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LOCALIZATION OF $\widehat{\mathfrak{g}}$–MODULES ON THE AFFINE GRASSMANNIAN

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Abstract. We consider the category of modules over the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ of critical level with regular central character. In our previous paper [FG2] we conjectured that this category is equivalent to the category of Hecke eigen-D-modules on the affine Grassmannian $G((t))/G[[t]]$. This conjecture was motivated by our proposal for a local geometric Langlands correspondence. In this paper we prove this conjecture for the corresponding I^0-equivariant categories, where I^0 is the radical of the Iwahori subgroup of $G((t))$. Our result may be viewed as an affine analogue of the equivalence of categories of $\mathfrak{g}$-modules and D-modules on the flag variety G/B, due to Beilinson-Bernstein and Brylinski-Kashiwara.

Introduction

0.1. Let G be a simple complex algebraic group and B its Borel subgroup. Consider the category $\mathrm{D}(G/B)$–mod of left D-modules on the flag variety G/B. The Lie algebra $\mathfrak{g}$ of G, and hence its universal enveloping algebra $U(\mathfrak{g})$, acts on the space $\Gamma(G/B, \mathcal{F})$ of global sections of any D-module $\mathcal{F}$. The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on $\Gamma(G/B, \mathcal{F})$ via the augmentation character $\chi_0 : Z(\mathfrak{g}) \to \mathbb{C}$. Let $\mathfrak{g}\text{–mod}_{\chi_0}$ be the category of $\mathfrak{g}$-modules on which $Z(\mathfrak{g})$ acts via the character χ_0. Thus, we obtain a functor

$$\Gamma : \mathrm{D}(G/B)\text{–mod} \to \mathfrak{g}\text{–mod}_{\chi_0} .$$

In [BB] A. Beilinson and J. Bernstein proved that this functor is an equivalence of categories. Moreover, they generalized this equivalence to the case of twisted D-modules, for twistings that correspond to dominant weights $\lambda \in \mathfrak{h}^*$.

Let N be the unipotent radical of B. We can consider the N–equivariant subcategories on both sides of the above equivalence. On the D-module side this is the category $\mathrm{D}(G/B)\text{–mod}^N$ of N–equivariant D-modules on G/B, and on the $\mathfrak{g}$-module side this is the block of the category $\mathcal{O}$ corresponding to the central character χ_0. The resulting equivalence of categories, which follows from [BB], and which was proved independently by J.-L. Brylinski and M. Kashiwara [BK], is very important in applications to representation theory of $\mathfrak{g}$.

Now let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra, the universal central extension of the formal loop agebra $\mathfrak{g}((t))$. Representations of $\widehat{\mathfrak{g}}$ have a parameter, an invariant bilinear form on $\mathfrak{g}$, which is called the level. There is a unique inner product κ_{can} which is normalized so that the square length of the maximal root of $\mathfrak{g}$ is equal to 2. Any other inner product is equal to $\kappa = k \cdot \kappa_{\mathrm{can}}$, where $k \in \mathbb{C}$, and so a level corresponds to a complex number k. In particular, it makes sense to speak of *integral levels*. Representations, corresponding to the bilinear form which is equal to minus one half of the Killing form (for which $k = -h^\vee$, minus the dual Coxeter number of $\mathfrak{g}$) are called representations of *critical level*. This is really the "middle point" amongst all levels (and not the zero level, as one might naively expect).

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There are several analogues of the flag variety in the affine case. In this paper (except in the Appendix) we will consider exclusively the *affine Grassmannian* $\mathrm{Gr}_G = G((t))/G[[t]]$.

Another possibility is to consider the affine flag scheme $\mathrm{Fl}_G = G((t))/I$, where I is the Iwahori subgroup of $G((t))$. Most of the results of this paper that concern the critical level have conjectural counterparts for the affine flag variety, but they are more difficult to formulate. In particular, one inevitably has to consider derived categories, whereas for the affine Grassmannian abelian categories suffice. We refer the reader to the Introduction of our previous paper [FG2] for more details.

There is a canonical line bundle $\mathcal{L}_{\mathrm{can}}$ on Gr_G such that the action of $\mathfrak{g}((t))$ on Gr_G lifts to an action of $\widehat{\mathfrak{g}}_{\kappa_{\mathrm{can}}}$ on $\mathcal{L}_{\mathrm{can}}$. For each level κ we can consider the category $\mathrm{D}(\mathrm{Gr}_G)_\kappa\text{–mod}$ of right D-modules on Gr_G twisted by $\mathcal{L}_{\mathrm{can}}^{\otimes k}$, where $\kappa = k \cdot \kappa_{\mathrm{can}}$. (Recall that although the line bundle $\mathcal{L}_{\mathrm{can}}^{\otimes k}$ only makes sense when k is integral, the corresponding category of twisted D-module is well-defined for an arbitrary k.) Since Gr_G is an ind-scheme, the definition of these categories requires some care (see [BD] and [FG1]).

Let $\widehat{\mathfrak{g}}_\kappa\text{–mod}$ be the category of (discrete) modules over the affine Kac-Moody algebra of level κ (see Sect. 1.1). Using the fact that the action of $\mathfrak{g}((t))$ on Gr_G lifts to an action of $\widehat{\mathfrak{g}}_{\kappa_{\mathrm{can}}}$ on $\mathcal{L}_{\mathrm{can}}$, we obtain that for each level κ we have a naturally defined functor of global sections:

$$\Gamma : \mathrm{D}(\mathrm{Gr}_G)_\kappa\text{–mod} \to \widehat{\mathfrak{g}}_\kappa\text{–mod}\,. \tag{1}$$

The question that we would like to address in this paper is whether this functor is an equivalence of categories, as in the finite-dimensional case.

0.2. The first results in this direction were obtained in [BD, FG1]. Namely, in *loc. cit.* it was shown that if κ is such that $\kappa = k \cdot \kappa_{\mathrm{can}}$ with $k + h^\vee \notin \mathbb{Q}^{>0}$, then the functor Γ of (1) is exact and faithful. (In contrast, it is known that this functor is not exact for $k + h^\vee \in \mathbb{Q}^{>0}$.) The condition $k + h^\vee \notin \mathbb{Q}^{>0}$ is analogous to the dominant weight condition of [BB].

Let us call κ *negative* if $k + h^\vee \notin \mathbb{Q}^{\geq 0}$. In this case one can show that the functor of (1) is fully faithful. In fact, in this case it makes more sense to consider H-monodromic twisted D-modules on the enhanced affine flag scheme $\widetilde{\mathrm{Fl}}_G = G((t))/I^0$, rather than simply twisted D-modules on Gr_G, and the corresponding functor Γ to $\widehat{\mathfrak{g}}_\kappa\text{–mod}$. The above exactness and fully-faithfulness assertions are still valid in this context. However, the above functor is not an equivalence of categories. Namely, the RHS of (1) has "many more" objects than the LHS.

When κ is integral, A. Beilinson has proposed a conjectural intrinsic description of the image of the category $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}$ inside $\widehat{\mathfrak{g}}_\kappa\text{–mod}$ (see Remark (ii) in the Introduction of [Be]). As far as we know, no such description was proposed when κ is not integral.

It is possible, however, to establish a partial result in this direction. Namely, let $I^0 \subset I$ be the unipotent radical of the Iwahori subgroup I. We can consider the category $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{I^0}$ of I^0-equivariant twisted D-modules on $\widetilde{\mathrm{Fl}}_G$. The corresponding functor Γ of global sections takes values in the affine version of category $\mathcal{O}$, i.e., in the subcategory $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{I^0} \subset \widehat{\mathfrak{g}}_\kappa\text{–mod}$, whose objects are $\widehat{\mathfrak{g}}_\kappa$-modules on which the action of the Lie algebra $\mathrm{Lie}(I^0) \subset \widehat{\mathfrak{g}}_\kappa$ integrates to an algebraic action of the group I^0.

One can show that the functor Γ induces an equivalence between an appropriately defined subcategory of H-monodromic objects of $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{I^0}$ and a specific block of $\widehat{\mathfrak{g}}_\kappa\text{–mod}^{I^0}$. This result, which is well-known to specialists, is not available in the published literature. For the sake of completeness, we sketch one of the possible proofs in the Appendix of this paper.

0.3. In this paper we shall concentrate on the case of the critical level, when $k = -h^\vee$. We will see that this case is dramatically different from the cases considered above. In [FG2] we made a precise conjecture describing the relationship between the corresponding categories $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod and $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod. We shall now review the statement of this conjecture.

First, let us note that the image of the functor Γ is in a certain subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod, singled out by the condition on the action of the center.

Let $\mathfrak{Z}_{\mathfrak{g}}$ denote the center of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod (which is the same as the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$). The fact that this center is non-trivial is what distinguishes the critical level from all other levels. Let $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ denote the quotient of $\mathfrak{Z}_{\mathfrak{g}}$, through which it acts on the vacuum module $\mathbb{V}_{\mathrm{crit}} := \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(\mathbb{C})$.

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$ be the full subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod, whose objects are $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules on which the action of the center $\mathfrak{Z}_{\mathfrak{g}}$ factors through $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$. It is known (see [FG1]) that for any $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod, the space of global sections $\Gamma(\mathrm{Gr}_G, \mathcal{F})$ is an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$. (Here and below we write $M \in \mathcal{C}$ if M is an object of a category $\mathcal{C}$.) Thus, $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$ is the category that may be viewed as an analogue of the category $\mathfrak{g}$–mod_{χ_0} appearing on the representation theory side of the Beilinson-Bernstein equivalence.

However, the functor of global sections $\Gamma : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ is not full, and therefore cannot possibly be an equivalence. The origin of the non-fullness of Γ is two-fold, with one ingredient rather elementary, and another less so.

First, the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$ has a large center, namely, the algebra $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ itself, while the center of the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod is the group algebra of the finite group $\pi_1(G)$ (i.e., it has a basis enumerated by the connected components of Gr_G).

Second, the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod carries an additional symmetry, namely, an action of the tensor category $\mathrm{Rep}(\check{G})$ of the Langlands dual group $\check{G}$, and this action trivializes under the functor Γ.

In more detail, let us recall that, according to [FF, F], we have a canonical isomorphism between $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ and the space $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$ of $\check{\mathfrak{g}}$-opers on the formal disc $\mathcal{D}$ (we refer the reader to Sect. 1 of [FG2] for the definition and a detailed review of opers). By construction, over the scheme $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$ there exists a canonical principal $\check{G}$-bundle, denoted by $\mathcal{P}_{\check{G},\mathrm{Op}}$. Let $\mathcal{P}_{\check{G},\mathfrak{Z}}$ bethe $\check{G}$-bundle over $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ corresponding to it under the above isomorphism. For an object $V \in \mathrm{Rep}(\check{G})$ let us denote by $\mathcal{V}_{\mathfrak{Z}}$ the associated vector bundle over $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, i.e., $\mathcal{V}_{\mathfrak{Z}} = \mathcal{P}_{\check{G},\mathfrak{Z}} \underset{\check{G}}{\times} V$.

Consider now the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]}$. By [MV], this category has a canonical tensor structure, and as such it is equivalent to the category $\mathrm{Rep}(\check{G})$ of algebraic representations of $\check{G}$; we shall denote by

$$V \mapsto \mathcal{F}_V : \mathrm{Rep}(\check{G}) \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$$

the corresponding functor. Moreover, we have a canonical action of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–$\mathrm{mod}^{G[[t]]}$ as a tensor category on $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod by convolution functors, $\mathcal{F} \mapsto \mathcal{F} \star \mathcal{F}_V$.

A. Beilinson and V. Drinfeld [BD] have proved that there are functorial isomorphisms

$$\Gamma(\mathrm{Gr}_G, \mathcal{F} \star \mathcal{F}_V) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{V}_{\mathfrak{Z}}, \qquad V \in \mathrm{Rep}(\check{G}),$$

compatible with the tensor structure. Thus, we see that there are non-isomorphic objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod that go under the functor Γ to isomorphic objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$.

0.4. In [FG2] we showed how to modify the category $D(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, by simultaneously "adding" to it $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ as a center, and "dividing" it by the above $\mathrm{Rep}(\check{G})$-action, in order to obtain a category that can be equivalent to $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$.

This procedure amounts to replacing $D(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$ by the appropriate category of *Hecke eigen-objects*, denoted $D(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$.

By definition, an object of $D(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ is an object $\mathcal{F} \in D(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, equipped with an action of the algebra $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ by endomorphisms and a system of isomorphisms

$$\alpha_V : \mathcal{F} \star \mathcal{F}_V \xrightarrow{\sim} \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{F}, \qquad V \in \mathrm{Rep}(\check{G}),$$

compatible with the tensor structure.

We claim that the functor $\Gamma : D(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ naturally gives rise to a functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : D(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$.

This is in fact a general property. Suppose for simplicity that we have an abelian category $\mathcal{C}$ which is acted upon by the tensor category $\mathrm{Rep}(H)$, where H is an algebraic group; we denote this functor by $\mathcal{F} \mapsto \mathcal{F}\star V, V \in \mathrm{Rep}(H)$. Let $\mathcal{C}^{\mathrm{Hecke}}$ be the category whose objects are collections $(\mathcal{F}, \{\alpha_V\}_{V \in \mathrm{Rep}(H)})$, where $\mathcal{F} \in \mathcal{C}$ and $\{\alpha_V\}$ is a compatible system of isomorphisms

$$\alpha_V : \mathcal{F} \star V \xrightarrow{\sim} \underline{V} \underset{\mathbb{C}}{\otimes} \mathcal{F}, \qquad V \in \mathrm{Rep}(H),$$

where $\underline{V}$ is the vector space underlying V. One may think of $\mathcal{C}^{\mathrm{Hecke}}$ as the "de-equivariantized" category $\mathcal{C}$ with respect to the action of H. It carries a natural action of the group H: for $h \in H$, we have $h \cdot (\mathcal{F}, \{\alpha_V\}_{V \in \mathrm{Rep}(H)}) = (\mathcal{F}, \{(h \otimes \mathrm{id}_{\mathcal{F}}) \circ \alpha_V\}_{V \in \mathrm{Rep}(H)})$. The category $\mathcal{C}$ may be reconstructed as the category of H-equivariant objects of $\mathcal{C}^{\mathrm{Hecke}}$ with respect to this action, see [Ga].

Suppose that we have a functor $\mathsf{G} : \mathcal{C} \to \mathcal{C}'$, such that we have functorial isomorphisms

$$\mathsf{G}(\mathcal{F} \star V) \simeq \mathsf{G}(\mathcal{F}) \underset{\mathbb{C}}{\otimes} \underline{V}, \qquad V \in \mathrm{Rep}(H), \tag{2}$$

compatible with the tensor structure. Then, according to [AG], there exists a functor $\mathsf{G}^{\mathrm{Hecke}} : \mathcal{C}^{\mathrm{Hecke}} \to \mathcal{C}'$ such that $\mathsf{G} \simeq \mathsf{G}^{\mathrm{Hecke}} \circ \mathrm{Ind}$, where the functor $\mathrm{Ind} : \mathcal{C} \to \mathcal{C}^{\mathrm{Hecke}}$ sends $\mathcal{F}$ to $\mathcal{F} \star \mathcal{O}_H$, where $\mathcal{O}_H$ is the regular representation of H. The functor $\mathsf{G}^{\mathrm{Hecke}}$ may be explicitly described as follows: the isomorphisms α_V and (2) give rise to an action of the algebra $\mathcal{O}_H$ on $\mathsf{G}(\mathcal{F})$, and $\mathsf{G}^{\mathrm{Hecke}}(\mathcal{F})$ is obtained by taking the fiber of $\mathsf{G}(\mathcal{F})$ at $1 \in H$.

