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A COROLLARY OF THE B-FUNCTION LEMMA

A. BEILINSON AND D. GAITSGORY

1. The statement

1.1. Let X be a smooth algebraic variety over an algebraically closed field k of characteristic 0. Let f be a function on X; let Y be the locus of zeros of f, and $j : U \hookrightarrow X$ the open embedding of the complement of Y. Let D_X be the sheaf of differential operators on X, and let $\mathcal{M}$ be a holonomic (left) D-module on U.

Let us tensor D_X with the ring of polynomials in one variable $k[s]$. I.e., let us consider the sheaf $\mathrm{D}_X[s]$, and the corresponding category of (left) $\mathrm{D}_X[s]$-modules (we follow the conventions in the theory of D-modules, where we only consider sheaves of D_X- or $\mathrm{D}_X[s]$-modules that are quasi-coherent as sheaves of $\mathcal{O}_X$-modules).

Consider now the $\mathrm{D}_U[s]$-module "f^s". By definition, as $\mathcal{O}_U[s]$ module, it is free of rank one with the generator that we denote f^s, and vector fields acting on it by the formula

$$\xi(f^s) = s \cdot \xi(f) \cdot f^{s-1},$$

where $f^{s-1} := f^{-1} \cdot f^s$.

Consider the $\mathrm{D}_U[s]$-module $\mathcal{M} \otimes$ "f^s" $:= \mathcal{M} \underset{\mathcal{O}_U}{\otimes}$ "f^s", and the $\mathrm{D}_X[s]$-module

$$j_*(\mathcal{M} \otimes \text{“}f^s\text{”}).$$

It is easy to see that in general $j_*(\mathcal{M} \otimes \text{“}f^s\text{”})$ is not finitely generated as a $\mathrm{D}_X[s]$-module:

Example. Consider $X = \mathbb{A}^1 := \mathrm{Spec}(k[t])$, $f = t$, $\mathcal{M} = \mathcal{O}_X$. Let $\widetilde{\mathcal{M}}$ be the $\mathrm{D}_X[s]$-submodule of $j_*(\text{“}f^s\text{”})$, generated by the section f^s. It is easy to see that we have an isomorphism

$$j_*(\text{“}f^s\text{”})/\widetilde{\mathcal{M}} \simeq \underset{n=0,1,2,\ldots}{\oplus} \big(\delta_0 \otimes (k[s]/s-n)\big),$$

where δ_0 is the δ-function at $0 \in \mathbb{A}^1$, thought of as a left D-module on $\mathbb{A}^1$, and $n \in \mathbb{N}$ is regarded as a point of $k \subset \mathrm{Spec}(k[s])$.

1.2. The goal of this note is to describe the set $\mathbf{V}(\mathcal{M})$ of all $\mathrm{D}_X[s]$-submodules $\widetilde{\mathcal{M}} \subset j_*(\mathcal{M} \otimes \text{“}f^s\text{”})$, such that $j^*(\widetilde{\mathcal{M}}) = \mathcal{M} \otimes \text{“}f^s\text{”}$, and the subset $\mathbf{V}_f(\mathcal{M}) \subset \mathbf{V}(\mathcal{M})$ that corresponds to those $\widetilde{\mathcal{M}}$ that are finitely generated as $\mathrm{D}_X[s]$-modules.

For $\widetilde{\mathcal{M}} \in \mathbf{V}(\mathcal{M})$ and a point $\lambda \in k \subset \mathrm{Spec}(k[s])$ consider the D_X-module $\widetilde{\mathcal{M}}_\lambda := \widetilde{\mathcal{M}}/(s-\lambda)$. We have the canonical maps

$$j_!(\mathcal{M} \otimes \text{“}f^\lambda\text{”}) \to \widetilde{\mathcal{M}}_\lambda \to j_*(\mathcal{M} \otimes \text{“}f^\lambda\text{”}),$$

where $\mathcal{M} \otimes \text{“}f^\lambda\text{”} := \mathcal{M} \underset{\mathcal{O}_U}{\otimes} \text{“}f^\lambda\text{”}$ denotes the corresponding D-module over U.

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To state our main result, we shall adopt the following conventions. By an arithmetic progression in k we shall mean a coset of k modulo $\mathbb{Z}$. Let $\Lambda \subset k$ be a subset equal to union of finitely many arithmetic progressions. We say that some property of an element of Λ holds for $\lambda \gg 0$ (resp., $\lambda \ll 0$), if it holds for elements of the form $\lambda_0 + n$ for any fixed $\lambda_0 \in \Lambda$, whenever $n \in \mathbb{Z}$ is sufficiently large (resp., small).

