op}} \mathrm{Hom}(\mathsf{D}_2 \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_i, \mathsf{D}_1) = \lim_{i \in \mathcal{I}^{op}} \mathrm{Hom}(\mathsf{D}_2, \mathsf{D}_1 \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}_i^\vee)$$

as desired.

For (3), note that each $\mathsf{C}_i^\vee \to \mathsf{C}_j^\vee$ then admits a left adjoint, and the limit defining C can be computed as the colimit of these categories. Now the hypothesis (2) is obviously satisfied.

$\square$

19.13. We will need the following notion in what follows:

A *pushforward structure* on a 1-affine morphism $f : \mathcal{Y} \to \mathcal{Z}$ is a morphism:

$$\varepsilon_f : f_*(\mathsf{QCoh}_{\mathcal{Y}}) \to \mathsf{QCoh}_{\mathcal{Z}} \in \mathsf{ShvCat}_{/\mathcal{Z}}.$$

We have a corresponding category PreStk^{pf} of prestacks under 1-affine morphisms equipped with pushforward structures. That is, objects are prestacks, morphisms $\mathcal{Y} \to \mathcal{Z}$ are pairs (f, ε_f) of a 1-affine morphism $\mathcal{Y} \to \mathcal{Z}$ with a pushforward structure ε_f, and compositions $\mathcal{W} \xrightarrow{(g,\varepsilon_g)} \mathcal{Y} \xrightarrow{(f,\varepsilon_f)} \mathcal{Z}$ are computed by the map $f \circ g$ with the pushforward structure:

$$f_* g_*(\mathsf{QCoh}_{\mathcal{W}}) \xrightarrow{f_*(\varepsilon_g)} f_*(\mathsf{QCoh}_{\mathcal{Y}}) \xrightarrow{\varepsilon_f} \mathsf{QCoh}_{\mathcal{W}}.$$

We have the obvious forgetful functor $\mathsf{PreStk}^{pf} \to \mathsf{PreStk}$.

Remark 19.13.1. Suppose that $f : \mathcal{Y} \to \mathcal{Z}$ is a 1-affine morphism with a pushforward structure ε_f. Let $\mathcal{W} \to \mathcal{Z}$ be an arbitrary map. Then the base-change $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{W} \to \mathcal{W}$ inherits a canonical pushforward structure from the base-change property of Proposition 19.9.1 (2).

19.14. Next, we wish to discuss the preservation of dualizability under pushforwards of sheaves of categories.

Definition 19.14.1. A pushforward structure ε_f on a 1-affine map $f : \mathcal{Y} \to \mathcal{Z}$ is *dual-passing* if for every dualizable $\mathsf{C} \in \mathsf{ShvCat}_{/\mathcal{Y}}$, the upper horizontal arrow in the commutative diagram:

$$\begin{tikzcd} f_*(\mathsf{C}) \underset{\mathsf{QCoh}_{\mathcal{Z}}}{\otimes} f_*(\mathsf{C}^\vee) \arrow[r] \arrow[d] & \mathsf{QCoh}_{\mathcal{Z}} \\ f_*(\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{C}^\vee) \arrow[r] & f_*(\mathsf{QCoh}_{\mathcal{Y}}) \arrow[u, "\varepsilon_f"'] \end{tikzcd} \tag{19.14.1}$$

realizes $f_*(\mathsf{C})$ as dual to $f_*(\mathsf{C}^\vee)$.

We say that a map f is *dual-passing* if f is 1-affine and equipped with a dual-passing pushforward structure.

Remark 19.14.2. If f is dual-passing, then in particular f_* preserves dualizable sheaves of categories, and we have functorial identifications $f_*(\mathsf{C})^\vee \simeq f_*(\mathsf{C}^\vee)$.

Remark 19.14.3. Suppose $\mathcal{W} \xrightarrow{g} \mathcal{Y} \xrightarrow{f} \mathcal{Z}$ are dual-passing morphisms of prestacks, then the composition of these morphisms in PreStk^{pf} is readily seen to be dual-passing as well.

Therefore, we obtain the nonfull subcategory $\mathsf{PreStk}^{dp} \subseteq \mathsf{PreStk}^{pf}$ of prestacks under dual-passing morphisms (but 2-morphisms, etc. are the same in PreStk^{dp} as in PreStk^{pf}).

19.15. We now discuss the existence of dual-passing morphisms.

Proposition 19.15.1. *Suppose $\mathcal{Y}$ is a 1-affine prestack with $\mathsf{QCoh}(\mathcal{Y})$ rigid monoidal. Then the map:*

$$\Gamma(\mathcal{Y}, -) : \mathsf{QCoh}(\mathcal{Y}) \to \mathsf{Vect}$$

(necessarily continuous by rigidity) is a dual-passing pushforward structure on the structure map $\mathcal{Y} \to \mathrm{Spec}(k)$.