We take $\mathcal{C} = D(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, $\mathcal{C}' = \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$, and $\mathsf{G} = \Gamma$. The only difference is that now we are working over the base $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$, which we have to take into account.

0.5. The conjecture suggested in [FG2] states that the resulting functor

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : D(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}} . \tag{3}$$

is an equivalence. In *loc. cit.* we have shown that the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$, when extended to the derived category, is fully faithful.

This conjecture has a number of interesting corollaries pertaining to the structure of the category of representations at the critical level:

Let us fix a point $\chi \in \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, and let us choose a trivialization of the fiber $\mathcal{P}_{\check{G},\chi}$ of $\mathcal{P}_{\check{G},\mathfrak{Z}}$ at χ. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi$ be the subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}$, consisting of objects, on which the center acts according to the character corresponding to χ.

Let $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ be the category, obtained from $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$, by the procedure $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{Hecke}}$ for $H = \check{G}$, described above. Our conjecture implies that we have an equivalence

$$\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod} \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi . \tag{4}$$

In particular, we obtain that for every two points $\chi, \chi' \in \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ and an isomorphism of $\check{G}$-torsors $\mathcal{P}_{\check{G},\chi} \simeq \mathcal{P}_{\check{G},\chi'}$ there exists a canonical equivalence $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\chi'}$. This may be viewed as an analogue of the translation principle that compares the subcategories $\mathfrak{g}\text{–mod}_\chi \subset \mathfrak{g}\text{–mod}$ for various central characters $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$ in the finite-dimensional case.

By taking $\chi = \chi'$, we obtain that the group $\check{G}$, or, rather, its twist with respect to $\mathcal{P}_{\check{G},\chi}$, acts on $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi$.

As we explained in the Introduction to [FG2], the conjectural equivalence of (4) fits into the general picture of local geometric Langlands correspondence.

Namely, for a point $\chi \in \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$ as above, both sides of the equivalence (4) are natural candidates for the conjectural Langlands category associated to the trivial $\check{G}$-local system on the disc $\mathcal{D}$. This category, equipped with an action of the loop group $G((t))$, should be thought of as a "categorification" of an irreducible unramified representation of the group G over a local non-archimedian field. Proving this conjecture would therefore be the first test of the local geometric Langlands correspondence proposed in [FG2].

0.6. Unfortunately, at the moment we are unable to prove the equivalence (3) in general. In this paper we will treat the following particular case:

Recall that I^0 denotes the unipotent radical of the Iwahori subgroup, and let us consider the corresponding I^0-equivariant subcategories on both sides of (3).

On the D-module side, we obtain the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$, defined in the same way as $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$, but with the requirement that the underlying D-module $\mathcal{F}$ be strongly I^0-equivariant.

On the representation side, we obtain the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$, corresponding to the condition that the action of $\mathrm{Lie}(I^0) \subset \widehat{\mathfrak{g}}_{\mathrm{crit}}$ integrates to an algebraic action of I^0.

We shall prove that the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ defines an equivalence of categories

$$\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}} . \tag{5}$$

This equivalence implies an equivalence

$$\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}^{I^0} \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_\chi \tag{6}$$

for any fixed character $\chi \in \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ and a trivialization of $\mathcal{P}_{\check{G},\chi}$ as above. In particular, we obtain the corollaries concerning the translation principle and the action of $\check{G}$ on $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_\chi$.

We remark that from the point of view of the local geometric Langlands correspondence the categories appearing in the equivalence (6) should be viewed as "categorifications" of the space of I-invariant vectors in an irreducible unramified representation of the group G over a local non-archimedian field (which is a module over the corresponding affine Hecke algebra).

Let us briefly describe the strategy of the proof. Due to the fact [FG2] that the functor in one direction in (5) is fully-faithful at the level of the derived categories, the statement of the theorem is essentially equivalent to the fact that for every object $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$ there exists an object $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ and a non-zero map $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}) \to \mathcal{M}$. We explain this in detail in Sect. 3.

We exhibit a collection of objects $\mathbb{M}_{w,\mathrm{reg}}$, numbered by elements $w \in W$, where W is the Weyl group of $\mathfrak{g}$, which are quotients of Verma modules over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, such that for every $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}^{I^0}$ we have $\mathrm{Hom}(\mathbb{M}_{w,\mathrm{reg}}, \mathcal{M}) \neq 0$ for at least one $w \in W$.

We then show (see Theorem 3.2) that each such $\mathbb{M}_{w,\mathrm{reg}}$ is isomorphic to $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_w)$ for some explicit object $\mathcal{F}^{\mathfrak{Z}}_w \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}^{I^0}$, thereby proving the equivalence (5).

0.7. It is instructive to put our results in the context of other closely related equivalences of categories.

Using the (tautological) equivalence:

$$\mathrm{D}(\mathrm{Gr}_G)\text{-mod}^{I^0} \simeq \mathrm{D}(\widetilde{\mathrm{Fl}}_G)\text{-mod}^{G[[t]]}$$

(here and below we omit the subscript κ when $\kappa = 0$) and the equivalence of Theorem 5.5, we obtain that for every negative integral level $\kappa = k \cdot \kappa_{\mathrm{can}}$ there exists an equivalence between $\mathrm{D}(\mathrm{Gr}_G)\text{-mod}^{I^0}$ and the regular block of the category $\widehat{\mathfrak{g}}_\kappa\text{-mod}^{G[[t]]}$, studied in [KL]. The latter category is equivalent, according to *loc. cit.*, to the category of modules over the quantum group $U_q^{\mathrm{res}}(\mathfrak{g})$, where $q = \exp \pi i/(k + h^\vee)$.

Using these equivalences, it was shown in [AG] that the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}\text{-mod}^{I^0}$, defined as above, is equivalent to the regular block $u_q(\mathfrak{g})\text{-mod}_0$ of the category of modules over the small quantum group $u_q(\mathfrak{g})$. The tensor product by the line bundle $\mathcal{L}_{\mathrm{can}}^{-h^\vee}$ defines an equivalence

$$\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}\text{-mod}^{I^0} \to \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod}^{I^0}$$

(but this equivalence does not, of course, respect the functor of global sections). Combining this with the equivalence of (6), we obtain the following diagram of equivalent categories:

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi^{I^0} \xleftarrow{\sim} \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod}^{I^0} \xrightarrow{\sim} u_q(\mathfrak{g})\text{-mod}_0 \,. \tag{7}$$

Recall in addition that in [ABBGM] it was shown that the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}\text{-mod}^{I^0}$ is equivalent to an appropriately defined category $\mathrm{D}(\mathcal{F}l^{\frac{\infty}{2}})^{I^0}$ of I^0-equivariant D-modules on the semi-infinite flag variety (it is defined in terms of the Drinfeld compactification $\overline{\mathrm{Bun}}_N$). Hence, we obtain another diagram of equivalent categories:

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi^{I^0} \xleftarrow{\sim} \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod}^{I^0} \xrightarrow{\sim} \mathrm{D}(\mathcal{F}l^{\frac{\infty}{2}})^{I^0}. \tag{8}$$

In particular, we obtain a functor

$$\mathrm{D}(\mathcal{F}l^{\frac{\infty}{2}})^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi^{I^0},$$

which is, moreover, an equivalence. Its existence had been predicted by B. Feigin and the first named author.

In fact, one would like to be able to define the category $\mathrm{D}(\mathcal{F}l^{\frac{\infty}{2}})$ without imposing the I^0-equivariance condition, and extend the equivalence of [ABBGM] to this more general context. Together with the equivalence of (3), this would imply the existence of the diagram

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi \xleftarrow{\sim} \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod} \xrightarrow{\sim} \mathrm{D}(\mathcal{F}l^{\frac{\infty}{2}}),$$

but we are far from being able to achieve this goal at present.

Finally, let us mention one more closely related category, namely, the derived category $D\big(\mathrm{QCoh}((\check{G}/\check{B})^{DG}\text{-mod})\big)$ of complexes of quasi-coherent sheaves over the DG-scheme

$$(\check{G}/\check{B})^{DG} := \mathrm{Spec}\Big(\mathrm{Sym}_{\mathcal{O}_{\check{G}/\check{B}}}(\Omega^1(\check{G}/\check{B})[1])\Big).$$

The above DG-scheme can be realized as the derived Cartesian product $\widetilde{\check{\mathfrak{g}}} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt}$, where $\mathrm{pt} \to \check{\mathfrak{g}}$ corresponds to the point $0 \in \check{\mathfrak{g}}$, and $\widetilde{\check{\mathfrak{g}}} = \{(x, \check{\mathfrak{b}}) | x \in \check{\mathfrak{b}} \subset \check{\mathfrak{g}}\}$ is Grothendieck's alteration.

From the results of [ABG] one can obtain an equivalence of the derived categories

$$D^b\big(\mathrm{QCoh}((\check{G}/\check{B})^{DG}\text{–mod})\big) \simeq D^b\big(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}\text{–mod}\big)^{I^0}.$$

Hence we obtain an equivalence:

$$D^b\big(\mathrm{QCoh}((\check{G}/\check{B})^{DG}\text{–mod})\big) \simeq D^b\big(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_\chi\big)^{I^0}. \tag{9}$$

The existence of such an equivalence follows from the Main Conjecture 6.11 of [FG2]. Note that, unlike the other equivalences mentioned above, it does not preserve the t-structures, and so is inherently an equivalence of derived categories.

0.8. **Contents.** Let us briefly describe how this paper is organized:

In Sect. 1, after recalling some previous results, we state the main result of this paper, Theorem 1.7. In Sect. 2 we review representation-theoretic corollaries of Theorem 1.7. In Sect. 3 we show how to derive Theorem 1.7 from Theorem 3.2, and in Sect. 4 we prove Theorem 3.2.

Finally, in the Appendix, we prove a partial localization result at the negative level mentioned in Sect. 0.2.

The notation in this paper follows that of [FG2].

1. The Hecke category

In this section we recall the main definitions and state our main result. We will rely on the concepts introduced in our previous paper [FG2].

1.1. **Recollections.** Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and G the connected algebraic group of adjoint type with the Lie algebra $\mathfrak{g}$. We shall fix a Borel subgroup $B \subset G$. Let $\check{G}$ denote the Langlands dual group of G, and by $\check{\mathfrak{g}}$ its Lie algebra.

Let $\mathrm{Gr}_G = G((t))/G[[t]]$ be the affine Grassmannian associated to G. We denote by $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$ the category of critically twisted right D-modules on the affine Grassmannian and by $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$ the corresponding $G[[t]]$-equivariant category. Recall that via the geometric Satake equivalence (see [MV]) the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$ has a natural structure of tensor category under convolution, and as such it is equivalent to $\mathrm{Rep}(\check{G})$. We shall denote by $V \mapsto \mathcal{F}_V$ the corresponding tensor functor $\mathrm{Rep}(\check{G}) \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$.

We have the convolution product functors

$$\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}, \mathcal{F}_V \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]} \mapsto \mathcal{F} \star \mathcal{F}_V \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}.$$

These functors define an action of $\mathrm{Rep}(\check{G})$, on the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$. Thus, in the terminology of [Ga], $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$ has the structure of category over the stack $\mathrm{pt}/\check{G}$.

Now let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ be the *affine Kac-Moody algebra* associated to the critical inner product $-h^\vee \kappa_{\mathrm{can}}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}$ the category of discrete $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules (see [FG2]). Its objects are $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules in which every vector is annihilated by the Lie subalgebra $\mathfrak{g} \otimes t^n\mathbb{C}[[t]]$ for sufficiently large n. Let $\mathbb{V}_{\mathrm{crit}} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}$ be the vacuum module $\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(\mathbb{C})$. Denote by $\mathfrak{Z}_\mathfrak{g}$ the topological commutative algebra that is the center of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}$. Let $\mathfrak{Z}^{\mathrm{reg}}_\mathfrak{g}$ denote its "regular" quotient, i.e., the quotient modulo the annihilator of $\mathbb{V}_{\mathrm{crit}}$. We denote by $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ the full subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}$, consisting of objects, on which the action of the center $\mathfrak{Z}_\mathfrak{g}$ factors through $\mathfrak{Z}^{\mathrm{reg}}_\mathfrak{g}$.

Recall that via the Feigin-Frenkel isomorphism [FF, F], the algebra $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ is identified with the algebra of regular functions on the scheme $\mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$ of $\check{\mathfrak{g}}$-opers on the formal disc $\mathcal{D}$. In particular, $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ carries a canonical $\check{G}$-torsor, denoted $\mathcal{P}_{\check{G},\mathfrak{Z}}$, whose fiber $\mathcal{P}_{\check{G},\chi}$ at $\chi \in \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$ is the fiber of the $\check{G}$-torsor underlying the oper χ at the origin of the disc $\mathcal{D}$. The $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ gives rise to a morphism $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathrm{pt}/\check{G}$. We shall denote by

$$V \mapsto \mathcal{V}_{\mathfrak{Z}}$$

the resulting tensor functor from $\mathrm{Rep}(\check{G})$ to the category of locally free $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-modules.

We define $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{-mod}$ as the fiber product category

$$\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}),$$

in the terminology of [Ga].

Explicitly, $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{-mod}$ has as objects the data of $(\mathcal{F}, \alpha_V,\ \forall\, V \in \mathrm{Rep}(\check{G}))$, where $\mathcal{F}$ is an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$, endowed with an action of the algebra $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ by endomorphisms, and α_V are isomorphisms of D-modules

$$\mathcal{F} \star \mathcal{F}_V \simeq \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F},$$

compatible with the action of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ on both sides, and such that the following two conditions are satisfied:

- For V being the trivial representations $\mathbb{C}$, the morphism α_V is the identity map.
- For $V, W \in \mathrm{Rep}(\check{G})$ and $U := V \otimes W$, the diagram

$$\begin{CD}
(\mathcal{F} \star \mathcal{F}_V) \star \mathcal{F}_W @>{\sim}>> \mathcal{F} \star \mathcal{F}_U \\
@V{\alpha_V \star \mathrm{id}_{\mathcal{F}_W}}VV @V{\alpha_U}VV \\
(\mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F}) \star \mathcal{F}_W @. \mathcal{U}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F} \\
@V{\sim}VV @V{\sim}VV \\
\mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} (\mathcal{F} \star \mathcal{F}_W) @>{\mathrm{id}_{\mathcal{V}_{\mathfrak{Z}}} \otimes \alpha_W}>> \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{W}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{F}
\end{CD}$$

  is commutative.

Morphisms in this category between $(\mathcal{F}, \alpha_V)$ and $(\mathcal{F}', \alpha'_V)$ are maps of D-modules $\phi : \mathcal{F} \to \mathcal{F}'$ that are compatible with the actions of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ on both sides, and such that

$$(\mathrm{id}_{\mathcal{V}_{\mathfrak{Z}}} \otimes \phi) \circ \alpha_V = \alpha'_V \circ (\phi \star \mathrm{id}_{\mathcal{F}_V}).$$

1.2. **Definition of the functor.** Recall that according to [FG1], the functor of global sections

$$\mathcal{F} \mapsto \Gamma(\mathrm{Gr}_G, \mathcal{F})$$

defines an exact and faithful functor $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$. Let us recall, following [FG2], the construction of the functor

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}.$$

First, let us recall the following result of [BD] (combined with an observation of [FG2], Lemma 8.4.3):

Theorem 1.3.