We now are ready to state our theorem:

Theorem 1. *There exist a subset $\Lambda \subset k$ equal to the union of finitely many arithmetic progressions such that for any $\widetilde{\mathcal{M}} \in \mathbf{V}_f(\mathcal{M})$ we have:*

(1) *For $\lambda \notin \Lambda$ the maps*

$$j_!(\mathcal{M} \otimes \text{“}f^\lambda\text{”}) \to \widetilde{\mathcal{M}}_\lambda \to j_*(\mathcal{M} \otimes \text{“}f^\lambda\text{”})$$

are isomorphisms. In particular, $\widetilde{\mathcal{M}}_\lambda \simeq j_{!*}(\mathcal{M} \otimes \text{“}f^\lambda\text{”})$.

(2) *For $\lambda \in \Lambda$ with $\lambda \ll 0$, the map $\widetilde{\mathcal{M}}_\lambda \to j_*(\mathcal{M} \otimes \text{“}f^\lambda\text{”})$ is an isomorphism.*

(3) *For $\lambda \in \Lambda$ with $\lambda \gg 0$, the map $j_!(\mathcal{M} \otimes \text{“}f^\lambda\text{”}) \to \widetilde{\mathcal{M}}_\lambda$ is an isomorphism.*

Note that assertion of the theorem provides an algorithm for computing $j_!(\mathcal{M})$. Namely, we must pick any finitely generated submodule $\widetilde{\mathcal{M}} \subset j_*(\mathcal{M} \otimes \text{“}f^s\text{”})$, such that $j^*(\widetilde{\mathcal{M}}) \simeq \mathcal{M} \otimes \text{“}f^s\text{”}$, and

$$j_!(\mathcal{M}) \simeq \widetilde{\mathcal{M}}/s - n$$

for a sufficiently large integer n.

2. A reformulation

2.1. We shall derive Theorem 1 from a slightly more precise assertion. Before stating it, let us recall the following result, which is a well-known consequence of the b-function lemma (the proof will be recalled for completeness in the next section).

In what follows, if P is a module over $k[s]$ and λ is an element of $k \subset \mathrm{Spec}(k[s])$, we shall denote by $P_{(\lambda)}$ the localization of P at the corresponding maximal ideal, i.e., $s - \lambda$.

We are going to study $\mathrm{D}_X[s]_{(\lambda)}$-submodules $\widetilde{\mathcal{M}}_{(\lambda)} \subset j_*(\mathcal{M} \otimes \text{“}f^s\text{”})_{(\lambda)}$ such that $j^*(\widetilde{\mathcal{M}}_{(\lambda)}) = (\mathcal{M} \otimes \text{“}f^s\text{”})_{(\lambda)}$. We shall denote this set by $\mathbf{V}(\mathcal{M}_{(\lambda)})$.

Theorem 2. *For any $\lambda \in k$ the following holds:*

(A) *The $\mathrm{D}_X[s]_{(\lambda)}$-module $j_*(\mathcal{M} \otimes \text{“}f^s\text{”})_{(\lambda)}$ is finitely generated. Denote it $\widetilde{\mathcal{M}}^{max}_{(\lambda)}$.*

(B) *The set $\mathbf{V}(\mathcal{M}_{(\lambda)})$ contains the minimal element. Denote it $\widetilde{\mathcal{M}}^{min}_{(\lambda)}$. Moreover, we have:*

(B.1) *The quotient $\widetilde{\mathcal{M}}^{max}_{(\lambda)}/\widetilde{\mathcal{M}}^{min}_{(\lambda)}$ is $(s - \lambda)$-torsion.*

(B.2) *The natural map $j_!(\mathcal{M} \otimes \text{“}f^\lambda\text{”}) \to (\widetilde{\mathcal{M}}^{min}_{(\lambda)})/s - \lambda$ is an isomorphism.*

(C) *There exists a subset $\Lambda \subset k$ equal to the union of finitely many arithmetic progressions such for $\lambda \notin \Lambda$, $\widetilde{\mathcal{M}}^{min}_{(\lambda)} = \widetilde{\mathcal{M}}^{max}_{(\lambda)}$.*

2.2. The strengthening of Theorem 1 mentioned above reads as follows:

Theorem 3. *Let Λ be as above, and let $\widetilde{\mathcal{M}}$ be an element of $\mathbf{V}(\mathcal{M})$.*

(I) *For $\lambda \notin \Lambda$, the maps*

$$\widetilde{\mathcal{M}}^{min}_{(\lambda)} \to \widetilde{\mathcal{M}}_{(\lambda)} \to \widetilde{\mathcal{M}}^{max}_{(\lambda)}$$

are isomorphisms.