Proof. This is a general result about modules for rigid monoidal categories and is explained in [Gai12a].

□

Remark 19.15.2. In particular, the hypotheses of Proposition 19.15.1 are satisfied if X is a quasi-compact quasi-separated scheme.

Similarly, we have the following result.

Proposition 19.15.3. *For $f : \mathcal{Y} \to \mathcal{Z}$ a quasi-compact quasi-separated schematic morphism, the pushforward functor (c.f. Proposition 19.9.1* (3)*):*

$$f_*(\mathsf{QCoh}_{\mathcal{Y}}) \to \mathsf{QCoh}_{\mathcal{Z}}$$

is a dual-passing structure on f.

Proof. We immediately reduce to the case where $\mathcal{Z}$ is an affine scheme, where it again follows from [Gai12a].

□

Corollary 19.15.4. *Let PreStk_{qcqs} denote the category of prestacks under quasi-compact quasi-separated schematic morphisms. Then we obtain a canonical map $\mathsf{PreStk}_{qcqs} \to \mathsf{PreStk}^{dp}$ that is a (partially-defined) section of the map $\mathsf{PreStk}^{dp} \to \mathsf{PreStk}$.*

This follows because the pushforward structures $f_*(\mathsf{QCoh}_{\mathcal{Y}}) \to \mathsf{QCoh}_{\mathcal{Z}}$ are right adjoints to the tautological maps $\mathsf{QCoh}_{\mathcal{Z}} \to f_*(\mathsf{QCoh}_{\mathcal{Y}})$.

19.16. Let $i : S \hookrightarrow T$ be a closed embedding of quasi-compact quasi-separated schemes with quasi-compact complement. Let $T_S^\wedge$ be the formal completion of T along S. Recall from Theorem 19.8.1 that $T_S^\wedge$ is 1-affine.

Let $\widehat{i}$ denote the canonical map of prestacks $T_S^\wedge \to T$. Note that $\widehat{i}$ is a 1-affine morphism (this follows either directly from Theorem 19.8.1 or from Proposition 19.10.6).

According to [GR14], the restriction functor:

$$\widehat{i}^* : \mathsf{QCoh}(T) \to \mathsf{QCoh}(T_S^\wedge)$$

admits a fully-faithful *left* adjoint. We follow *loc. cit.* in denoting this functor by $\widehat{i}_?$.

By rigidity of $\mathsf{QCoh}(T)$, the functor $\widehat{i}_?$ is a morphism of $\mathsf{QCoh}(T)$-module categories. Therefore, we obtain a pushforward structure:

$$\varepsilon_{T_S^\wedge} : \widehat{i}_*(\mathsf{QCoh}_{T_S^\wedge}) \to \mathsf{QCoh}_T$$

given on global sections by $\widehat{i}_?$.

Proposition 19.16.1. *The pushforward structure $\varepsilon_{T^{\wedge}_S}$ is dual-passing.*

Proof. Recall that $\mathsf{QCoh}(T^{\wedge}_S)$ is compactly generated in this case. Therefore, we have:

$$\mathsf{QCoh}(T^{\wedge}_S) \underset{\mathsf{QCoh}(T)}{\otimes} \mathsf{QCoh}(T^{\wedge}_S) \xrightarrow{\simeq} \mathsf{QCoh}(T^{\wedge}_S \underset{T}{\times} T^{\wedge}_S) = \mathsf{QCoh}(T^{\wedge}_S).$$

For any pair $\mathcal{C}, \mathcal{D}$ of $\mathsf{QCoh}(T^{\wedge}_S)$-module categories in DGCat_{cont}, we claim that the canonical functor:

$$\mathcal{C} \underset{\mathsf{QCoh}(T)}{\otimes} \mathcal{D} \to \mathcal{C} \underset{\mathsf{QCoh}(T^{\wedge}_S)}{\otimes} \mathcal{D}$$

is an equivalence. Indeed, we immediately reduce to the case where $\mathcal{C} = \mathcal{D} = \mathsf{QCoh}(T^{\wedge}_S)$, where it follows from the above.