(1) *For* $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod *and* $V \in \mathrm{Rep}(\check{G})$ *we have a functorial isomorphism*

$$\beta_V : \Gamma(\mathrm{Gr}_G, \mathcal{F} \star \mathcal{F}_V) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}_{\mathfrak{Z}}.$$

(2) *For* $\mathcal{F}, V$ *as above and* $W \in \mathrm{Rep}(\check{G})$*,* $U := V \otimes W$ *the diagram*

$$\begin{array}{ccc}
\Gamma(\mathrm{Gr}_G, (\mathcal{F} \star \mathcal{F}_V) \star \mathcal{F}_W) & \xrightarrow{\sim} & \Gamma(\mathrm{Gr}_G, \mathcal{F} \star (\mathcal{F}_V \star \mathcal{F}_W)) \\
{\scriptstyle \beta_W}\big\downarrow & & \big\downarrow{\scriptstyle \sim} \\
\Gamma(\mathrm{Gr}_G, (\mathcal{F} \star \mathcal{F}_V)) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{W}_{\mathfrak{Z}} & & \Gamma(\mathrm{Gr}_G, \mathcal{F} \star \mathcal{F}_U) \\
{\scriptstyle \beta_V}\big\downarrow & & \big\downarrow{\scriptstyle \beta_U} \\
\Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{W}_{\mathfrak{Z}} & \xrightarrow{\sim} & \Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{U}_{\mathfrak{Z}}
\end{array}$$

is commutative.

Consider the scheme $\mathsf{Isom}_{\mathfrak{Z}} : \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$. Let $\mathbf{1}_{\mathsf{Isom}_{\mathfrak{Z}}}$ denote the unit section $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathsf{Isom}_{\mathfrak{Z}}$.

Let us denote by $R_{\mathfrak{Z}}$ the direct image of the structure sheaf under $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathrm{pt}/\check{G}$, viewed as an object of $\mathrm{Rep}(\check{G})$. It carries an action of $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ by endomorphisms. Let $\mathcal{R}_{\mathfrak{Z}}$ be the associated (infinite-dimensional) vector bundle over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$; by definition, we have a canonical isomorphism

$$\mathcal{R}_{\mathfrak{Z}} \simeq \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}).$$

We will think of the projection $p_r : \mathsf{Isom}_{\mathfrak{Z}} \to \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ as corresponding to the original $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-action on $R_{\mathfrak{Z}}$, and hence on $\mathcal{R}_{\mathfrak{Z}}$, by the transport of structure. We will think of the other projection $p_l : \mathsf{Isom}_{\mathfrak{Z}} \to \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$, as corresponding to the $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-module structure on $\mathcal{R}_{\mathfrak{Z}}$ coming from the fact that this is a vector bundle associated to a $\check{G}$-representation.

We claim that for every object $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$–mod, the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module $\Gamma(\mathrm{Gr}_G, \mathcal{F}^H)$ carries a natural action of the algebra $\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})$ by endomorphisms.

First, note that $\Gamma(\mathrm{Gr}_G, \mathcal{F}^H)$ is a $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-bimodule: we shall refer to the $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-action coming from its action on any object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–$\mathrm{mod}_{\mathrm{reg}}$ as "right", and to the one. coming from the $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-action on $\mathcal{F}^H$ as "left".

On the one hand, we have:

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}^H \star \mathcal{F}_{R_{\mathfrak{Z}}}) \overset{\beta_{R_{\mathfrak{Z}}}}{\simeq} \Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \underset{r, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}, l}{\otimes} \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}),$$

and on the other hand,

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}^H \star \mathcal{F}_{R_{\mathfrak{Z}}}) \overset{\alpha_{R_{\mathfrak{Z}}}}{\simeq} \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}) \underset{l, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}, l}{\otimes} \Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \otimes \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}).$$

By composing we obtain the desired action map

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{r, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}, l}{\otimes} \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}) \xrightarrow{\alpha_{R_{\mathfrak{Z}}} \circ \beta_{R_{\mathfrak{Z}}}^{-1}} \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}) \underset{l, \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}, l}{\otimes} \Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \xrightarrow{\mathbf{1}_{\mathsf{Isom}_{\mathfrak{Z}}}^*} \Gamma(\mathrm{Gr}_G, \mathcal{F}^H).$$

The fact that it is associative follows from the second condition on α_V and Theorem 1.3(2).

We define the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ by

$$\mathcal{F}^H \mapsto \Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \underset{\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}), \mathbf{1}^*_{\mathsf{Isom}_{\mathfrak{Z}}}}{\otimes} \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}.$$

Since the functor Γ is exact, the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ is evidently right-exact, and we will denote by $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ its left derived functor $D^-(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}) \to D^-(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}})$

The following was established in [FG2], Theorem 8.7.1:

Theorem 1.4. *The functor* $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$*, restricted to* $D^b(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod})$*, is fully faithful.*

In [FG2] we formulated the following

Conjecture 1.5. *The functor* $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ *is exact and defines an equivalence of categories* $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ *and* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$.

1.6. **The statement of the main result.** Recall that both categories $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ carry a natural action of the group $G((t))$ (see [FG2], Sect. 22, where this is discussed in detail). Let $I \subset G[[t]]$ be the Iwahori subgroup, the preimage of the Borel subgroup $B \subset G$ in $G[[t]]$ under the evaluation map $G[[t]] \to G$. Let I^0 be the unipotent radical of I. Let us denote by $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$ the corresponding categories if I-equivariant objects. Since I^0 is connected, these are full subcategories in $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$, respectively.

The functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$ induces a functor $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$. The goal of the present paper is to prove the following:

Theorem 1.7.
(1) *For any* $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ *we have* $L^i\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H) = 0$ *for all* $i > 0$.
(2) *The functor*

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$$

is an equivalence of categories.

This is a special case of Conjecture 1.5.

2. Corollaries of the main theorem

We shall now discuss some applications of Theorem 1.7. Note that both sides of the equivalence stated in Theorem 1.7 are categories over the algebra $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$.

2.1. **Specialization to a fixed central character.** Let us fix a point $\chi \in \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, i.e., a character of $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$, and consider the subcategories on both sides of the equivalence of Theorem 1.7(2), corresponding to objects on which the center acts according to this character. Let us denote the resulting subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$ by $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\chi}$. The resulting subcategory of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ can be described as follows.

Let us denote by $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ the category, whose objects are the data of $(\mathcal{F}, \alpha_V)$, where $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$ and α_V are isomorphisms of D-modules defined for every $V \in \mathrm{Rep}(\check{G})$

$$\mathcal{F} \star \mathcal{F}_V \simeq \underline{V} \underset{\mathbb{C}}{\otimes} \mathcal{F},$$

where $\underline{V}$ denotes the vector space underlying the representation V. These isomorphisms must be compatible with tensor products of objects of $\mathrm{Rep}(\check{G})$ in the same sense as in the definition of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$.

Note that $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ carries a natural weak action of the algebraic group $\check{G}$: [1] Given an S-point $\mathbf{g}$ of $\check{G}$ and an S-family of objects $(\mathcal{F},\alpha_V)$ of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ we obtain a new S-family by keeping $\mathcal{F}$ the same, but replacing α_V by $\mathbf{g}\cdot\alpha_V$, where $\mathbf{g}$ acts naturally on $\underline{V}\otimes\mathcal{O}_S$.

In addition, $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ carries a commuting Harish-Chandra action of the group $G((t))$; in particular, the subcategory $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}^{I^0}$ makes sense.

Let $\mathcal{P}_{\check{G},\chi}$ be the fiber of the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ at χ. Tautologically we have:

Lemma 2.2.
(1) *For every trivialization $\gamma:\mathcal{P}_{\check{G},\chi}\simeq\mathcal{P}^0_{\check{G}}$ there exists a canonical equivalence respecting the action of $G((t))$*

$$\left(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}\right)_\chi\simeq\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod},$$

where the LHS denotes the fiber of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ at χ.

(2) *If $\gamma'=\mathbf{g}\cdot\gamma$ for $\mathbf{g}\in\check{G}$, the above equivalence is modified by the self-functor of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$, given by the action of $\mathbf{g}$.*

Hence, from Theorem 1.7 we obtain:

Corollary 2.3. *For every trivialization $\gamma:\mathcal{P}_{\check{G},\chi}\simeq\mathcal{P}^0_{\check{G}}$ there exists a canonical equivalence*

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_\chi\simeq\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}^{I^0}.$$

From Corollary 2.3 we obtain:

Corollary 2.4.
(1) *For any two points $\chi_1,\chi_2\in\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ and an isomorphism of $\check{G}$-torsors $\mathcal{P}_{\check{G},\chi_1}\simeq\mathcal{P}_{\check{G},\chi_2}$ there exists a canonical equivalence*

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\chi_1}\simeq\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\chi_2}.$$

(2) *For every $\chi\in\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, the group of automorphisms of the $\check{G}$-torsor $\mathcal{P}_{\check{G},\chi}$ acts on the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_\chi$.*

More generally, let S be an affine scheme, and let $\chi_{1,S}$ and $\chi_{2,S}$ be two S-points of $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{S,1}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{S,2}$ be the corresponding base-changed categories.

By definition, the objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{i,S}$ are the objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$, endowed with an action of $\mathcal{O}_S$ compatible with the initial action of $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ on $\mathcal{M}$ via the homomorphism $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}\to\mathcal{O}_S$, corresponding to $\chi_{i,S}$. Morphisms in this category are $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-morphisms compatible with the action of $\mathcal{O}_S$.

We obtain:

Corollary 2.5. *For every lift of the map*

$$(\chi_{1,S}\times\chi_{2,S}):S\to\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})\times\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$$

to a map $S\to\mathsf{Isom}_{\mathfrak{Z}}$, there exists a canonical equivalence

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{S,1}\simeq\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{S,2}.$$

[1] We refer the reader to [FG2], Sect. 20.1, where this notion is introduced.

2.6. **Description of irreducibles.** Corollary 2.3 allows to describe explicitly the set of irreducible objects in $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}^{I^0}$. For that we will need to recall some more notation related to the categories $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{-mod}$ and $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{-mod}$.

Consider the forgetful functor $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{-mod} \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$. It admits a left adjoint, denoted $\mathrm{Ind}^{\mathrm{Hecke}}$, which can be described as follows.

Let R be the object of $\mathrm{Rep}(\check{G})$ equal to $\mathcal{O}_{\check{G}}$ under the left regular action; let $\mathcal{F}_R$ denote the corresponding object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}^{G[[t]]}$. Then for $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$, the convolution $\mathcal{F}\star\mathcal{F}_R$ is naturally an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{-mod}$, and it is easy to see that $\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}) := \mathcal{F}\star\mathcal{F}_R$ is the desired left adjoint.

Similarly, the forgetful functor $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{-mod} \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$ admits a left adjoint functor $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ given by $\mathcal{F} \mapsto \mathcal{F}\star\mathcal{F}_{R_{\mathfrak{Z}}}$. The next assertion follows from the definitions:

Lemma 2.7.

(1) *For* $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$ *there exist canonical isomorphisms:*

$$\Gamma(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})) \simeq \Gamma(\mathrm{Gr}, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}),$$

where $\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})$ *is a module over* $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ *via one of the projections* $\mathsf{Isom}_{\mathfrak{Z}} \to \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$.

(2) *For* $\mathcal{F}$ *as above,*

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}\left(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})\right) \simeq \Gamma(\mathrm{Gr}, \mathcal{F}).$$

Let us now recall the description of irreducible objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{-mod}^{I^0}$, established in [ABBGM], Corollary 1.3.10.

Recall that I-orbits on Gr_G are parameterized by the set W_{aff}/W, where W_{aff} denotes the extended affine Weyl group. For an element $\widetilde{w} \in W_{\mathrm{aff}}$ let us denote by $\mathrm{IC}_{\widetilde{w},\mathrm{Gr}_G}$ the corresponding irreducible object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}^I$.

For an element $w \in W$, let $\check{\lambda}_w \in W_{\mathrm{aff}}$ denote the unique dominant coweight satisfying:

$$\begin{cases} \langle \alpha_\imath, \check{\lambda}\rangle = 0 \text{ if } w(\alpha_\imath) \text{ is positive, and} \\ \langle \alpha_\imath, \check{\lambda}\rangle = 1 \text{ if } w(\alpha_\imath) \text{ is negative,} \end{cases}$$

for $\imath$ running over the set of vertices of the Dynkin diagram.

It was shown in *loc. cit.* that the objects $\mathrm{Ind}^{\mathrm{Hecke}}(\mathrm{IC}_{w\cdot\lambda_w})$ for $w \in W$ are the irreducibles of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{-mod}^{I^0}$.

Combining this with Lemma 2.7 and Corollary 2.3, we obtain:

Theorem 2.8. *Isomorphism classes of irreducible objects of* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}^{I^0}$ *are parameterized by pairs* $(\chi \in \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}), w \in W)$. *For each such pair the corresponding irreducible object is given by*

$$\Gamma(\mathrm{Gr}_G, \mathrm{IC}_{w\cdot\lambda_w}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathbb{C}_\chi.$$

2.9. **The algebroid action.** Let $\mathsf{isom}_{\mathfrak{Z}}$ be the Lie algebroid of the groupoid $\mathsf{Isom}_{\mathfrak{Z}}$. According to [BD] (see also [FG2], Sect. 7.4 for a review), we have a canonical action of $\mathsf{isom}_{\mathfrak{Z}}$ on $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})$ by outer derivations, where $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})$ is the topological associative algebra corresponding to the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ and its tautological forgetful functor to vector spaces.

In more detail, there exists a topological associative algebra, denoted $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, and called the renormalized universal enveloping algebra at the critical level. It is endowed with a natural filtration, with the 0-th term $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_0$ being $U^{\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, and

$$U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_1/U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_0 \simeq U^{\mathrm{reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\widehat{\otimes}} \mathsf{isom}_{\mathfrak{Z}}.$$

The action of $\mathsf{isom}_{\mathfrak{Z}}$ on $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}})$ is given by the adjoint action of $\mathsf{isom}_{\mathfrak{Z}}$, regarded as a subset of $\subset U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_1/U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})_0$.

Let S be an affine scheme, and let χ_S be an S-point of $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$. Let ξ_S be a section of $\mathsf{isom}_{\mathfrak{Z}}|_S$. Set $S' := S \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$; then the image of ξ_S in $T(\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}))|_S$ gives rise to an S'-point, denoted, χ'_S, of $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$.