(II) *The map $\widetilde{\mathcal{M}}_{(\lambda)} \to \widetilde{\mathcal{M}}^{max}_{(\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$ that are $\ll 0$.*

(III) *The element $\widetilde{\mathcal{M}}$ belongs to $\mathbf{V}_f(\mathcal{M})$ if and only if the map $\widetilde{\mathcal{M}}^{min}_{(\lambda)} \to \widetilde{\mathcal{M}}_{(\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$ that are $\gg 0$.*

2.3. Let us first see some obvious implications. First, point (C) of Theorem 2 implies point (I) of Theorem 3. Combined with point (B.2) of Theorem 2, point (I) of Theorem 3 implies point (1) of Theorem 1.

Point (II) of Theorem 3 implies point (2) of Theorem 1. Point (III) of Theorem 3, combined with point (B.2) of Theorem 2 implies point (3) of Theorem 1.

Finally, the "only if" direction Theorem 3(III), combined with point (A) of Theorem 2, implies the "if" direction.

Furthermore, we have the following corollaries:

Corollary 1. *Specifying an element $\widetilde{\mathcal{M}} \in \mathbf{V}(\mathcal{M})$ is equivalent to specifying, for each $\lambda \in \Lambda$, of an element $\widetilde{\mathcal{M}}_{(\lambda)} \in \mathbf{V}(\mathcal{M}_{(\lambda)})$, such that $\widetilde{\mathcal{M}}_{(\lambda)} = \widetilde{\mathcal{M}}^{max}_{(\lambda)}$ for all λ that are $\ll 0$.*

Corollary 2. *Let $\widetilde{\mathcal{M}}^1$ and $\widetilde{\mathcal{M}}^2$ be elements of $\mathbf{V}_f(\mathcal{M})$. Then the localizations $\widetilde{\mathcal{M}}^1_{(\lambda)}$ and $\widetilde{\mathcal{M}}^2_{(\lambda)}$ coincide for all but finitely many elements $\lambda \in k$.*

2.4. We shall now give a description of the set $\mathbf{V}(\mathcal{M}_{(\lambda)})$, appearing in Corollary 1, in terms of a vanishing cycles datum. With no restriction of generality, we can assume that $\lambda = 0$.

Recall that Sect. 4.2 of [2] identifies the quotient $\widetilde{\mathcal{M}}^{max}_{(0)}/\widetilde{\mathcal{M}}^{min}_{(0)}$, which is a $\mathrm{D}_X[s]_{(0)}$-module set-theoritically supported on $Y = X - U$, with the D-module $\Psi^{nilp}(\mathcal{M})$ of nilpotent nearby cycles of $\mathcal{M}$, with the action of s on it being the nilpotent "logarithm of monodromy" operator.

Thus, elements $\mathcal{N}$ of $\mathbf{V}(\mathcal{M}_{(0)})$ are in bijection with s-stable D_X-submodules

$$\mathcal{K} \subset \Psi^{nilp}(\mathcal{M}).$$

For each $\mathcal{K}$ as above, let us describe more explicitly the corresponding D_X-module $\mathcal{N}_0 := \mathcal{N}/s$. By [1], $\mathcal{N}_0$ is completely determined by the corresponding D-module of vanishing cycles $\Phi^{nilp}(\mathcal{N}_0)$, together with maps

$$\Psi^{nilp}(\mathcal{M}) \xrightarrow{\mathbf{c}} \Phi^{nilp}(\mathcal{N}_0) \xrightarrow{\mathbf{v}} \Psi^{nilp}(\mathcal{M}),$$

such that the composition $\mathbf{v} \circ \mathbf{c} : \Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M})$ equals s.