From here it is easy to see that for $\mathcal{C}$ dualizable the map:

$$\mathsf{QCoh}(T) \xrightarrow{\widehat{i}^*} \mathsf{QCoh}(T^{\wedge}_S) \to \mathcal{C} \underset{\mathsf{QCoh}(T^{\wedge}_S)}{\otimes} \mathcal{C}^{\vee} = \mathcal{C} \underset{\mathsf{QCoh}(T)}{\otimes} \mathcal{C}^{\vee}$$

is the desired coevaluation map to the proposed evaluation map:

$$\mathcal{C} \underset{\mathsf{QCoh}(T)}{\otimes} \mathcal{C}^{\vee} \xrightarrow{\simeq} \mathcal{C} \underset{\mathsf{QCoh}(T^{\wedge}_S)}{\otimes} \mathcal{C}^{\vee} \to \mathsf{QCoh}(T^{\wedge}_S) \xrightarrow{\widehat{i}_?} \mathsf{QCoh}(T).$$

□

19.17. We will use the following somewhat technical lemma in what follows.

Lemma 19.17.1. *Let $\mathcal{Y}_i$ be an $\mathcal{I}$-shaped diagram of prestacks with $\mathcal{Y} = \operatorname{colim} \mathcal{Y}_i$. Suppose that the structure maps $\mathcal{Y}_i \to \mathcal{Y}_j$ and $\varphi_i : \mathcal{Y}_i \to \mathcal{Y}$ have been given compatible dual-passing structures, i.e., we have a lift of the corresponding $\mathcal{I}^{\triangleright}$-shaped diagram to PreStk^{dp}.*

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a map in PreStk with a pushforward structure ε_f such that the induced pushforward structure $\varepsilon_{f \circ \pi}$ on the map $f \circ \pi : \mathcal{U} \to \mathcal{Z}$ is dual-passing.

Then ε_f is dual-passing.

Proof. Let $\mathsf{E} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ be arbitrary. Then we have an obvious identification:

$$\mathsf{E} \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\varphi_i^*(\mathsf{E}) = \lim_{i \in \mathcal{I}^{op}} \mathsf{E} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \varphi_{i,*}(\mathsf{QCoh}_{\mathcal{Y}_i}). \tag{19.17.1}$$

Applying (19.17.1) repeatedly, for $\mathsf{C}, \mathsf{D} \in \mathsf{ShvCat}_{/\mathcal{Y}}$ arbitrary, we obtain:

$$\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D} \xrightarrow{\simeq} \Big(\lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\varphi_i^*(\mathsf{C})\Big) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D}$$

$$\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D} \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\varphi_i^*(\mathsf{C} \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D}) = \lim_{i \in \mathcal{I}^{op}} \Big(\varphi_{i,*}\varphi_i^*(\mathsf{C}) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D}\Big)$$

with last equality the projection formula. Therefore, we deduce:

$$\Big(\lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\varphi_i^*(\mathsf{C})\Big) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D} \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \Big(\varphi_{i,*}\varphi_i^*(\mathsf{C}) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} \mathsf{D}\Big).$$

Suppose that C is dualizable with dual $\mathsf{C}^\vee$. Because each φ_i is assumed dual-passing, each $\mathsf{C}_i := \varphi_{i,*}\varphi_i^*(\mathsf{C})$ is dualizable with natural identifications $\mathsf{C}_i^\vee = \varphi_{i,*}\varphi_i^*(\mathsf{C})$. We see from Proposition 19.12.4 that we have a canonical identification:

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathsf{C}_i^\vee \xrightarrow{\simeq} \mathsf{C}^\vee$$

where the structure maps are the dual functors to $\mathsf{C} \to \mathsf{C}_i$.

Now let $\mathsf{D} \in \mathsf{ShvCat}_{/\mathcal{Z}}$ be fixed. Applying the projection formula repeatedly, we obtain:

$$\lim_{i \in \mathcal{I}^{op}} \Big(f_*\varphi_{i,*}\mathsf{C}_i\Big) \underset{\mathsf{QCoh}_{\mathcal{Z}}}{\otimes} \mathsf{D} = f_*\Big(\lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\mathsf{C}_i\Big) \underset{\mathsf{QCoh}_{\mathcal{Z}}}{\otimes} \mathsf{D} = f_*\Big((\lim_{i \in \mathcal{I}^{op}} \varphi_{i,*}\mathsf{C}_i) \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} f^*(\mathsf{D})\Big) =$$
$$f_*\Big(\lim_{i \in \mathcal{I}^{op}} (\varphi_{i,*}\mathsf{C}_i \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} f^*(\mathsf{D}))\Big) = \lim_{i \in \mathcal{I}^{op}} \Big(f_*(\varphi_{i,*}\mathsf{C}_i \underset{\mathsf{QCoh}_{\mathcal{Y}}}{\otimes} f^*(\mathsf{D}))\Big) = \lim_{i \in \mathcal{I}^{op}} \Big((f_*\varphi_{i,*}\mathsf{C}_i) \underset{\mathsf{QCoh}_{\mathcal{Z}}}{\otimes} \mathsf{D}\Big).$$

Therefore, we see from Proposition 19.12.4 that $f_*(\mathsf{C})$ is dualizable, and it is immediate to see from Construction 19.12.3 that the evaluation pairing is computed using the pushforward structure ε_f, as desired.