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S$ (resp., $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}$) be the corresponding base-changed category, where the latter identifies with the category of discrete modules over $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{O}_S$ (resp., $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{O}_{S'}$). Then the above action of $\mathsf{isom}_{\mathfrak{Z}}$ on $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ gives rise to the following construction:

To every $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S$ we can functorially attach an extension

$$0 \to \mathcal{M} \to \mathcal{M}' \to \mathcal{M} \to 0, \qquad \mathcal{M}' \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}. \tag{10}$$

The module $\mathcal{M}'$ is defined as follows. The above action of $\mathsf{isom}_{\mathfrak{Z}}$ by outer derivations of $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}})$ allows to lift ξ_S to an isomorphism

$$A(\xi_S) : \widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}},\chi_{S'}}{\otimes} \mathcal{O}_{S'} \to \widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}},\chi_S}{\otimes} \mathcal{O}_S[\epsilon]/\epsilon^2.$$

We set $\mathcal{M}'$ to be the $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}},\chi_{S'}}{\otimes} \mathcal{O}_{S'}$-module, corresponding via $A(\xi_S)$ to $\mathcal{M}[\epsilon]/\epsilon^2$.

The isomorphism $A(\xi_S)$ is defined up to conjugation by an element of the form $1+\epsilon\cdot u$, $u \in \widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}},\chi_S}{\otimes} \mathcal{O}_S$. Since this automorphism can be canonically lifted onto $\mathcal{M}[\epsilon]/\epsilon^2$, we obtain that $\mathcal{M}'$ is well-defined.

By construction, the functor $\mathcal{M} \mapsto \mathcal{M}'$ respects the Harish-Chandra $G((t))$-actions on the categories $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}$, respectively.

Let us note now that a data $(\chi_S : S \to \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}), \xi_S \in \mathsf{isom}_{\mathfrak{Z}}|_S)$ as above can be regarded as a map $S' \to \mathsf{Isom}_{\mathfrak{Z}}$, where first and second projections

$$S' \to \mathsf{Isom}_{\mathfrak{Z}} \rightrightarrows \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$$

are equal to

$$S' \to S \overset{\chi_S}{\to} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \text{ and } S' \overset{\chi'_S}{\to} \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}),$$

respectively.

Hence, Corollary 2.5 gives rise to an equivalence

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S^{I^0} \otimes \mathbb{C}[\epsilon]/\epsilon^2 \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}^{I^0},$$

and, in particular, to a functor

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}^{I^0}. \tag{11}$$

Proposition 2.10. *The functor*

$$\mathcal{M} \mapsto \mathcal{M}' : \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S'}$$

of (10)*, restricted to* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S^{I^0}$*, is canonically isomorphic to the above functor* (11).

Proof. The assertion follows from the fact that for $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$, the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-action on $\Gamma(\mathrm{Gr}_G, \mathcal{F})$ lifts canonically to an action of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ (see [FG2], Sect 7.4), so that for (S, χ_S, ξ_S) as above we have a canonical trivialization

$$\gamma_{\mathcal{F}} : \Gamma(\mathrm{Gr}_G, \mathcal{F})' \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F})[\epsilon]/\epsilon^2,$$

in the notation of (10). Moreover, this functorial isomorphism is compatible with that of Theorem 1.3 in the sense that for every $V \in \mathrm{Rep}(\check{G})$, the diagram

$$\begin{tikzcd}
\Gamma(\mathrm{Gr}_G, \mathcal{F} \star \mathcal{F}_V)' \arrow[r, "\gamma_{\mathcal{F}\star\mathcal{F}_V}"] \arrow[d, "\beta_V"'] & \Gamma(\mathrm{Gr}_G, \mathcal{F} \star \mathcal{F}_V)[\epsilon]/\epsilon^2 \arrow[d, "\beta_V \otimes \mathrm{id}"'] \\
\Big(\Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}\Big)' \arrow[r, "\gamma_{\mathcal{F}} \otimes \xi_S"] & \Big(\Gamma(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{V}\Big)[\epsilon]/\epsilon^2,
\end{tikzcd}$$

commutes, where the bottom arrow comprises the isomorphism $\gamma_{\mathcal{F}}$ and the canonical action of ξ_S on $\mathcal{V}_{\mathfrak{Z}}$. The latter compatibility follows assertion (b) in Theorem 8.4.2 of [FG2].

□

2.11. **Relation to semi-infinite cohomology.** Let us consider the functor of semi-infinite cohomology on the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}^{I^0}$:

$$\mathcal{M} \mapsto H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M} \otimes \Psi_0)$$

(see [FG2], Sect. 18 for details concerning this functor).

For an S-point χ_S of $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$ and $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_S$, each $H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M} \otimes \Psi_0)$ is naturally an $\mathcal{O}_S$-module.

Let now $(\chi_{1,S}, \chi_{2,S})$ be a pair of S-points of $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}})$, equipped with a lift $S \to \mathsf{Isom}_{\mathfrak{Z}}$, and let $\mathcal{M}_1 \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S,1}^{I^0}$ and $\mathcal{M}_2 \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{S,2}^{I^0}$ be two objects corresponding to each other under the equivalence of Corollary 2.5.

Proposition 2.12. *Under the above circumstances the* $\mathcal{O}_S$*-modules*

$$H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M}_1 \otimes \Psi_0) \text{ and } H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M}_2 \otimes \Psi_0)$$

are canonically isomorphic.

Proof. The assertion of the proposition can be tautologically translated as follows:

The functor

$$\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod} \xrightarrow{\Gamma} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}} \xrightarrow{H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ?\otimes\Psi_0)} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\text{-mod}$$

factors through a functor

$$H_{\check{G}}^{\frac{\infty}{2}+i} : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod} \to \mathrm{Rep}(\check{G}),$$

followed by the pull-back functor, corresponding to the morphism $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \to \mathrm{pt}/\check{G}$. Moreover, for $V \in \mathrm{Rep}(\check{G})$ we have a functorial isomorphism

$$H_{\check{G}}^{\frac{\infty}{2}+i}(\mathcal{F} \star \mathcal{F}_V) \simeq H_{\check{G}}^{\frac{\infty}{2}+i}(\mathcal{F}) \otimes V, \tag{12}$$

compatible with the isomorphism of Theorem 1.3(1).

The sought-after functor $H_{\check{G}}^{\frac{\infty}{2}+i}$ has been essentially constructed in [FG2], Sect. 18.3. Namely,

$$\mathrm{Hom}_{\check{G}}\big(V^{\check{\lambda}}, H_{\check{G}}^{\frac{\infty}{2}+i}(\mathcal{F})\big) := H^i(N((t)), \mathcal{F}|_{N((t))\cdot t^{\check{\lambda}}} \otimes \Psi_0),$$

in the notation of *loc. cit.* The isomorphisms (12) follow from the definitions.

□

Finally, we would like to compare the isomorphisms of Proposition 2.12 and Proposition 2.10. Let $\mathcal{M}$ be an object of $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}^{I^0}$; let χ_S be an S-point of $\text{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\text{reg}})$ and ξ_S a section of $\mathsf{isom}_{\mathfrak{Z}}|_S$.

On the one hand, in Proposition 18.3.2 of [FG2] we have shown that there exists a canonical isomorphism:

$$\mathbf{a}_{\mathcal{M}} : H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M}' \otimes \Psi_0) \simeq H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M} \otimes \Psi_0)[\epsilon]/\epsilon^2,$$

valid for any $\mathcal{M} \in \widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}$.

On the other hand, combining Proposition 2.10 and Proposition 2.12 we obtain another isomorphism

$$\mathbf{b}_{\mathcal{M}} : H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M}' \otimes \Psi_0) \simeq H^{\frac{\infty}{2}+i}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathcal{M} \otimes \Psi_0)[\epsilon]/\epsilon^2.$$

Unraveling the two constructions, we obtain the following:

Lemma 2.13. *The isomorphisms $\mathbf{a}_{\mathcal{M}}$ and $\mathbf{b}_{\mathcal{M}}$ coincide.*

3. Proof of the main theorem

In Sect. 1.6 we have constructed a functor

$$\Gamma^{\text{Hecke}_{\mathfrak{Z}}} : \text{D}(\text{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{Z}}}\text{-mod}^{I^0} \to \widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}^{I^0}.$$

Now we wish to show that this functor is an equivalence of categories. This will prove Theorem 1.7.

We start by constructing in Sect. 3.1 certain objects $\mathcal{F}_w^{\mathfrak{Z}}, w \in W$, of the category $\text{D}(\text{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{Z}}}\text{-mod}^{I^0}$ such that $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\mathcal{F}_w^{\mathfrak{Z}}) \simeq \mathbb{M}_{w,\text{reg}}$, the "standard modules" of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}^{I^0}$. The main result of Sect. 3.1, Theorem 3.2, will be proved in Sect. 4. Next, in Sect. 3.4 we prove part (1) of Theorem 1.7 that the functor $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}$ is exact. We then outline in Sect. 3.9 a general framework for proving that it is an equivalence. Using this framework, we prove Theorem 1.7 modulo Theorem 3.2.

In Sect. 3.14 we explain what needs to be done in order to prove our stronger Conjecture 1.5. Finally, in Sects. 3.16–3.19 we give an alternative proof of part (1) of Theorem 1.7.

3.1. **Standard modules.** For an element $w \in W$, let $\mathbb{M}_w$ be the Verma module over $\widehat{\mathfrak{g}}$,

$$\mathbb{M}_w = \text{Ind}_{\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_{\text{crit}}}(M_{w(\rho)-\rho}),$$

where for a weight λ we denote by M_λ the Verma module over $\mathfrak{g}$ with highest weight λ.

Let $\mathbb{M}_{w,\text{reg}}$ be the maximal quotient module that belongs to $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}$, i.e., $\mathbb{M}_{w,\text{reg}} = \mathbb{M}_w \underset{\mathfrak{Z}_{\mathfrak{g}}}{\otimes} \mathfrak{Z}_{\mathfrak{g}}^{\text{reg}}$. In fact, it was shown in [FG2], Corollary 13.3.2, that as modules over $\mathfrak{Z}_{\mathfrak{g}}$, all $\mathbb{M}_w$ are supported over a quotient algebra $\mathfrak{Z}_{\mathfrak{g}}^{\text{nilp}}$, and are flat as $\mathfrak{Z}_{\mathfrak{g}}^{\text{nilp}}$-modules. The subscheme $\text{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\text{reg}}) \subset \text{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is contained in $\text{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\text{nilp}})$, so the definition of $\mathbb{M}_{w,\text{reg}}$ does not neglect any lower cohomology.

The main ingredient in the remaining steps of our proof of Theorem 1.7 is the following:

Theorem 3.2. *For each $w \in W$ there exists an object $\mathcal{F}_w^{\mathfrak{Z}} \in \text{D}(\text{Gr}_G)_{\text{crit}}^{\text{Hecke}_{\mathfrak{Z}}}\text{-mod}^{I^0}$, such that $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, \mathcal{F}_w)$ is isomorphic to $\mathbb{M}_{w,\text{reg}}$.*

The proof of this theorem will consist of an explicit construction of the objects $\mathcal{F}^{\mathfrak{Z}}_w$, which will be carried out in Sect. 4.

The proof of Theorem 1.7 will only use a part of the assertion of Theorem 3.2: namely, that there exist objects $\mathcal{F}^{\mathfrak{Z}}_w \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}^{I^0}$, endowed with a surjection

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_w) \twoheadrightarrow \mathbb{M}_{w,\mathrm{reg}}. \tag{13}$$

What we will actually use is the following corollary of this statement:

Corollary 3.3. *For every* $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{I^0}_{\mathrm{reg}}$ *there exists an object* $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}^{I^0}$ *and a non-zero map* $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H) \to \mathcal{M}$.

Proof. By [FG2], Lemma 7.8.1, for every object $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{I^0}_{\mathrm{reg}}$ there exists $w \in W$ and a non-zero map $\mathbb{M}_{w,\mathrm{reg}} \to \mathcal{M}$.

□

3.4. **Exactness.** Let us recall from Sect. 2.6 the left adjoint functor $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ to the obvious forgetful functor $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod} \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$.

It is clear that every object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ can be covered by one of the form $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})$. From Lemma 2.7(1) we obtain that we can use bounded from above complexes, whose terms consist of objects of the form $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})$, in order to compute $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}$. Thus, we obtain:

Lemma 3.5. *For* $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$,

$$\mathrm{L}^i\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H) \simeq \mathrm{Tor}_i^{\mathsf{Fun}(\mathsf{Isom}_{\mathfrak{Z}})}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}^H), \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}\big).$$

We shall call an object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ *finitely generated* if it can be obtained as a quotient of an object of the form $\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})$, where $\mathcal{F}$ is a finitely generated object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$.

It is easy to see that an object $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ is finitely generated if and only if the functor $\mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}}(\mathcal{F}^H, \cdot)$ commutes with direct sums.

We shall call an object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ *finitely presented*, if it is isomorphic to $\mathrm{coker}\big(\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}_1) \to \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}_2)\big)$, where $\mathcal{F}_1, \mathcal{F}_2$ are both finitely generated objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$. The following lemma is straightforward.

Lemma 3.6.

(1) *An object* $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ *is finitely presented if and only if the functor* $\mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}}(\mathcal{F}^H, \cdot)$ *commutes with filtering direct limits.*

(2) *Every object of* $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ *is isomorphic to a filtering direct limit of finitely presented ones.*

The proof of the following proposition will be given in Sect. 3.13.

Proposition 3.7. *For every finitely presented object of* $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$*, the corresponding object* $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H) \in D^-(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$ *belongs to* $D^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$.

The crucial step in the proof of part (1) of Theorem 1.7 is the following:

Proposition 3.8. *If* $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}^{I^0}$ *is such that* $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H)$ *belongs to* $D^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})^{I^0}$*, then*

$$\mathrm{L}^i\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H)\big) = 0, \qquad i > 0.$$

Proof. Let $\mathcal{M}$ be the lowest cohomology of $\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H)$, which lives, say, in degree $-k$. By Corollary 3.3 there exists another object $\mathcal{F}^H_1 \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ and a non-zero map $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H_1) \to \mathcal{M}$. Hence, we obtain a non-zero map in $D^-(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}})$

$$\mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H_1)[k] \to \mathrm{L}\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H).$$

But by Theorem 1.4, such map comes from a map $\mathcal{F}^H_1[k] \to \mathcal{F}^H$, which is impossible if $k > 0$. □

Proof of part (1) of Theorem 1.7. Combining Proposition 3.7 and Proposition 3.8, we obtain that $\mathrm{L}^i\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H) = 0$ for any $i > 0$ and any $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$, which is finitely presented.

However, by Lemma 3.5, the functors

$$\mathcal{F}^H \mapsto \mathrm{L}^i\,\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^H)$$

commute with direct limits, and our assertion follows from Lemma 3.6(2). □

3.9. **Proof of the equivalence.** Consider the following general categorical framework. Let $\mathsf{G} : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor between abelian categories. Assume that for $X, Y \in \mathcal{C}_1$ the maps

$$\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(\mathsf{G}(X), \mathsf{G}(Y)) \text{ and } \mathrm{Ext}^1_{\mathcal{C}_1}(X, Y) \to \mathrm{Ext}^1_{\mathcal{C}_2}(\mathsf{G}(X), \mathsf{G}(Y))$$

are isomorphisms.

Lemma 3.10. *If* G *admits a right adjoint functor* F *which is conservative, then* G *is an equivalence.* [2]

Proof. The fully faithfulness assumption on G implies that the adjunction map induces an isomorphism between the composition $\mathsf{F} \circ \mathsf{G}$ and the identity functor on $\mathcal{C}_1$. We have to show that the second adjunction map is also an isomorphism.