It is easy to see that $\Phi^{nilp}(\mathcal{N}_0)$ is given in terms of $\mathcal{K}$ by either of the following two expressions:

$$\operatorname{coker}\left(\mathcal{K} \xrightarrow{\iota \oplus s} \Psi^{nilp}(\mathcal{M}) \oplus \mathcal{K}\right)$$

or

$$\ker\left(\Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}(\mathcal{M}) \xrightarrow{s\oplus\pi} \Psi^{nilp}(\mathcal{M})/\mathcal{K}\right),$$

where $\iota : \mathcal{K} \hookrightarrow \Psi^{nilp}(\mathcal{M})$ and $\pi : \Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M})/\mathcal{K}$ are the natural embedding and projection, respectively. The above kernel and co-kernel are identified by means of the map $\Psi^{nilp}(\mathcal{M}) \oplus \mathcal{K} \to \Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}$ which has the following non-zero components:

$$-s : \Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M});\ \iota : \mathcal{K} \to \Psi^{nilp}(\mathcal{M});\ \pi : \Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M})/\mathcal{K}.$$

The map $\mathbf{c}$ is the composition

$$\Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M}) \oplus \mathcal{K} \to \Phi^{nilp}(\mathcal{N}_0),$$

and the map $\mathbf{v}$ is the composition

$$\Phi^{nilp}(\mathcal{N}_0) \to \Psi^{nilp}(\mathcal{M})/\mathcal{K} \oplus \Psi^{nilp}(\mathcal{M}) \to \Psi^{nilp}(\mathcal{M}).$$

We note that the !-restriction of $\mathcal{N}_0$ to Y is then

$$\mathrm{Cone}(\Psi^{nilp}(\mathcal{M})/\mathcal{K} \xrightarrow{s} \Psi^{nilp}(\mathcal{M})/\mathcal{K})[-1],$$

and the *-restriction of $\mathcal{N}_0$ to Y is $\mathrm{Cone}(\mathcal{K} \xrightarrow{s} \mathcal{K})$.

3. Proofs

3.1. As all statements are local, we can assume that X is affine. First, let us recall the statement of the usual b-function lemma:

Lemma 1. (J. Bernstein) *Let $\mathcal{M}$ be as in Sect. 1.1, and let $m_1, ..., m_n$ be generators of $\mathcal{M}$ as a D_U-module. Then there exist elements $P_{i,j} \in \mathrm{D}_X[s]$ and an element $\mathbf{b} \in k[s]$ such that for every i*

$$\Sigma_j\, P_{i,j}(m_j \otimes f^s) = \mathbf{b} \cdot (m_i \otimes f^{s-1}).$$

Let us deduce some of the statements of Theorems 2 and 3:

3.2. First, it is clear that for $\lambda \in k$ and $n \in \mathbb{Z}$ such that

$$\Big((\lambda - n) - \mathbb{N}\Big) \cap \mathrm{roots}(\mathbf{b}) = \emptyset,$$

the elements $m_i \otimes f^{s-n}$ generate $j_*(\mathcal{M} \otimes \text{“}f^s\text{”})_{(\lambda)}$ as a $\mathrm{D}_X[s]_{(\lambda)}$-module. This implies point (A) of Theorem 2.

Set

$$\Lambda = \mathbb{Z} + \mathrm{roots}(\mathbf{b}).$$

Point (C) of Theorem 2 and point (II) of Theorem 3 follow as well.

3.3. Note that we also obtain that the $\mathrm{D}_X \otimes k(s)$-module $j_*(\mathcal{M} \otimes \text{“}f^s\text{”}) \underset{k[s]}{\otimes} k(s)$ does not have proper submodules, whose restriction to U is $(\mathcal{M} \otimes \text{“}f^s\text{”}) \underset{k[s]}{\otimes} k(s)$.

This proves point (B.1) of Theorem 2 modulo the existence of $\widetilde{\mathcal{M}}^{min}_{(\lambda)}$.

3.4. To prove point (B) of Theorem 2 and the remaining "only if" direction of Theorem 3(III), we shall use a duality argument.

Let A be a localization of a smooth k-algebra (we shall take A to be either $k[s]$ or $k[s]_{(\lambda)}$, or $k(s)$). Let $n = \dim(X)$. Consider the ring $\mathrm{D}_X \otimes A$.

Let $D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod})$ (resp., $D^b_{coh}(\text{mod-}\,\mathrm{D}_X \otimes A)$) denote the bounded derived category of left (resp., right) $\mathrm{D}_X \otimes A$-modules with coherent cohomologies.