□

19.18. Next, we discuss pushforward structures in the de Rham setting.

Theorem 19.18.1. *Let S and T be two schemes of almost finite type and let $f : S_{dR} \to T_{dR}$ be a map. The corresponding pushforward structure $\varepsilon_{f,dR}$ defined by de Rham cohomology is dual-passing.*

Proof.

Step 1. First, we treat the case that $f = i_{dR}$ for $i : S \to T$ a closed embedding.

Applying base-change by any map $\varphi : T' \to T_{dR}$ from an almost finite type affine scheme, we land in the situation of Proposition 19.16.1. I.e., if $S' = T'^{,cl,red} \times_T S$ we have the Cartesian diagram:

$$\begin{tikzcd} T^{\wedge,\prime}_{S'} \arrow[r, "\widehat{i}'"] \arrow[d, "\psi"] & T' \arrow[d, "\varphi"] \\ S_{dR} \arrow[r, "i_{dR}"] & T_{dR}. \end{tikzcd} \tag{19.18.1}$$

It suffices to show that the induced pushforward structure obtained by basechange coincides with the one from Proposition 19.16.1. We will check this below, though it is surely well-known.

We will use "quasi-coherent" notation everywhere, recalling that e.g. $i_{dR}^* : \mathsf{QCoh}(T_{dR}) \simeq D(T) \to D(S) \simeq \mathsf{QCoh}(S_{dR})$ is the upper-! functor in the D-module setting. We still use the notation $i_{dR,*}$ for its left adjoint.

The Cartesian square (19.18.1) gives a base-change morphism:

$$\widehat{i}'_? \psi^* \to \varphi^* i_{dR,*} \tag{19.18.2}$$

of functors $\mathsf{QCoh}(S_{dR}) \to \mathsf{QCoh}(T')$, which we need to show is an equivalence.

Let $j : U \subseteq T'$ denote the (open) complement to S', let $\mathcal{F} \in \mathsf{QCoh}(S_{dR})$ and let $\mathcal{G} \in \mathsf{QCoh}(U)$. We see that:

$$\mathrm{Hom}_{\mathsf{QCoh}(T')}(\varphi^* i_{dR,*}(\mathcal{F}), j_*(\mathcal{G})) = \mathrm{Hom}_{\mathsf{QCoh}(U)}(j^* \varphi^* i_{dR,*}(\mathcal{F}), \mathcal{G}).$$

We have $j^*\varphi^* i_{dR,*}(\mathcal{F}) = 0$, since it is obtained by forgetting $D(U) = \mathsf{QCoh}(U_{dR}) \to \mathsf{QCoh}(U)$ of an object that is obviously zero.

Therefore, we see that $\varphi^* i_{dR,*}$ maps into the left orthogonal to $\mathsf{QCoh}(U) \subseteq \mathsf{QCoh}(T')$. This is well-known (c.f. [GR14]) to coincide with $\widehat{i}'_?(\mathsf{QCoh}(T'^{,\wedge}_{S'}))$. Therefore, by fully-faithfulness of $\widehat{i}'_?$, it suffices to show that the map (19.18.2) is an equivalence after applying $\widehat{i}'_?$, but this is obvious.

Step 2. Next, we prove the result in the case where $T = \mathrm{Spec}(k)$.

By Lemma 19.17.1 and Toen's descent theorem for sheaves of categories, we reduce to the case where S is affine (by taking a Zariski covering of S by affine schemes).

We can then take a closed embedding of S into a smooth scheme (specifically, into affine space) and then by Step 1 we reduce to the case where S is smooth.

For an integer n, let $\mathrm{DR}^n(S)$ denote the formal completion of S inside of the $(n+1)$-fold product S^{n+1}, so $[n] \mapsto \mathrm{DR}^n(S)$ is the de Rham groupoid of S (i.e., it is the Cech groupoid associated with the map $S \to S_{dR}$). Let $\psi_n : \mathrm{DR}^n(S) \to S_{dR}$ denote the canonical maps.