For $X' \in \mathcal{C}_2$ let Y' and Z' be the kernel and cokernel, respectively, of the adjunction map

$$\mathsf{G} \circ \mathsf{F}(X') \to X'.$$

Being a right adjoint functor, F is left-exact, hence we obtain an exact sequence

$$0 \to \mathsf{F}(Y') \to \mathsf{F} \circ \mathsf{G} \circ \mathsf{F}(X') \to \mathsf{F}(X').$$

But since $\mathsf{F}(X') \to \mathsf{F} \circ \mathsf{G}(\mathsf{F}(X'))$ is an isomorphism, we obtain that $\mathsf{F}(Y') = 0$. Since F is conservative, this implies that $Y' = 0$.

Suppose that $Z' \neq 0$. Since $\mathsf{F}(Z') \neq 0$, there exists an object $Z \in \mathcal{C}_1$ with a non-zero map $\mathsf{G}(Z) \to Z'$. Consider the induced extension

$$0 \to \mathsf{G} \circ \mathsf{F}(X') \to W' \to \mathsf{G}(Z) \to 0.$$

Since G induces a bijection on Ext^1, this extension can be obtained from an extension

$$0 \to \mathsf{F}(X') \to W \to Z \to 0$$

in $\mathcal{C}_1$. In other words, we obtain a map $\mathsf{G}(W) \to X'$, which does not factor through $\mathsf{G} \circ \mathsf{F}(X') \subset X'$, which contradicts the (G, F) adjunction.

□

[2] Recall that a functor F is called *conservative* if for any $X \neq 0$ we have $\mathsf{F}(X) \neq 0$.

Thus, in order to prove of part (2) of Theorem 1.7 it remains to show that the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}^{I^0}_{\mathrm{reg}}$ admits a right adjoint. (The fact that it is conservative will then follow immediately from Corollary 3.3.)

Recall from [FG2], Sect. 20.7, that the tautological functor $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0} \hookrightarrow \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ admits a right adjoint, given by Av_{I^0}. Hence, it suffices to prove the following:

Proposition 3.11. *The functor* $\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}} : \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ *admits a right adjoint.*

Proof. First, we will show the following:

Lemma 3.12. *The functor* $\Gamma : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$ *admits a right adjoint.*

Proof. We will prove that for any level k the functor $\Gamma : \mathrm{D}(\mathrm{Gr}_G)_k\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_k$ admits a right adjoint (see the Introduction for the definition of these categories). I.e., we have to prove the representability of the functor

$$\mathcal{F} \mapsto \mathrm{Hom}_{\widehat{\mathfrak{g}}_k\text{–mod}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}), \mathcal{M}\big) \tag{14}$$

for every given $\mathcal{M} \in \widehat{\mathfrak{g}}_k\text{–mod}$.

Consider the following general set-up. Let $\mathcal{C}$ be an abelian category, and let $\mathcal{C}^0$ be a full (but not necessarily abelian) subcategory, such that the following holds:

- $\mathcal{C}^0$ is equivalent to a small category.
- The cokernel of any surjection $X'' \twoheadrightarrow X'$ with $X', X'' \in \mathcal{C}^0$, also belongs to $\mathcal{C}^0$.
- $\mathcal{C}$ is closed under filtering direct limits.
- For $X \in \mathcal{C}^0$, the functor $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)$ commutes with filtering direct limits.
- Every object of $\mathcal{C}$ is isomorphic to a filtering direct limit of objects of $\mathcal{C}^0$.

Then we claim that any contravariant left-exact functor $\mathsf{F} \to \mathrm{Vect}$, which maps direct sums to direct products (and, hence, direct limits to inverse limits, by the previous assumption), is representable.

Indeed, given such F, consider the category of pairs (X, f), where $X \in \mathcal{C}^0$ and $f \in \mathsf{F}(X)$. Morphisms between (X, f) and (X', f') are maps $\phi : X \to X'$, such that $\phi^*(f') = f$. By the first assumption on $\mathcal{C}^0$, this category is small. By the second assumption on $\mathcal{C}^0$ and the left-exactness of F, this category is filtering. It is easy to see that the object

$$\varinjlim_{(X,f)} X.$$

represents the functor F.

We apply this lemma to $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)_k\text{–mod}$ with $\mathcal{C}^0$ being the subcategory of finitely-generated D-modules. We set F to be the functor (14), and the representability assertion follows.

Note that we could have applied the above general principle to $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}$ and $\mathcal{C}^0$ being the subcategory of finitely presented objects, and obtain the assertion of Proposition 3.11 right away.

□

Thus, for $\mathcal{M}$, let $\mathcal{F}$ be the object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}$ that represents the functor

$$\mathcal{F}_1 \mapsto \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}_1), \mathcal{M}\big)$$

for a given $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$. We claim that $\mathcal{F}$ is naturally an object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{z}}}_{\mathrm{crit}}\text{-mod}$ and that it represents the functor

$$\mathcal{F}_1^H \mapsto \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}}\big(\Gamma^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathrm{Gr}_G, \mathcal{F}_1^H), \mathcal{M}\big). \tag{15}$$

First, since the algebra $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ acts on $\mathcal{M}$ by endomorphisms, the object $\mathcal{F}$ carries an action of $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ by functoriality. Let us now construct the morphisms α_V. Evidently, it is sufficient to do so for V finite-dimensional. Let V^* denote its dual.

For a test object $\mathcal{F}_1 \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$ we have:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}(\mathcal{F}_1, \mathcal{F} \star \mathcal{F}_V) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}(\mathcal{F}_1 \star \mathcal{F}_{V^*}, \mathcal{F}) \simeq \\
&\simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}_1 \star \mathcal{F}_{V^*}), \mathcal{M}\big) \simeq \\
&\simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}_1) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{V}^*_{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}, \mathcal{M}\big) \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}_1), \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M}\big),
\end{aligned}$$

where the last isomorphism takes place since $\mathcal{V}_{\mathfrak{Z}}$ is locally free. For the same reason,

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}(\mathcal{F}_1, \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{F}) \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}_1), \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M}\big),$$

which implies that there exists a canonical isomorphism α_V

$$\mathcal{F} \star \mathcal{F}_V \simeq \mathcal{V}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{F},$$

as required. That these isomorphisms are compatible with tensor products of objects of $\mathrm{Rep}(\check{G})$ follows from Theorem 1.3(2).

Finally, the fact that $(\mathcal{F}, \alpha_V)$, thus defined, represents the functor (15), follows from the construction. This completes the proof of Proposition 3.11.

□

Thus, we obtain that the functor $\Gamma^{\mathrm{Hecke}_{\mathfrak{z}}}$ admits a right adjoint functor. Moreover, this right adjoint functor is conservative by Corollary 3.3. Therefore part (2) of Theorem 1.7 now follows from part (1), proved in Sect. 3.4, and Lemma 3.10, modulo Proposition 3.7 and Theorem 3.2. It remains to prove those two statements. Proposition 3.7 will be proved in the next subsection and Theorem 3.2 will be proved in Sect. 4.

3.13. **Proof of Proposition 3.7.** Recall the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod}$, introduced in Sect. 2.6. Recall also that the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ on $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ is non-canonically trivial, and let us fix such a trivialization. This choice identifies the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{z}}}_{\mathrm{crit}}\text{-mod}$ with $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$, i.e., with the category of objects of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod}$ endowed with an action of $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ by endomorphisms.

Under this equivalence, the functor $\mathcal{F} \mapsto \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathcal{F})$ goes over to

$$\mathcal{F} \mapsto \mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}) \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}.$$

Note also that the trivialization of $\mathcal{P}_{\check{G},\mathfrak{Z}}$ identifies $\mathsf{Isom}_{\mathfrak{Z}}$ with $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}) \times \check{G} \times \mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, so that the map $\mathbf{1}_{\mathsf{Isom}_{\mathfrak{Z}}}$ corresponds to $\Delta_{\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})} \times \mathbf{1}_{\check{G}}$. For $\mathcal{F}$ as above, we have an identification

$$\Gamma\big(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathcal{F})\big) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F}) \otimes \mathcal{O}_{\check{G}} \otimes \mathfrak{Z}^{\mathrm{reg}}.$$

Let $\mathcal{F}^H$ be a finitely presented object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{z}}}_{\mathrm{crit}}\text{-mod}$ equal to the cokernel of a map

$$\phi : \mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}_1) \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} \to \mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}_2) \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}.$$

Recall that $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ is isomorphic to a polynomial algebra $\mathbb{C}[x_1,...,x_n,...]$. Since $\mathcal{F}_1$ was assumed finitely generated, a map as above has the form $\phi_m \otimes \mathrm{id}_{\mathbb{C}[x_{m+1},x_{m+2},...]}$, where ϕ_m is a map

$$\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}_1) \otimes \mathbb{C}[x_1,...,x_m] \to \mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}_2) \otimes \mathbb{C}[x_1,...,x_m]$$

defined for some m.

Hence, as a module over $\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}}) \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \otimes \mathcal{O}_{\check{G}} \otimes \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$,

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \simeq \mathcal{L} \otimes \mathbb{C}[x_{m+1}, x_{m+2},...], \tag{16}$$

where $\mathcal{L}$ is some module over $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \otimes \mathcal{O}_{\check{G}} \otimes \mathbb{C}[x_1,...,x_m]$.

We can compute

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}^H) \underset{\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})}{\overset{L}{\otimes}} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$$

in two steps, by first restricting to the preimage of the diagonal under

$$\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \times \check{G} \times \mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}) \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[x_{m+1}, x_{m+2},...]) \times \mathrm{Spec}(\mathbb{C}[x_{m+1}, x_{m+2},...]),$$

and then by further restriction to $\mathrm{Spec}(\mathbb{C}[x_1,...,x_m]) \times \mathrm{Spec}(\mathbb{C}[x_{m+1}, x_{m+2},...])$ sitting inside

$$\mathrm{Spec}(\mathbb{C}[x_1,...,x_m]) \times \check{G} \times \mathrm{Spec}(\mathbb{C}[x_1,...,x_m]) \times \mathrm{Spec}(\mathbb{C}[x_{m+1}, x_{m+2},...]).$$

When we apply the first step to the module appearing in (16), it is acyclic off cohomological degree 0. The second step has a cohomological amplitude bounded by $m + \dim(\check{G})$.

Hence,

$$\mathrm{Tor}_i^{\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}^H), \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\big) = 0$$

for $i > m + \dim(\check{G})$, which is what we had to show.

This completes the proof of Proposition 3.7. Therefore the proof of Theorem 1.7 is now complete modulo Theorem 3.2.

3.14. **A remark on the general case.** Let us note that the proof of Theorem 1.7 presented above would enable us to prove the general Conjecture 1.5 if we could show that the functor

$$\mathrm{Loc} : \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}} \to \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod},$$

right adjoint to the functor $\Gamma : \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ is conservative. In other words, in order to prove Conjecture 1.5 we need to know that for every $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ there exists a critically twisted D-module $\mathcal{F}$ on Gr_G with a non-zero map $\Gamma(\mathrm{Gr}_G, \mathcal{F}) \to \mathcal{M}$. This, in turn, can be reformulated as follows:

Let $\mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}}$ be the *-sheaf of critically twisted differential operators on Gr_G. This is a pro-object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$, defined by the property that

$$\mathrm{Hom}(\mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}}, \mathcal{F}) \simeq \Gamma(\mathrm{Gr}_G, \mathcal{F})$$

functorially in $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$.

Explicitly, let us write Gr_G as $\text{"}\underset{\overrightarrow{\mathcal{Y}}}{\lim}\text{"}\, \mathcal{Y}$, where $\mathcal{Y} \subset \mathrm{Gr}_G$ are closed sub-schemes. For each such $\mathcal{Y}$, let $\mathrm{Dist}(\mathcal{Y})_{\mathrm{crit}} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}$ be the twisted D-module of distributions on $\mathcal{Y}$, i.e., the object $\mathrm{Ind}_{\mathrm{QCoh}(\mathrm{Gr}_G)}^{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}(\mathcal{O}_{\mathcal{Y}})$, which means by definition that

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}\Big(\mathrm{Ind}_{\mathrm{QCoh}(\mathrm{Gr}_G)}^{\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}}(\mathcal{O}_{\mathcal{Y}}), \mathcal{F}\Big) = \mathrm{Hom}_{\mathrm{QCoh}(\mathrm{Gr}_G)}\Big(\mathcal{O}_Y, \mathcal{F}\Big).$$

Then

$$\mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}} := \text{"}\underset{\overleftarrow{\mathcal{Y}}}{\lim}\text{"}\ \mathrm{Dist}(\mathcal{Y})_{\mathrm{crit}} \in \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{-mod}).$$

Let $\Gamma(\mathrm{Gr}_G, \mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}})$ be the corresponding object of $\mathrm{Pro}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$.

We obtain:

Corollary 3.15. *The following assertions are equivalent:*

(1) *Conjecture 1.5 holds.*

(2) *The object* $\Gamma(\mathrm{Gr}_G, \mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}})$ *is a pro-projective generator of* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$.

(3) *The functor on* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$

$$\mathcal{M} \mapsto \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}}\Big(\mathrm{Gr}_G, \mathrm{Diff}(\mathrm{Gr}_G)_{\mathrm{crit}}, \mathcal{M}\Big)$$

is conservative.

3.16. **Another proof of exactness.** In this subsection we give shall present an alternative proof of part (1) of Theorem 1.7.

According to Lemma 3.5, proving the exactness property stated in part (1) of Theorem 1.7 is equivalent to proving that

$$\mathrm{Tor}_i^{\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})}\big(\Gamma(\mathrm{Gr}_G, \mathcal{F}^H), \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}\big) = 0 \tag{17}$$

for all $i > 0$ and $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–mod}^{I^0}$. We will derive this from the following weaker statement:

Proposition 3.17. *For every* $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{I^0}$*, the space of sections* $\Gamma(\mathrm{Gr}_G, \mathcal{F})$ *is flat as a* $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$*-module.*

Note that our general conjecture (1.5) predicts that both (17) and the assertion of Proposition 3.17 should hold without the I^0-equivariance assumption. However, at the moment we can neither prove the corresponding generalization of Proposition 3.17 nor derive (17) from it.

Let us first show how Proposition 3.17 implies (17) on the I^0-equivariant category.

Proposition 3.18. *Every finitely generated object of* $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–mod}^{I^0}$ *admits a finite filtration, whose subquotients are of the form*

$$\mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}, \tag{18}$$

where $\mathcal{L}$ *is a* $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$*-module.*

Let us deduce (17) from this proposition.

Proof. It is enough to show that (17) holds for finitely presented objects of the category $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–mod}^{I^0}$. By Proposition 3.18, we conclude that it is enough to consider objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–mod}^{I^0}$ of the form given by (18).

We have:

$$\Gamma\big(\mathrm{Gr}_G, \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\otimes} \mathcal{L}\big) \underset{\mathrm{Fun}(\mathsf{Isom}_{\mathfrak{Z}})}{\overset{L}{\otimes}} \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \simeq \Gamma\big(\mathrm{Gr}_G, \mathcal{F}\big) \underset{\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}}{\overset{L}{\otimes}} \mathcal{L},$$

and the assertion follows from Proposition 3.17.

□

Let us now prove Proposition 3.18.

Proof. Choosing a trivialization of $\mathcal{P}_{\check{G},\mathfrak{Z}}$ as in the previous subsection, we can identify $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}\text{–mod}^{I^0}$ with $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}^{I^0} \otimes \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$.