Consider the contravariant functor

$$\mathbb{D}_A : D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod}) \to D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod}),$$

defined by composing the contravariant functor

$$\mathcal{M} \mapsto \mathrm{RHom}(\mathcal{M}, \mathrm{D}_X \otimes A),$$

which maps

$$D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod}) \to D^b_{coh}(\text{mod-}\,\mathrm{D}_X \otimes A),$$

followed by tensor product with $\omega_X^{-1}[n]$ that maps $D^b_{coh}(\text{mod-}\,\mathrm{D}_X \otimes A)$ back to $D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod})$. The same argument as in the case of usual D-modules shows that $\mathbb{D}_A \circ \mathbb{D}_A \simeq \mathrm{Id}$.

We have the following basic property of the functor $\mathbb{D}_A$: let $A \to B$ be a homomorphism between k-algebras, and let $\mathcal{N}$ be an object of $D^b_{coh}(\mathrm{D}_X \otimes A\text{-mod})$. We have:

$$\mathbb{D}_B \left(B \overset{L}{\underset{A}{\otimes}} \mathcal{N} \right) \simeq B \overset{L}{\underset{A}{\otimes}} \mathbb{D}_A(\mathcal{N}). \tag{1}$$

In particular, for $\mathcal{M} \in D^b_{coh}(\mathrm{D}_X\text{-mod})$, we have $\mathbb{D}_A(\mathcal{M} \otimes A) \simeq \mathbb{D}(\mathcal{M}) \otimes A$, where $\mathbb{D}$ denotes the usual duality on $D^b_{coh}(\mathrm{D}_X\text{-mod})$.

3.5. First, let us note that $\mathbb{D}_{k[s]}(\mathcal{M} \otimes \text{“}f^s\text{”})$ is acyclic off cohomological degree 0, and

$$\mathbb{D}_{k[s]}(\mathcal{M} \otimes \text{“}f^s\text{”}) \overset{\sigma}{\simeq} \mathbb{D}(\mathcal{M}) \otimes \text{“}f^s\text{”},$$

where σ means that the action of $k[s]$ on the two sides differs by the automorphism $\sigma : k[s] \to k[s], \sigma(s) = -s$.

Let now $\mathcal{N}$ be an element of $\mathbf{V}(\mathcal{M}_{(\lambda)})$; in particular, $\mathcal{N}$ is finitely generated over $\mathrm{D}_X[s]_{(\lambda)}$ by Theorem 2(A). We shall prove:

Lemma 2.

(a) *The* $\mathrm{D}_X[s]_{(\lambda)}$*-module* $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ *is concentrated in cohomological degree zero.*

(b) *The canonical map*

$$\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N}) \to j_*\Big(\mathbb{D}_{k[s]_{(\lambda)}}\left((\mathcal{M} \otimes \text{“}f^s\text{”})_{(\lambda)}\right)\Big) \overset{\sigma}{\simeq} j_*(\mathbb{D}(\mathcal{M}) \otimes \text{“}f^s\text{”})_{(-\lambda)}$$

is an injection.

For the proof of the lemma see Sect. 3.7 below.

3.6. **End of proofs of the theorems.** The above lemma implies point (B) of Theorem 2 and the "if" direction in Theorem 3(III):

For point (B) of Theorem 2, the sought-for submodule $\widetilde{\mathcal{M}}^{min}_{(\lambda)}$ is given by

$$\mathbb{D}_{k[s]_{(\lambda)}}\left(j_*(\mathbb{D}(\mathcal{M})\otimes \text{“}f^s\text{”})_{(-\lambda)}\right).$$

Point (B.2) follows from equation (1).

For a finitely generated submodule $\widetilde{\mathcal{M}}$ as in point (III) of Theorem 3, the map

$$\widetilde{\mathcal{M}}^{min}_{(\lambda)} \to \widetilde{\mathcal{M}}_{(\lambda)}$$

is an isomorphism whenever the corresponding map

$$(\mathbb{D}_{k[s]}(\widetilde{\mathcal{M}}))_{(-\lambda)} \to j_*(\mathbb{D}(\mathcal{M})\otimes \text{“}f^s\text{”})_{(-\lambda)}$$

is an isomorphism.