Then for $\mathsf{C} \in \mathsf{ShvCat}_{/S_{dR}}$, we have:

$$\Gamma(S_{dR}, \mathsf{C}) \simeq \lim_{[n]\in\mathbf{\Delta}} \Gamma(\mathrm{DR}^n(S), \psi_n^*(\mathsf{C})) = \lim_{[n]\in\mathbf{\Delta}^{inj}} \Gamma(\mathrm{DR}^n(S), \psi_n^*(\mathsf{C})) \tag{19.18.3}$$

with $\mathbf{\Delta}^{inj}$ the semisimplicial category.

By smoothness of S, we have the equivalence of augmented cosimplicial categories:

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathsf{QCoh}(\mathrm{DR}^2(S)) \arrow[d, "\Upsilon_{\mathrm{DR}^2(S)}"', "\simeq"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathsf{QCoh}(\mathrm{DR}^1(S)) \arrow[d, "\Upsilon_{\mathrm{DR}^1(S)}"', "\simeq"] \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathsf{QCoh}(S) \arrow[d, "\Upsilon_S"', "\simeq"] \arrow[r] & \mathsf{QCoh}(S_{dR}) \arrow[d, "\Upsilon_{S_{dR}}"', "\simeq"] \\ \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathsf{IndCoh}(\mathrm{DR}^2(S)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathsf{IndCoh}(\mathrm{DR}^1(S)) \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathsf{IndCoh}(S) \arrow[r] & \mathsf{IndCoh}(S_{dR}) \end{tikzcd} \tag{19.18.4}$$

where on the bottom we use upper-! functors. This is an equivalence of $\mathsf{QCoh}(\mathrm{DR}^{\bullet}(S))$-module categories.

Moreover, each of the bottom arrows (in the corresponding cosemisimplicial diagram) on the bottom row of (19.18.4) admits a left adjoint given by IndCoh-pushforward by indproperness.

Recall from [GR14] that the functors Υ intertwine the self-duality of $\mathsf{QCoh}(\mathrm{DR}^n(S))$ from Proposition 19.16.1 with Serre duality on IndCoh.

For C as above, we see that (19.18.3) is given by tensoring with the upper row, so each of the maps in the semisimplicial limit in (19.18.3) admits a left adjoint. Therefore, by [Gai12a] Lemma 2.2.2., $\Gamma(S_{dR}, \mathsf{C})$ is dualizable with dual given by the de Rham groupoid and Construction 19.12.3.

From here we immediately check that the duality is given by the pushforward structure, as desired.

Step 3. In the general case, factor $f : S_{dR} \to T_{dR}$ through its graph:

$$S_{dR} \to S_{dR} \times T_{dR} \to T_{dR}.$$

The former map is treated in Step 1, and the latter by base-change from Step 2.

□

20. The twisted arrow construction and correspondences

20.1. This appendix explains how to map into a category $\mathcal{C}_{corr}$ of correspondences in $\mathcal{C}$. The desired answer is that giving a functor $\mathcal{D} \to \mathcal{C}_{corr}$ is the same as giving a functor from the *twisted arrow category* $\mathsf{Tw}(\mathcal{D})$ of $\mathcal{D}$ to $\mathcal{C}$ with a certain property (formulated in §20.9 below).

However, there is a slight annoyance here: such a result should be formulated as an adjunction, and the domain and codomain of these functors needs to be treated carefully: correspondences are defined only for categories with fiber products, while $\mathsf{Tw}(\mathcal{C})$ generally does not have fiber products, even if $\mathcal{C}$ does (it needs to have pushouts as well).

Fortunately, this problem is essentially solved in [GR14]. We describe their solution and construct this adjunction in what follows.

Presumably this material is well-known to specialists, but we are unaware of a reference. The main construction of this section was found independently by Nick Rozenblyum.

Remark 20.1.1. This material plays a purely technical role; it is only used in the main construction of §14.

20.2. **Twisted arrows.** Let $\mathcal{C}$ be a category.

We define a simplicial groupoid $[n] \mapsto \mathsf{Tw}_{[n]}(\mathcal{C})$ by taking n-simplices the groupoid of diagrams:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & X_n \arrow[d] \\ Y_0 & Y_1 \arrow[l] & \dots \arrow[l] & Y_n \arrow[l] \end{tikzcd}$$

in $\mathcal{C}$, as equipped with its obvious simplicial structure.

More precisely: for a finite totally ordered set I, let I^{op} denote the same set with the opposite ordering. We have a functor:

$$\mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}$$

$$I \mapsto I * I^{op}$$

with the operation $*$ being the join (alias: concatenation) of two ordered sets.

The twisted arrow construction is more often given as composition with this endofunctor. This construction defines a complete Segal space $\mathsf{Tw}(\mathcal{C})$.