Similarly to the case of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}_{\mathfrak{Z}}}$–mod, we shall call an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}$–mod finitely generated if it is isomorphic to a quotient of some $\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F})$ for a finitely generated $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod.

Let us recall from [ABBGM], Corollary 1.3.10(1), that every finitely generated object in $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}^{I^0}$ has a finite length. Therefore, every finitely generated object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}^{I^0} \otimes \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ admits a finite filtration, whose subquotients are quotients of modules of the form $\mathcal{F}^H \otimes \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ with $\mathcal{F}^H \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}^{I^0}$ being irreducible. However, every such quotient has the form $\mathcal{F}^H \otimes \mathcal{L}$ for some $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-module $\mathcal{L}$.

Moreover, as was mentioned in Sect. 2.6, by [ABBGM], Corollary 1.3.10(2), every irreducible in $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}^{\mathrm{Hecke}}\text{–mod}^{I^0}$ is of the form $\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{I^0}$. This implies the assertion of the proposition.

□

3.19. **Proof of Proposition 3.17.** We can assume that our object $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{I^0}$ is finitely generated, which automatically implies that it has a finite length. This reduces us to the case when $\mathcal{F}$ is irreducible.

It is easy to see that any irreducible object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{I^0}$ is equivariant also with respect to $\mathbb{G}_m$, which acts on $G((t))$, and hence on Gr_G, by rescalings $t \mapsto at$. Moreover, the grading arising on its space of sections is bounded from above. (Our conventions are such that $\mathbb{V}_{\mathrm{crit}}$ is *negatively* graded.)

Recall now that the action of $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})$ on a module of the form $\Gamma(\mathrm{Gr}_G, \mathcal{F})$ for an object $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod canonically extends to an action of the renormalized algebra $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$. Recall also that $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$ contains a $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ sub-bimodule and a Lie subalgebra $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})^\sharp$, which is an extension

$$0 \to \widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}}) \to \widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})^\sharp \to \mathsf{isom}_{\mathfrak{Z}} \to 0.$$

(The resulting action of $\mathsf{isom}_{\mathfrak{Z}}$ by outer derivations on $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})$ is the one discussed in Sect. 2.9.)

We will prove the following general assertion, which implies Proposition 3.17:

Lemma 3.20. *Let $\mathcal{M}$ be an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{reg}}$, such that the action of $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})$ on it extends to an action of $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$. Assume also that $\mathcal{M}$ is endowed with a grading, compatible with the one on $U^{\mathrm{ren,reg}}(\widehat{\mathfrak{g}}_{\mathrm{crit}})$, given by rescalings $t \mapsto at$. Finally, assume that the grading on $\mathcal{M}$ is bounded from above. Then $\mathcal{M}$ is flat as a $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$-module.*

The proof is a variation of the argument used in [BD], Sect. 6.2.2:

Proof. We can identify $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$ with a polynomial algebra $\mathbb{C}[x_1, ..., x_n, ...]$. Moreover, we can do so in a grading-preserving fashion, in which case each generator x_i will be homogeneous of a *negative* degree.

It is enough to show that $\mathcal{M}$ is flat over each subalgebra $\mathbb{C}[x_1, ..., x_m] \subset \mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}$. We will prove the following assertion:

For every vector $\mathbf{v} \in \mathbb{A}^m := \mathrm{Spec}(\mathbb{C}[x_1, ..., x_m])$, the $\mathbb{C}[x_1, ..., x_m]$-module $\mathcal{M}$ is (non-canonically) isomorphic to its translate by means of $\mathbf{v}$.

Clearly, a module over $\mathbb{C}[x_1, ..., x_m]$ having this property is flat. To prove the above claim we proceed as follows. Choose a section ξ of $\mathsf{isom}_{\mathfrak{Z}}$, which projects onto $\mathbf{v}$ under $\mathsf{isom}_{\mathfrak{Z}} \to T(\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}}))$, where we think of $\mathbf{v}$ as a constant vector field on $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{reg}} \simeq \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n, ...])$. Let us further lift ξ to an element ξ' of $\widetilde{U}_{\mathrm{crit}}^{\mathrm{reg}}(\widehat{\mathfrak{g}})^\sharp$.

Since the grading on the x_i's is positive, we can choose ξ' to belong to the (completion of the) sum of strictly positive graded components of $\widetilde{U}^{\mathrm{reg}}_{\mathrm{crit}}(\widehat{\mathfrak{g}})^\sharp$.

Then the assumption that the grading on $\mathcal{M}$ is bounded from above, implies that $\exp(\xi')$ is a well-defined automorphism of $\mathcal{M}$ as a vector space. This automorphism covers the automorphism $\exp(\mathbf{v})$ of $\mathbb{C}[x_1, ..., x_m]$, and the latter is the same as the translation by $\mathbf{v}$.

$\square$

4. Proof of Theorem 3.2

In this section we construct the objects $\mathcal{F}^{\mathfrak{Z}}_w$ of the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{–mod}^{I^0}$ whose existence is stated in Theorem 3.2.

4.1. We first describe the analogues of these objects in the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}^{I^0}$. These objects, which we will denote by $\mathcal{F}_w$, were studied in [ABBGM] under the name "baby co-Verma modules".

First, we consider the case $w = w_0$. Recall that the Langlands dual group comes equipped with a standard Borel subgroup $\check{B} \subset \check{G}$; we shall denote by $\check{H}$ the Cartan quotient of $\check{B}$.

Let $\check{B}^- \subset \check{G}$ be a Borel subgroup in the generic relative position with respect to $\check{B}$. The latter means that $\check{B} \cap \check{B}^-$ is *a* Cartan subgroup; we shall identify it with $\check{H}$ by means of the projection

$$\check{B} \cap \check{B}^- \hookrightarrow \check{B} \twoheadrightarrow \check{H}.$$

For $\check{\lambda} \in \check{\Lambda}^+$ let $\ell^{\check{\lambda}}$ be the line of coinvariants $(V^{\check{\lambda}})_{\check{N}^-}$, where $V^{\check{\lambda}}$ denotes the standard irreducible $\check{G}$-representation of highest weight $\check{\lambda}$ with respect to $\check{B}$.

The assignment $\check{\lambda} \mapsto \ell^{\check{\lambda}}$ is an $\check{H}$-torsor, and we obtain a collection of maps

$$V^{\check{\lambda}} \overset{\kappa^{\check{\lambda}}}{\twoheadrightarrow} \ell^{\check{\lambda}}, \tag{19}$$

satisfying the Plücker relations, i.e., for any two dominant coweights $\check{\lambda}$ and $\check{\mu}$, the diagram

$$\begin{array}{ccc} V^{\check{\lambda}} \otimes V^{\check{\mu}} & \xrightarrow{\kappa^{\check{\lambda}} \otimes \kappa^{\check{\mu}}} & \ell^{\check{\lambda}} \otimes \ell^{\check{\mu}} \\ \Big\downarrow & & \sim\Big\downarrow \\ V^{\check{\lambda}+\check{\mu}} & \xrightarrow{\kappa^{\check{\lambda}+\check{\mu}}} & \ell^{\check{\lambda}+\check{\mu}} \end{array} \tag{20}$$

commutes.

Let $\mathrm{Fl}_G = G((t))/I$ be the affine flag variety. We have the category $\mathrm{D}(\mathrm{Fl}_G)_{\mathrm{crit}}\text{–mod}$ of right critically twisted D-modules on Fl_G and the corresponding Iwahori equivariant category $\mathrm{D}(\mathrm{Fl}_G)_{\mathrm{crit}}\text{–mod}^I$. Given $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_G)_\kappa\text{–mod}^I$ and $\mathcal{M} \in \mathrm{D}(\mathrm{Fl}_G)_{\mathrm{crit}}\text{–mod}^I$, we can form their convolution, denoted by $\mathcal{M} \underset{I}{\star} \mathcal{F}$, which is an object of $D^b(\mathrm{D}(\mathrm{Fl}_G)_{\mathrm{crit}}\text{–mod})^I$ (see [FG2] for details).

For a dominant map $\check{\lambda}$ let $j_{\check{\lambda},*}$ denote the $*$-extension of the critically twisted D-module corresponding to the constant sheaf on the Iwahori orbit of the point $t^{\check{\lambda}} \in \mathrm{Fl}_G$. Let $j_{\check{\lambda},\mathrm{Gr}_G,*} \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^I$ be $j_{\check{\lambda},*} \underset{I}{\star} \delta_{1,\mathrm{Gr}_G}$; in other words it is the $*$-extension of the constant D-module on the Iwahori orbit of the point $t^{\check{\lambda}} \in \mathrm{Gr}_G$. Note that for $\check{\mu} \in \check{\Lambda}^+$ we have a canonical map

$$j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{V^{\check{\mu}}} \to j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*},$$

obtained by identifying $\mathcal{F}_{V^{\check{\mu}}}$ with $\mathrm{IC}_{\overline{\mathrm{Gr}}^{\check{\mu}}}$.

Consider the object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{–mod}$ equal to the direct sum

$$\widetilde{\mathcal{F}}_{w_0} := \underset{\check{\lambda}\in\check{\Lambda}^+}{\oplus} \mathrm{Ind}^{\mathrm{Hecke}}\big(j_{\check{\lambda},\mathrm{Gr}_G,*}\big) \otimes \ell^{-\check{\lambda}}.$$

For a dominant coweight $\check{\mu}$ we have an evident map

$$j_{\check{\mu},*} \underset{I}{\star} \widetilde{\mathcal{F}}_{w_0} \to \ell^{\check{\mu}} \otimes \widetilde{\mathcal{F}}_{w_0}. \tag{21}$$

We obtain two maps $\widetilde{\mathcal{F}}_{w_0} \star \mathcal{F}_{V^{\check{\mu}}} \rightrightarrows \widetilde{\mathcal{F}}_{w_0} \otimes \ell^{\check{\mu}}$ that correspond to the two circuits of the following non-commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{F}}_{w_0} \star \mathcal{F}_{V^{\check{\mu}}} \arrow[r, "\alpha_V"] \arrow[d] & \underline{V}^{\check{\mu}} \otimes \widetilde{\mathcal{F}}_{w_0} \arrow[d, "\kappa^{\check{\mu}}"'] \\
j_{\check{\mu},*} \underset{I}{\star} \widetilde{\mathcal{F}}_{w_0} \arrow[r] & \ell^{\check{\mu}} \otimes \widetilde{\mathcal{F}}_{w_0},
\end{tikzcd}$$

where the left vertical arrow comes from the following map, defined for each $\check{\lambda}$:

$$j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_R \star \mathcal{F}_{V^{\check{\mu}}} \simeq j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{V^{\check{\mu}}} \star \mathcal{F}_R \to j_{\check{\lambda}+\check{\mu},\mathrm{Gr}_G,*} \star \mathcal{F}_R.$$

Here we are using the object $\mathcal{F}_R$ of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^{G[[t]]}$ introduced in Sect. 2.6, so that $\mathrm{Ind}^{\mathrm{Hecke}}(\mathcal{F}) \simeq \mathcal{F} \star \mathcal{F}_R$.

We set $\mathcal{F}_{w_0}$ to be the maximal quotient of $\widetilde{\mathcal{F}}_{w_0}$, which co-equalizes the resulting two maps

$$\ell^{-\check{\mu}} \otimes \widetilde{\mathcal{F}}_{w_0} \star \mathcal{F}_{V^{\check{\mu}}} \rightrightarrows \widetilde{\mathcal{F}}_{w_0}$$

for every $\check{\mu} \in \check{\Lambda}^+$. Note that the map (21) gives rise to a map

$$j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}_{w_0} \to \ell^{\check{\mu}} \otimes \mathcal{F}_{w_0}. \tag{22}$$

By construction, $\mathcal{F}_{w_0}$ has the following universal property:

Let $\mathcal{F}^H$ be an object of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}\text{–mod}^I$, endowed with a system of morphisms

$$j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}^H \to \ell^{\check{\mu}} \otimes \mathcal{F}^H, \tag{23}$$

compatible with the isomorphisms

$$j_{\check{\mu},*} \underset{I}{\star} j_{\check{\mu}',*} \simeq j_{\check{\mu}+\check{\mu}',*} \tag{24}$$

and $\ell^{\check{\mu}} \otimes \ell^{\check{\mu}'} \simeq \ell^{\check{\mu}+\check{\mu}'}$.

Let $\phi : \mathcal{F}_R \to \mathcal{F}^H$ be a map, such that for every $\check{\mu} \in \check{\Lambda}$ the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{F}_R \star \mathcal{F}_{V^{\check{\mu}}} \arrow[r, "\alpha_V"] \arrow[d, "\sim"'] & \underline{V}^{\check{\mu}} \otimes R \arrow[r, "\mathrm{id}_{\underline{V}^{\check{\lambda}}} \otimes \phi"] & \underline{V}^{\check{\mu}} \otimes \mathcal{F}^H \arrow[r, "\kappa^{\check{\mu}}"] & \ell^{\check{\mu}} \otimes \mathcal{F}^H \\
\mathcal{F}_{V^{\check{\mu}}} \star \mathcal{F}_R \arrow[r] & j_{\check{\mu},\mathrm{Gr}_G,*} \star \mathcal{F}_R \arrow[r, "\sim"] & j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}_R \arrow[r, "\mathrm{id}_{j_{\check{\mu},*}} \star \phi"] & j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}^H. \arrow[u]
\end{tikzcd}$$

Lemma 4.2. *Under the above circumstances, there exists a unique map $\mathcal{F}_{w_0} \to \mathcal{F}^H$, extending ϕ, and which intertwines the maps (21) and (23).*

4.3. We shall now establish the equivalence between the present definition of $\mathcal{F}_{w_0}$ and the objects defined in [ABBGM].

For a weight $\check{\nu} \in \check{\Lambda}$ consider the inductive system of objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod, parameterized by pairs of elements $\check{\lambda}, \check{\mu} \in \check{\Lambda}^+ \,|\, \check{\lambda} - \check{\mu} = \check{\nu}$, and whose terms are given by

$$j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{(V^{\check{\mu}})^*} \otimes \ell^{-\check{\lambda}+\check{\mu}}.$$

The maps in this inductive system are defined whenever two pairs $(\check{\lambda}', \check{\mu}')$ and $(\check{\lambda}, \check{\mu})$ are such that $\check{\lambda}' - \check{\lambda} = \check{\mu}' - \check{\mu} =: \check{\eta} \in \check{\Lambda}^+$, and the corresponding map equals the composition

$$\begin{aligned} &j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{(V^{\check{\mu}})^*} \otimes \ell^{-\check{\lambda}+\check{\mu}} \to j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{V^{\check{\eta}}} \star \mathcal{F}_{(V^{\check{\eta}})^*} \star \mathcal{F}_{(V^{\check{\mu}})^*} \otimes \ell^{-\check{\lambda}+\check{\mu}} \to \\ &\to j_{\check{\lambda}+\check{\eta},\mathrm{Gr}_G,*} \star \mathcal{F}_{(V^{\check{\mu}+\check{\eta}})^*} \otimes \ell^{-\check{\lambda}-\check{\eta}+(\check{\mu}+\check{\eta})}. \end{aligned}$$

Let $\mathcal{F}'_{w_0}(\check{\nu}) \in \mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod be the direct limit of the above system. We endow $\mathcal{F}'_{w_0} := \underset{\check{\nu}\in\check{\Lambda}}{\oplus} \mathcal{F}'_{w_0}(\check{\nu})$ with the structure of an object of $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}$–mod as in Sect. 3.2.1 of [ABBGM].