3.7. **Proof of Lemma 2.** We shall use the following corollary of Lemma 1, established in [3]:

Corollary 3. *The* $\mathrm{D}_X \otimes k(s)$*-module* $j_*(\mathcal{M}\otimes \text{“}f^s\text{”}) \underset{k[s]}{\otimes} k(s)$ *is holonomic.*

From the corollary, we obtain that non-zero cohomologies of $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ are s-torsion. Hence, to prove point (a), it is enough to show that

$$k \overset{L}{\underset{k[s]_{(\lambda)}}{\otimes}} \mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N}) \tag{2}$$

is acyclic off cohomological degree 0.

This acyclicity would also imply that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ has no s-torsion. Combined with Sect. 3.3, this would imply point (b) of the lemma as well.

Using isomorphism (1), the acyclicity of (2) is equivalent to $k \overset{L}{\underset{k[s]_{(\lambda)}}{\otimes}} \mathcal{N} =: \mathcal{N}_\lambda$ being holonomic. The latter is true for $\mathcal{N} = j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}$, since in this case $\mathcal{N}_\lambda \simeq j_*(\mathcal{M}\otimes \text{“}f^\lambda\text{”})$, which is known to be holonomic.

For any $\mathcal{N}$ we argue as follows. We note that $j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}/\mathcal{N}$, being finitely generated over $\mathrm{D}_X \otimes k[s]_{(\lambda)}$ and $(s-\lambda)$-torsion, is finitely generated over D_X. Since $(j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}/\mathcal{N})/s-\lambda$ is holonomic, being a quotient of $j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}/s-\lambda$, we obtain that $j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}/\mathcal{N}$ is itself holonomic as a D_X-module.

We have a map

$$\mathcal{N}_\lambda \to j_*(\mathcal{M}\otimes \text{“}f^\lambda\text{”}),$$

whose kernel and cokernel are subquotients of $j_*(\mathcal{M}\otimes \text{“}f^s\text{”})_{(\lambda)}/\mathcal{N}$, which implies that $\mathcal{N}_\lambda$ is holonomic as well.

□

3.8. **An alternative argument.** We can prove that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ lies in cohomological degree 0 directly, without quoting Corollary 3. Namely, we have the following general assertion that follows from the usual Nakayama lemma:

Lemma 3. *Let B be a filtered k-algebra such that $\mathrm{gr}(B)$ is a commutative finitely generated algebra over k. Let R be a localization of a commutative finitely generated k-algebra at a maximal ideal $\mathfrak{m}$. Then if $\mathcal{P}$ is a finitely generated $R \otimes B-$module, such that $\mathcal{P}/\mathfrak{m} \cdot \mathcal{P} = 0$, then $\mathcal{P} = 0$.*

Hence, Lemma 3 implies that the acyclicity of (2) implies that $\mathbb{D}_{k[s]_{(\lambda)}}(\mathcal{N})$ lies in cohomological degree 0, i.e., point (a) of Lemma 2.

In particular, we can apply Lemma 2(a) to $j_*(\mathcal{M} \otimes \text{“}f^s\text{”})$, and isomorphism (1) to the homomorphism $k[s] \to k(s)$. We conclude that $\mathbb{D}_{k(s)}\left(j_*(\mathcal{M} \otimes \text{“}f^s\text{”}) \underset{k[s]}{\otimes} k(s)\right)$ lies in cohomological degree 0, i.e., that $j_*(\mathcal{M} \otimes \text{“}f^s\text{”}) \underset{k[s]}{\otimes} k(s)$ is holonomic. This reproves Corollary 3.

References

[1] A. Beilinson, *How to glue perverse sheaves*, in: K-theory, arithmetic and geometry (Moscow, 1984–1986), pp. 42–51, Lecture Notes in Math., **1289**, Springer, Berlin, 1987.

[2] A. Beilinson, J. Bernstein, *A proof of Jantzen's conjectures*, in: I. M. Gel'fand Seminar, pp. 1–50, Adv. Soviet Math., **16**, Part 1, Amer. Math. Soc., Providence, RI, 1993.

[3] J. Bernstein, *Algebraic theory of D-modules*, lecture notes, available as .ps file at http://www.math.uchicago.edu/∼mitya/langlands.html.

A.B.: Dept. of Math, The Univ. of Chicago, 5734 University Ave., Chicago IL, 60637

D.G.: Dept. of Math., Harvard Univ., 1 Oxford str., Cambridge MA, 02138

E-mail address: `A.B.:sasha@math.uchicago.edu, D.G.:gaitsgde@math.harvard.edu`