Remark 20.2.1. One can show that $\mathsf{Tw}(\mathcal{C})$ coincides with the twisted arrow category of $\mathcal{C}$ as defined in [Lur11a].

Remark 20.2.2. Note that the groupoid $\mathsf{Tw}_{[n]}(\mathcal{C})$ is canonically equivalent to the groupoid of composable morphisms:

$$X_0 \to X_1 \to \ldots \to X_n \to Y_n \to Y_n \to \ldots \to Y_0$$

in $\mathcal{C}$.

20.3. **Categories with directions.** We will need the following notion from [GR14].

A *category with directions* is a category $\mathcal{C}$ equipped with two classes $(hor, vert)$ of morphisms in $\mathcal{C}$, called *horizontal* and *vertical* respectively, such that:

(1) Equivalences are both horizontal and vertical.
(2) Any morphism equivalent to a horizontal (resp. vertical) morphism is horizontal (resp. vertical).[49]
(3) Horizontal and vertical morphisms are closed under compositions.
(4) Given $X \to Y$ horizontal and $Z \to Y$ vertical, their Cartesian product $X \times_Y Z$ exists, with the map $X \times_Y Z \to Z$ (resp. $X \times_Y Z \to X$) horizontal (resp. vertical).

Categories with directions form a category Cat_{dir} with morphisms functors preserving horizontal and vertical arrows and preserving Cartesian products of diagrams $X \to Z \leftarrow Y$ with $X \to Y$ horizontal and $Z \to Y$ vertical.

Example 20.3.1. Any category can be regarded as a category with directions in which horizontal arrows are allowed to be arbitrary and vertical arrows are required to be equivalences. This construction defines a fully-faithful functor $\mathsf{Cat} \hookrightarrow \mathsf{Cat}_{dir}$.

Example 20.3.2. If $\mathcal{C}$ admits fiber products, we can take horizontal and vertical maps to both be arbitrary morphisms in $\mathcal{C}$.

[49]We include this condition for clarity, but due to the conventions of §2.6, this condition is forced by our framework.

20.4. Let $\mathcal{C}$ be a category. We will construct on $\mathsf{Tw}(\mathcal{C})$ a canonical structure of category with directions.

We say that a morphism:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ Y_0 & Y_1 \arrow[l] \end{tikzcd}$$

in $\mathsf{Tw}(\mathcal{C})$ is *horizontal* if $Y_1 \to Y_0$ is an equivalence, and *vertical* if $X_0 \to X_1$ is an equivalence.

We claim that such a choice of horizontal and vertical maps in $\mathsf{Tw}(\mathcal{C})$ define the structure of category with directions on $\mathcal{C}$.

The only non-trivial condition is the base-change one, so let us verify that one. Suppose that we are given a diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d] & X \arrow[d] & W \arrow[l] \arrow[d] \\ Z & Y \arrow[l] \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd}$$

in $\mathcal{C}$ (equivalently: morphisms $W \to X \to Y \to Z$), which we regard as a diagram:

$$\Big(X \to Z\Big) \xrightarrow{vert} \Big(X \to Y\Big) \xleftarrow{hor} \Big(W \to Y\Big)$$

in $\mathsf{Tw}(\mathcal{C})$. Then one immediately verifies that $W \to Z$ is the resulting fiber product.

Indeed, giving compatible maps $(A \to B)$ to $(X \to Z)$ and $(W \to Y)$ translates to giving a diagram:

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
X \arrow[r, "\mathrm{id}_X"] \arrow[d] & X \arrow[d] & W \arrow[l] \arrow[d] \\
Z \arrow[dr] & Y \arrow[l] \arrow[r, "\mathrm{id}_Y"] & Y \arrow[dl] \\
 & B &
\end{tikzcd}$$

which is obviously the same as giving compatible maps $A \to W$ and $Z \to B$.

We therefore see that Tw upgrades to a functor:

$$\mathsf{Tw} : \mathsf{Cat} \to \mathsf{Cat}_{dir}.$$

20.5. **Grids.** We now recall the construction of correspondences following [GR14].

Define the $(1,1)$-category $\mathsf{Grid}_{[n]}$ to be the category associated with the partially ordered set of convex subsets of $[n]$.

Explicitly: objects of $\mathsf{Grid}_{[n]}$ are indexed by pairs of integers (i, j) with $0 \leqslant i \leqslant j \leqslant n$, where i is the infimum of the corresponding subset of $[n]$ and j is its supremum. There is a (unique) morphism $(i, j) \to (i', j')$ if and only if $i' \leqslant i$ and $j \leqslant j'$.