Proposition 4.4. *There exists a natural isomorphism*

$$\mathcal{F}'_{w_0} \simeq \mathcal{F}_{w_0}.$$

Proof. The map $\mathcal{F}_{w_0} \to \mathcal{F}'_{w_0}$ is constructed using Lemma 4.2, and the corresponding property of $\mathcal{F}'_{w_0}$ established in [ABBGM], Corollary 3.2.3.

To show that this map is an isomorphism, we construct a map in the opposite direction $\mathcal{F}'_{w_0} \to \mathcal{F}_{w_0}$ (as mere objects of $\mathrm{D}(\mathrm{Gr}_G)_{\mathrm{crit}}$–mod) as follows:

For each $\check{\lambda}, \check{\mu} \in \check{\Lambda}^+$, we let $j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_{(V^{\check{\mu}})^*} \otimes \ell^{-\check{\lambda}+\check{\mu}}$ embed into $j_{\check{\lambda},\mathrm{Gr}_G,*} \star \mathcal{F}_R \otimes \ell^{-\check{\lambda}}$ by means of

$$\mathcal{F}_{(V^{\check{\mu}})^*} \otimes \ell^{\check{\mu}} \hookrightarrow \mathcal{F}_{(V^{\check{\mu}})^*} \otimes \underline{V}^{\check{\mu}} \hookrightarrow \mathcal{F}_R,$$

where the second arrow is given by

$$\ell^{\check{\mu}} \simeq (\underline{V}^{\check{\mu}})^{\check{N}} \hookrightarrow \underline{V}^{\check{\mu}}.$$

It is straightforward to check that this gives rise to a well-defined map from the inductive system corresponding to $\mathcal{F}'_{w_0}(\check{\nu})$, and that the above two maps $\mathcal{F}_{w_0} \leftrightarrows \mathcal{F}'_{w_0}$ are mutually inverse. □

Corollary 4.5. *The maps* (22) $j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}_{w_0} \to \ell^{\check{\mu}} \otimes \mathcal{F}_{w_0}$ *are isomorphisms.*

Proof. The assertion follows from the fact that the maps

$$j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}'_{w_0}(\check{\nu}) \to \ell^{\check{\mu}} \otimes \mathcal{F}'_{w_0}(\check{\nu}+\check{\mu})$$

are easily seen to be isomorphisms.

□

Let us now define the objects $\mathcal{F}_w$ for other elements $w \in W$. We set

$$\mathcal{F}_w := j_{w\cdot w_0,!} \underset{I}{\star} \mathcal{F}_{w_0}.$$

In other words, if $w_0 = w' \cdot w$, then

$$\mathcal{F}_{w_0} \simeq j_{w',*} \underset{I}{\star} \mathcal{F}_w.$$

From Proposition 4.4 it follows that $\mathcal{F}_w$ are D-modules, i.e., that no higher cohomologies appear.

4.6. Let us now define the sought-after objects $\mathcal{F}^{\mathfrak{Z}}_w$ of the category $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$.

Consider the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ over $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$. Recall from Sect. 1.1 that we have a canonical isomorphism $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}) \simeq \mathrm{Op}_{\check{\mathfrak{g}}}(\mathcal{D})$, under which $\mathcal{P}_{\check{G},\mathfrak{Z}}$ goes over the canonical $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathrm{Op}}$ on the space of opers (see [FG2], Sect. 8.3, for details). Thus, we obtain a canonical reduction of $\mathcal{P}_{\check{G},\mathfrak{Z}}$ to $\check{B}$ that we will denote by $\mathcal{P}_{\check{B},\mathfrak{Z}}$.

This $\check{B}$-reduction defines a $\check{B}^-$-reduction on $\mathcal{P}_{\check{G},\mathfrak{Z}}$. In order to define a $\check{B}^-$-reduction, we need to specify for each $\check{\lambda} \in \check{\Lambda}$ a line bundle, which we will denote by $\mathcal{L}^{\check{\lambda}}_{w_0}$, and for each $\check{\lambda} \in \check{\Lambda}^+$ a surjective homomorphism

$$\kappa^{\check{\lambda},\mathfrak{Z}} : \mathcal{V}^{\check{\lambda}}_{\mathfrak{Z}} \to \mathcal{L}^{\check{\lambda}}_{w_0}.$$

These line bundles should be equipped with isomorphisms $\mathcal{L}^{\check{\lambda}+\check{\mu}}_{w_0} \simeq \mathcal{L}^{\check{\lambda}}_{w_0} \otimes \mathcal{L}^{\check{\mu}}_{w_0}$, and hence give rise to a $\check{H}$-torsor on $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$, which we will denote by $\mathcal{P}_{\check{H},w_0}$. In addition, the maps $\kappa^{\check{\lambda},\mathfrak{Z}}$ should satisfy the Plücker relations, as in (20). Now observe that our $\check{B}$-reduction $\mathcal{P}_{\check{B},\mathfrak{Z}}$ gives rise to a collection of compatible line subbundles $\mathcal{L}^{\check{\lambda}}$ of $\mathcal{V}^{\check{\lambda}}_{\mathfrak{Z}}$. We then define $\mathcal{L}^{\check{\lambda}}_{w_0}$ as the dual of the line bundle $\mathcal{L}^{-w_0(\check{\lambda})} \hookrightarrow \mathcal{V}^{-w_0(\check{\lambda})}_{\mathfrak{Z}} \simeq (\mathcal{V}^{\check{\lambda}}_{\mathfrak{Z}})^*$.

It follows from the definition of opers (see [FG2], Sect. 1) that the line bundle $\mathcal{L}^{\check{\lambda}}_{w_0}$ over $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ is canonically isomorphic to the trivial line bundle tensored with the one-dimensional vector space $\omega_x^{\langle \rho, w_0(\check{\lambda})\rangle}$, where ω_x is the fiber of $\omega_{\mathcal{D}}$ at the closed point $x \in \mathcal{D}$.

We define the object $\widetilde{\mathcal{F}}^{\mathfrak{Z}}_{w_0} \in \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod}$ as a direct sum

$$\underset{\check{\lambda}\in\check{\Lambda}^+}{\oplus} \mathrm{Ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}\left(j_{\check{\lambda},\mathrm{Gr}_G,*}\right) \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{w_0}.$$

We define $\mathcal{F}^{\mathfrak{Z}}_{w_0}$ to be the quotient of $\widetilde{\mathcal{F}}^{\mathfrak{Z}}_{w_0}$ by the same relations as those defining $\mathcal{F}_{w_0}$ as a quotient of $\widetilde{\mathcal{F}}_{w_0}$.

If we choose a trivialization of the $\check{G}$-torsor $\mathcal{P}_{\check{G},\mathfrak{Z}}$ in such a way that $\mathcal{L}^{\check{\lambda}}_{w_0} \simeq \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} \otimes \ell^{\check{\lambda}}$ (such a trivialization exists), then under the equivalence

$$\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}_{\mathfrak{Z}}}_{\mathrm{crit}}\text{-mod} \simeq \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{Hecke}}_{\mathrm{crit}}\text{-mod} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}},$$

the object $\mathcal{F}^{\mathfrak{Z}}_{w_0}$ corresponds to $\mathcal{F}_{w_0}$.

By construction, we have a system of maps

$$j_{\check{\mu},*} \underset{I}{\star} \mathcal{F}^{\mathfrak{Z}}_{w_0} \simeq \mathcal{L}^{\check{\mu}}_{w_0} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{F}^{\mathfrak{Z}}_{w_0}, \tag{25}$$

which by Corollary 4.5 are in fact isomorphisms.

For other elements $w \in W$ we define

$$\mathcal{F}^{\mathfrak{Z}}_w := j_{w\cdot w_0,!} \underset{I}{\star} \mathcal{F}^{\mathfrak{Z}}_{w_0}.$$

4.7. Our present goal is to define the maps

$$\phi_w : \Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_w) \to \mathbb{M}_{w,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}. \tag{26}$$

Since $\mathbb{M}_{w,\mathrm{reg}} \simeq j_{w\cdot w_0,!} \underset{I}{\star} \mathbb{M}_{w_0,\mathrm{reg}}$, it is enough to define ϕ_w for $w = w_0$.

Let $\mathcal{M}$ be an object of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$. Assume that $\mathcal{M}$ is endowed with a system of maps

$$j_{\check{\mu},*} \underset{I}{\star} \mathcal{M} \to \mathcal{L}^{\check{\mu}}_{w_0} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M}, \tag{27}$$

defined for every $\check{\mu} \in \check{\Lambda}^+$, compatible with the isomorphisms (24) and $\mathcal{L}^{\check{\mu}}_{w_0} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{\check{\mu}'}_{w_0} \simeq \mathcal{L}^{\check{\mu}+\check{\mu}'}_{w_0}$.

Let ϕ be a map $\mathbb{V}_{\mathrm{crit}} \to \mathbb{M}$, such that for any $\check{\mu} \in \check{\Lambda}^+$ the diagram

$$\begin{array}{ccccccc} \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\mu}}}) & \xrightarrow{\beta_{V^{\check{\mu}}}} & \mathcal{V}^{\check{\mu}}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathbb{V}_{\mathrm{crit}} & \xrightarrow{\mathrm{id}_{\mathcal{V}^{\check{\mu}}_{\mathfrak{Z}}} \otimes \phi} & \mathcal{V}^{\check{\mu}}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M} & \xrightarrow{\kappa^{\check{\mu},\mathfrak{Z}}} & \mathcal{L}^{\check{\mu}}_{w_0} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{M} \\ \downarrow & & & & & & \uparrow \\ \Gamma(\mathrm{Gr}_G, j_{\check{\mu},\mathrm{Gr}_G,*}) & \xrightarrow{\sim} & j_{\check{\mu},\mathrm{Gr}_G,*} \star \mathbb{V}_{\mathrm{crit}} & \xrightarrow{\sim} & j_{\check{\mu},*} \underset{I}{\star} \mathbb{V}_{\mathrm{crit}} & \xrightarrow{\mathrm{id}_{j_{\check{\mu},*}} \star \phi} & j_{\check{\mu},*} \underset{I}{\star} \mathcal{M} \end{array} \tag{28}$$

is commutative.

By the construction of $\mathcal{F}^{\mathfrak{Z}}_{w_0}$, we have:

Lemma 4.8. *Under the above circumstances there exists a unique map*

$$\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_{w_0}) \to \mathcal{M},$$

which intertwines the maps (25) *and* (27).

Thus, to construct the map as in (26) for $w = w_0$ we need to verify that the module $\mathcal{M} := \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$ possesses the required structures.

First, the map

$$\mathbb{V}_{\mathrm{crit}} \to \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$$

was constructed in [FG2], Sect. 7.2.

4.9. To construct the data of (27) we need to recall some material from [FG2], Sect. 13.4. According to *loc. cit.* there exists some $\check{H}$-torsor $\{\check{\lambda} \mapsto \mathcal{L}'^{\check{\lambda}}_{w_0}\}$ on $\mathrm{Spec}(\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}})$ and a system of isomorphisms

$$j_{\check{\mu},*} \underset{I}{\star} \mathbb{M}_{w_0,\mathrm{reg}} \simeq \mathcal{L}'^{\check{\lambda}}_{w_0} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathbb{M}_{w_0,\mathrm{reg}}.$$

Thus, to construct the map ϕ_{w_0}, we need to prove the following assertion:

Lemma 4.10. *There exists an isomorphism of $\check{H}$-torsors*

$$\mathcal{L}^{\check{\mu}}_{w_0} \simeq \mathcal{L}'^{\check{\mu}}_{w_0}$$

which makes the diagram (28) *commutative for* $\mathcal{M} := \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$.

Below we will prove this assertion by a rather explicit calculation. In a future publication, we will discuss a more conceptual approach. The crucial step is the following statement:

Lemma 4.11. *The composition*

$$\Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\mu}}}) \to j_{\check{\mu},*} \underset{I}{\star} \mathbb{V}_{\mathrm{crit}} \overset{\mathrm{id}_{j_{\check{\mu},*}} \star \phi}{\to} j_{\check{\mu},*} \underset{I}{\star} \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$$

is non-zero.

This proposition will be proved in Sect. 4.12. Let us assume it and construct the required isomorphism $\mathcal{L}^{\check{\mu}}_{w_0} \simeq \mathcal{L}'^{\check{\mu}}_{w_0}$.

Proof of Lemma 4.10. Recall from [FG2], Corollary 13.4.2, that there exists an isomorphism, defined up to a scalar, $\mathcal{L}^{\check{\mu}}_{w_0} \simeq \mathcal{L}'^{\check{\mu}}_{w_0}$, compatible with the action of $\mathrm{Aut}(\mathcal{D})$. [3] We will show that any choice of such isomorphism makes the diagram (28) commutative, up to a non-zero scalar.

[3]Choosing a coordinate t on $\mathcal{D}$, we obtain a subgroup $\mathbb{G}_m \subset \mathrm{Aut}(\mathcal{D})$ of rescalings $t \mapsto at$.

Thus, we are dealing with two non-zero maps

$$\mathcal{V}^{\check{\mu}}_{\mathfrak{Z}} \underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes} \mathbb{V}_{\mathrm{crit}} \rightrightarrows \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle \rho, w_0(\check{\mu})+2\check{\rho}\rangle}.$$

Recall from [FG2], Sect. 17.2, that there exists an isomorphism

$$\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} \simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{V}_{\mathrm{crit}}, \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle \rho, 2\check{\rho}\rangle}),$$

compatible with the above $\mathbb{G}_m$-action. Thus, we are reduced to showing that the space grading-preserving maps of $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$-modules

$$\mathcal{V}^{\check{\mu}}_{\mathfrak{Z}} \to \omega_x^{\langle \rho, w_0(\check{\mu})\rangle} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$$

is 1-dimensional.

However, $\mathcal{V}^{\check{\mu}}_{\mathfrak{Z}}$ admits a canonical filtration, whose subquotients are isomorphic to $\omega_x^{\langle \rho, \check{\mu}'\rangle} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$, where $\check{\mu}'$ runs through the set weights of $V^{\check{\mu}}$ with multiplicities. For all $\check{\mu}' \neq w_0(\check{\mu})$, we have $\langle \rho, \check{\mu}'\rangle > \langle \rho, w_0(\check{\mu})\rangle$. Since the algebra $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$ is non-positively graded, the above inequality implies that the space of grading-preserving maps

$$\omega_x^{\langle \rho, \check{\mu}'\rangle} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} \to \omega_x^{\langle \rho, w_0(\check{\mu})\rangle} \otimes \mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}$$

is zero for $\check{\mu}' \neq w_0(\check{\mu})$, and 1-dimensional for $\check{\mu}' = w_0(\check{\mu})$. □

4.12. **Proof of Lemma 4.11.** It is clear that if $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$, with $\check{\mu}_1, \check{\mu}_2 \in \check{\Lambda}^+$, and the assertion of the proposition holds for $\check{\mu}$, then it also holds for $\check{\mu}_1$. Hence it is sufficient to consider the case of $\check{\mu}$ that are regular.