An inclusion $S \subseteq T \subseteq [n]$ is said to be *horizontal* if $\inf(S) = \inf(T)$ and *vertical* if $\sup(S) = \sup(T)$ (see (20.6.1) for the reason).

20.6. Fix a category with directions $(\mathcal{C}, hor, vert)$.

Define the groupoid $\mathsf{Grid}^{w}_{[n];hor,vert}(\mathcal{C})$ of *weak n-grids* in $\mathcal{C}$ as the groupoid of functors $\mathsf{Grid}^{op}_{[n]} \to \mathcal{C}$ sending horizontal arrows in $\mathsf{Grid}_{[n]}$ to horizontal arrows in $\mathcal{C}$, and similarly for vertical arrows.

Weak n-grids can be identified with diagrams:

$$
\begin{array}{ccccccccc}
X_{0,n} & \longrightarrow & X_{1,n} & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & X_{n,n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X_{0,n-1} & \longrightarrow & X_{1,n-1} & \longrightarrow & \dots & \longrightarrow & X_{n-1,n-1} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & & & & \\
\downarrow & & \downarrow & & & & & & \\
X_{0,1} & \longrightarrow & X_{1,1} & & & & & & \\
\downarrow & & & & & & & & \\
X_{0,0} & & & & & & & &
\end{array}
\tag{20.6.1}
$$

in $\mathcal{C}$ with the graphically horizontal arrows horizontal in $\mathcal{C}$ and similarly for vertical arrows.

We say that a weak n-grid is an *n-grid* if each of the $\big(1 + \ldots + (n-1)\big)$-commutative squares in (20.6.1) is Cartesian. We denote the groupoid of n-grids by $\mathsf{Grid}_{[n];hor,vert}(\mathcal{C})$.

As in [GR14], $[n] \mapsto \mathsf{Grid}_{[n]}(\mathcal{C})$ is a complete Segal space: the Segal condition is clear, and completeness translates to the statement that a correspondence is an equivalence if and only if each of its horizontal and vertical components are equivalences in $\mathcal{C}$. We will denote this category by $\mathcal{C}_{corr;hor,vert}$.

Example 20.6.1. In Example 20.3.1, we obtain the category $\mathcal{C}$ again. In Example 20.3.2, we obtain the category $\mathcal{C}_{corr}$.

20.7. Let $\mathcal{C}$ be a category with directions. We will construct a canonical functor:

$$
\mathsf{Tw}(\mathcal{C}_{corr;hor,vert}) \to \mathcal{C} \tag{20.7.1}
$$

of categories with directions.

We will do this at the level of Segal groupoids. As in Remark 20.2.2, the n-simplices of $\mathsf{Tw}(\mathcal{C}_{corr;hor,vert})$ are given by diagrams:

$$\begin{tikzcd}
X_{0,2n+1} \arrow[r] \arrow[d] & X_{1,2n+1} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & X_{2n+1,2n+1} \\
X_{0,2n} \arrow[r] \arrow[d] & X_{1,2n} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & X_{2n,2n} & \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots & & \\
X_{0,1} \arrow[r] \arrow[d] & X_{1,1} & & & \\
X_{0,0} & & & &
\end{tikzcd} \tag{20.7.2}$$

with all graphically horizontal arrows horizontal, similarly for vertical arrows, and all squares Cartesian. We then map this diagram to the n-composable arrows in $\mathcal{C}$:

$$X_{0,2n+1} \to X_{1,2n} \to \dots \to X_{n,n+1}.$$

One easily sees that this is compatible with simplicial structures as desired and therefore defines the desired functor (20.7.1).

Let us check that this functor is actually a functor of categories with directions.

An arrow:

$$\left(\begin{tikzcd} H_1 \arrow[r, "hor"] \arrow[d, "vert"] & Y_1 \\ X_1 & \end{tikzcd} \right) \to \left(\begin{tikzcd} H_2 \arrow[r, "hor"] \arrow[d, "vert"] & Y_2 \\ X_1 & \end{tikzcd} \right)$$

in $\mathsf{Tw}(\mathcal{C}_{corr;hor,vert})$ is the datum of a diagram:

$$
\begin{tikzcd}
X_1 & H_1 \arrow[l, "vert"'] \arrow[r, "hor"] & Y_1 \\
W \arrow[u, "vert"'] \arrow[d, "hor"] & & Z \arrow[u, "hor"'] \arrow[d, "vert"] \\
X_2 & H_2 \arrow[l, "vert"'] \arrow[r, "hor"] & Y_2
\end{tikzcd}
\tag{20.7.3}
$$

plus an isomorphism:

$$
H_1 \simeq W \times_{X_2} H_2 \times_{Y_2} Z \tag{20.7.4}
$$

as objects over both X_1 and Y_1.

We draw the diagram (20.7.3) as in (20.7.2):

$$
\begin{tikzcd}
H_1 \arrow[r] \arrow[d] & H_2 \underset{Y_2}{\times} Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Y_1. \\
W \underset{X_2}{\times} H_2 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & Y_2 & \\
W \arrow[r] \arrow[d] & X_2 & & \\
X_1 & & &
\end{tikzcd}
$$

We see that this diagram maps to the map $H_1 \to H_2$ in $\mathcal{C}$. Note that the map $H_1 \to H_2$ is defined by (20.7.4).

Now, the diagram (20.7.3) is horizontal if the correspondence Z is an equivalence, i.e., if both maps $Z \to Y_1$ and $Z \to Y_2$ are equivalences.

Then we have an isomorphism $H_1 \simeq W \times_{X_2} H_2$. Therefore, we see that the morphism $H_1 \to H_2$ is horizontal in this case, since $W \to X_2$ is horizontal and we are base-changing along the vertical map $H_2 \to X_2$.

20.8. Next, we will construct a canonical map:

$$\mathcal{C} \to \mathsf{Tw}(\mathcal{C})_{corr;hor,vert} \tag{20.8.1}$$

with *hor* and *vert* defined as in §20.4, i.e., for any twisted arrow category.

We map n-composable arrows:

$$X_0 \to X_1 \to \ldots \to X_n$$

in $\mathcal{C}$ to the diagram (20.6.1) with $X_{i,j}$ the induced morphism $\big(X_i \to X_j\big) \in \mathsf{Tw}(\mathcal{C})$, i.e., the diagram:

$$\begin{tikzcd}
\big(X_0 \to X_n\big) \arrow[r] \arrow[d] & \big(X_1 \to X_n\big) \arrow[r] \arrow[d] & \ldots \arrow[r] \arrow[d] & \ldots \arrow[r] \arrow[d] & \big(X_n \xrightarrow{\mathrm{id}} X_n\big) \\
\big(X_0 \to X_{n-1}\big) \arrow[r] \arrow[d] & \big(X_1 \to X_{n-1}\big) \arrow[r] \arrow[d] & \ldots \arrow[r] \arrow[d] & \big(X_{n-1} \xrightarrow{\mathrm{id}} X_{n-1}\big) & \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots & & \\
\big(X_0 \to X_1\big) \arrow[r] \arrow[d] & \big(X_1 \xrightarrow{\mathrm{id}} X_1\big) & & & \\
\big(X_0 \xrightarrow{\mathrm{id}} X_0\big) & & & &
\end{tikzcd}$$

in $\mathsf{Tw}(\mathcal{C})$. Note that all the graphically horizontal maps here are actually horizontal in $\mathsf{Tw}(\mathcal{C})$, and similarly for vertical maps.

This construction is compatible with simplicial structures and therefore defines the desired functor (20.8.1).

20.9. Note that the morphisms (20.7.1) and (20.8.1) are functorial in $\mathcal{C}$. One readily verifies that they define the unit and counit of an adjunction:

$$\mathsf{Cat} \underset{(-)_{corr;hor,vert}}{\overset{\mathsf{Tw}(-)}{\rightleftarrows}} \mathsf{Cat}_{dir}.$$

In particular, we see that for a category $\mathcal{C}$ with fiber products and a category $\mathcal{D}$, we have canonical identifications of the category of functors $\mathcal{D} \to \mathcal{C}_{corr}$ and the category of functors $\mathsf{Tw}(\mathcal{D}) \to \mathcal{C}$ such that, for every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}$, the square:

$$\begin{tikzcd}
(X \xrightarrow{g\circ f} Z) \arrow[r] \arrow[d] & (Y \xrightarrow{g} Z) \arrow[d] \\
(X \xrightarrow{f} Y) \arrow[r] & (Y \xrightarrow{\mathrm{id}_Y} Y)
\end{tikzcd}$$

in $\mathsf{Tw}(\mathcal{D})$ maps to a Cartesian square in $\mathcal{C}$. Indeed, unwinding the definitions, we find that this condition is equivalent to the requirement that those Cartesian squares in $\mathsf{Tw}(\mathcal{D})$ that are the base-change of a horizontal map by a vertical map should map to Cartesian squares in $\mathcal{C}$.

Remark 20.9.1. The functors obviously commute with products of categories (where the product of categories with directions is a category with directions in the obvious way), and therefore we have similar endofunctors e.g. of the category of symmetric monoidal categories, and a similar adjunction.

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