To prove the proposition we will use the semi-infinite cohomology functor, denoted by $H^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], ? \otimes \Psi_0)$, as in [FG2], Sect. 18. We will show that the composition

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\mu}}}) \otimes \Psi_0\big) \to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, j_{\check{\mu},\mathrm{Gr}_G,*}) \otimes \Psi_0\big) \to$$
$$\to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho, \check{\rho}\rangle} \otimes \Psi_0\big)$$

is non-zero (and, in fact, a surjection).

First, note that by [FG2], Sect. 18.3, the first arrow, i.e.,

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, \mathcal{F}_{V^{\check{\mu}}}) \otimes \Psi_0\big) \to H^{\frac{\infty}{2}}\big(\mathfrak{n}((t)), \mathfrak{n}[[t]], \Gamma(\mathrm{Gr}_G, j_{\check{\mu},\mathrm{Gr}_G,*}) \otimes \Psi_0\big)$$

is an isomorphism. Hence, it remains to analyze the second arrow. By [FG2], Proposition 18.1.1, this is equivalent to analyzing the arrow

$$H^{\frac{\infty}{2}}\big(\mathfrak{n}^-((t)), t\mathfrak{n}^-[[t]], j_{w_0\cdot\check{\rho},*} \underset{I}{\star} \Gamma(\mathrm{Gr}_G, j_{\check{\mu},\mathrm{Gr}_G,*}) \otimes \Psi_{-\check{\rho}}\big) \to$$
$$H^{\frac{\infty}{2}}\big(\mathfrak{n}^-((t)), t\mathfrak{n}^-[[t]], j_{w_0\cdot\check{\rho},*} \underset{I}{\star} \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho, \check{\rho}\rangle} \otimes \Psi_{-\check{\rho}}\big).$$

We claim that the corresponding map

$$j_{w_0\cdot\check{\rho},*} \underset{I}{\star} \Gamma(\mathrm{Gr}_G, j_{\check{\mu},\mathrm{Gr}_G,*}) \simeq j_{w_0\cdot\check{\rho},*} \underset{I}{\star} j_{\check{\mu},*} \underset{I}{\star} \mathbb{V}_{\mathrm{crit}} \to j_{w_0\cdot\check{\rho},*} \underset{I}{\star} j_{\check{\mu},*} \underset{I}{\star} \mathbb{M}_{w_0,\mathrm{reg}} \otimes \omega_x^{\langle 2\rho, \check{\rho}\rangle} \tag{29}$$

is surjective for $\check{\mu}$ regular. This would imply our claim, since the semi-infinite cohomology functor $H^{\frac{\infty}{2}}\big(\mathfrak{n}^-((t)), t\mathfrak{n}^-[[t]], ? \otimes \Psi_{-\check{\rho}}\big)$ is exact by Theorem 18.3.1 of [FG2].

Note that $j_{w_0\cdot\check\rho,*}\underset{I}{\star} j_{\check\mu,*}\simeq j_{w_0(\check\mu),*}\underset{I}{\star} j_{w_0\cdot\check\rho,*}$. Recall from [FG2], Sect. 17.2, that we have a commutative diagram

$$\begin{CD} j_{w_0\cdot\check\rho,*}\underset{I}{\star}\mathbb{V}_{\mathrm{crit}} @>{\mathrm{id}_{j_{w_0\cdot\check\rho,*}}\star\phi}>> j_{w_0\cdot\check\rho,*}\underset{I}{\star}\mathbb{M}_{w_0,\mathrm{reg}}\otimes\omega_x^{\langle 2\rho,\check\rho\rangle} \\ @V{\sim}VV @V{\sim}VV \\ \Gamma(\mathrm{Gr}_G, j_{w_0\cdot\check\rho,*}\underset{I}{\star}\delta_{1,\mathrm{gr}_G}) @>>> \mathbb{M}_{1,\mathrm{reg}}\otimes\omega_x^{\langle\rho,\check\rho\rangle}, \end{CD}$$

where the bottom arrow has the property that its cokernel, which we denote by $\mathcal{N}$, is *partially integrable*, i.e., it is admits a filtration with every subquotient integrable with respect to a sub-minimal parahoric Lie subalgebra corresponding to some vertex $\imath$ of the Dynkin diagram.

Thus, the map in (29) can be written as

$$j_{w_0(\check\mu),*}\underset{I}{\star}(j_{w_0\cdot\check\rho,*}\underset{I}{\star}\mathbb{V}_{\mathrm{crit}})\to j_{w_0(\check\mu),*}\underset{I}{\star}(\mathbb{M}_{1,\mathrm{reg}}\otimes\omega_x^{\langle\rho,\check\rho\rangle}),$$

and since the functor $j_{w_0(\check\mu),*}\underset{I}{\star}?$ is right-exact, it suffices to show that $j_{w_0(\check\mu),*}\underset{I}{\star}\mathcal{N}$ is supported in strictly negative cohomological degrees. In fact, we claim that this is true for any partially integrable I-integrable $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module and regular dominant coweight $\check\mu$.

Indeed, by devissage we may assume that $\mathcal{N}$ is integrable with respect to a sub-minimal parahoric corresponding to some vertex $\imath$ of the Dynkin diagram. Then $j_{s_\imath,*}\underset{I}{\star}\mathcal{N}$ lives in the cohomological degree -1. But since $\check\mu$ is regular, $j_{w_0(\check\mu),*}\underset{I}{\star} j_{s_\imath,!}\simeq j_{w_0(\check\mu)\cdot s_\imath,*}$, and hence,

$$j_{w_0(\check\mu),*}\underset{I}{\star}\mathcal{N}\simeq j_{w_0(\check\mu)\cdot s_\imath,*}\underset{I}{\star}(j_{s_\imath,*}\underset{I}{\star}\mathcal{N}),$$

and our assertion follows from the fact that the functor of convolution with $j_{w_0(\check\mu)\cdot s_\imath,*}$ is right-exact. □

4.13. **Proof of Corollary 3.3 and completion of the proof of Theorem 1.7.** Thus, we have proved Lemma 4.11 and therefore Lemma 4.10. By Lemma 4.8, this implies that we have a canonical map

$$\phi_{w_0}:\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G,\mathcal{F}^{\mathfrak{Z}}_{w_0})\to\mathbb{M}_{w,\mathrm{reg}}\otimes\omega_x^{\langle 2\rho,\check\rho\rangle}.$$

According to the remark after formula (26), we then obtain maps

$$\phi_{w}:\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G,\mathcal{F}^{\mathfrak{Z}}_{w})\to\mathbb{M}_{w,\mathrm{reg}}\otimes\omega_x^{\langle 2\rho,\check\rho\rangle}$$

for all $w\in W$ (as in formula (26)).

Proposition 4.14. *The map*

$$\phi_{1}:\Gamma^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{Gr}_G,\mathcal{F}^{\mathfrak{Z}}_{1})\to\mathbb{M}_{1,\mathrm{reg}}\otimes\omega_x^{\langle 2\rho,\check\rho\rangle}$$

is surjective.

Since the functors $j_{w,*}$ are right-exact, this proposition implies that the same surjectivity assertion holds for all $w\in W$. Hence, Proposition 4.14 implies Corollary 3.3 and Theorem 1.7.

Proof of Proposition 4.14. For $\check\lambda$, such that $\check\lambda-\check\rho$ is dominant and regular, let us consider the map

$$j_{w_0\cdot\check\lambda,*}\underset{I}{\star}\mathcal{F}_{R_{\mathfrak{Z}}}\underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes}\mathcal{L}^{-\check\lambda}_{w_0}\simeq j_{w_0,!}\underset{I}{\star}j_{\check\lambda,*}\underset{I}{\star}\mathcal{F}_{R_{\mathfrak{Z}}}\underset{\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}}}{\otimes}\mathcal{L}^{-\check\lambda}_{w_0}\to j_{w_0,!}\underset{I}{\star}\widetilde{\mathcal{F}}^{\mathfrak{Z}}_{w_0}\to j_{w_0,!}\underset{I}{\star}\mathcal{F}^{\mathfrak{Z}}_{w_0}\simeq\mathcal{F}^{\mathfrak{Z}}_{1},$$

and the resulting map

$$j_{w_0\cdot\check{\lambda},*} \underset{I}{\star} \mathbb{V}_{\text{crit}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\lambda}}_{w_0} \to \Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_1) \overset{\phi_1}{\to} \mathbb{M}_{1,\text{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}.$$

By construction, this map is obtained by applying the functor $j_{w_0\cdot\check{\lambda},*}\underset{I}{\star}?$ to the map

$$\mathbb{V}_{\text{crit}} \to \mathbb{M}_{w_0,\text{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle},$$

and it coincides with the map from (29) for $\check{\mu} = \check{\lambda} - \check{\rho}$. Hence, it is surjective by Sect. 4.12. □

4.15. **Completion of the proof of Theorem 3.2.** Thus, the proof of Theorem 1.7 is complete. Let us now finish the proof of the fact that the morphisms ϕ_w are actually isomorphisms and hence complete our proof of Theorem 3.2. Clearly, it is enough to do so for just one element of W. We shall give two proofs.

Proof 1. This argument will rely on Theorem 1.7. We will analyze the map ϕ_{w_0}. By [ABBGM], Proposition 3.2.5, the canonical map $\mathcal{F}_R \to \mathcal{F}_{w_0}$ identifies $\text{Ind}^{\text{Hecke}}(\delta_{1,\text{Gr}_G})$ with the co-socle of $\mathcal{F}_{w_0}$. Hence $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_{w_0})$ does not have sub-objects whose intersection with $\mathbb{V}_{\text{crit}} = \Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, R_{\mathfrak{Z}})$ is zero.

Therefore, to prove the injectivity of the map ϕ_{w_0}, it is enough to show that the composition

$$\mathbb{V}_{\text{crit}} \simeq \Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, R_{\mathfrak{Z}}) \to \Gamma^{\text{Hecke}_{\mathfrak{Z}}}(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_{w_0}) \overset{\phi_{w_0}}{\to} \mathbb{M}_{w_0,\text{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$$

is injective. However, the latter map is, by construction, the map $\mathbb{V}_{\text{crit}} \to \mathbb{M}_{w_0,\text{reg}} \otimes \omega_x^{\langle 2\rho,\check{\rho}\rangle}$ of [FG2], Sect. 17.2, which was injective by definition.

Proof 2. This argument will be independent of Theorem 1.7,(2). We will analyze the map ϕ_1. We have a canonical map

$$\text{IC}_{w_0\cdot\check{\rho},\text{Gr}} \star \mathcal{F}_{R_{\mathfrak{Z}}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\rho}}_{w_0} \to j_{w_0\cdot\check{\rho}} \underset{I}{\star} \mathcal{F}_{R_{\mathfrak{Z}}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\rho}}_{w_0} \to \mathcal{F}^{\mathfrak{Z}}_1,$$

and by [ABBGM], Propositions 3.2.6 and 3.2.10, its cokernel is partially integrable.

The composition

$$\Gamma(\text{Gr}_G, \text{IC}_{w_0\cdot\check{\rho},\text{Gr}}) \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\rho}}_{w_0} \simeq \Gamma^{\text{Hecke}_{\mathfrak{Z}}}\left(\text{Gr}_G, \text{IC}_{w_0\cdot\check{\rho},\text{Gr}} \star \mathcal{F}_{R_{\mathfrak{Z}}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \mathcal{L}^{-\check{\rho}}_{w_0}\right) \simeq \Gamma^{\text{Hecke}_{\mathfrak{Z}}}\left(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_1\right) \overset{\phi_1}{\to}$$
$$\to \mathbb{M}_{1,\text{reg}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \omega_x^{\langle 2\rho,\check{\rho}\rangle}$$

comes from the map

$$\Gamma(\text{Gr}_G, \text{IC}_{w_0\cdot\check{\rho},\text{Gr}}) \to \mathbb{M}_{1,\text{reg}} \underset{\mathfrak{Z}^{\text{reg}}_{\mathfrak{g}}}{\otimes} \omega_x^{\langle \rho,\check{\rho}\rangle},$$

of [FG2], Sect. 17.3, which is injective by *loc.cit.*

Hence, the kernel of the map ϕ_1 is partially integrable. But we claim that $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}\left(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_1\right)$ admits no partially integrable submodules.

Indeed, suppose that $\mathcal{N}$ is a submodule of $\Gamma^{\text{Hecke}_{\mathfrak{Z}}}\left(\text{Gr}_G, \mathcal{F}^{\mathfrak{Z}}_1\right)$, integrable with respect to a sub-minimal parahoric, corresponding to a vertex $\imath$ of the Dynkin diagram. Since the functor $j_{s_\imath,*}\underset{I}{\star}$ is invertible on the derived category, we would obtain a non-zero map:

$$j_{s_\imath,*} \underset{I}{\star} \mathcal{N} \to \text{L}\,\Gamma^{\text{Hecke}_{\mathfrak{Z}}}\left(\text{Gr}_G, j_{s_\imath,*} \underset{I}{\star} \mathcal{F}^{\mathfrak{Z}}_1\right).$$

But the LHS is supported in the cohomological degrees < 0, and the RHS is acyclic away from cohomological degree 0. [4] This is a contradiction.

[4] Here we are relying on part (1) of Theorem 1.7, which was proved independently.

This completes the proof of Theorem 3.2.

5. Appendix: an equivalence at the negative level

5.1. Let κ be a negative level, i.e., $\kappa = k \cdot \kappa_{\mathrm{can}}$ with $k + h^\vee \notin \mathbb{Q}^{\geq 0}$.

Let $\widetilde{\mathrm{Fl}}_G$ be the enhanced affine flag scheme, i.e, $G((t))/I^0$, and let $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa$–mod be the corresponding category of twisted D-modules.

Note that $\widetilde{\mathrm{Fl}}_G$ is acted on by the group $I/I^0 \simeq H$ by right multiplication. Let us denote by $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w}$ the corresponding category of weakly H-equivariant objects of $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa$–mod (see [FG2], Sect. 20.2).

For an object $\mathcal{F} \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w}$, consider $\Gamma(\widetilde{\mathrm{Fl}}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_\kappa$–mod. The weak H-equivariant structure on $\mathcal{F}$ endows $\Gamma(\widetilde{\mathrm{Fl}}_G, \mathcal{F})$ with a commuting action of H. We let

$$\Gamma^H : \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w} \to \widehat{\mathfrak{g}}_\kappa\text{–mod}$$

to be the composition of $\Gamma(\widetilde{\mathrm{Fl}}_G, \cdot)$, followed by the functor of H-invariants.

Recall from [FG2], Sect. 20.4, that every object of $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w}$ carries a canonical action of $\mathrm{Sym}(\mathfrak{h})$ by endomorphism, denoted $a^\sharp$.

For $\lambda \in \mathfrak{h}^*$ let

$$\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,\lambda} \subset \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w,\lambda}$$

be the full subcategories of $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w}$, corresponding to the condition that $a^\sharp(h) = \lambda(h)$ for $h \in \mathfrak{h}$ in the former case, and that $a^\sharp(h) - \lambda(h)$ acts locally nilpotently in the latter. Since the group H is connected, both of these categories are full subcategories in $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa$–mod.

We let $D(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod})^{H,w,\lambda} \subset D(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod})$ be the full subcategory consisting of complexes, whose cohomologies belong to $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w,\lambda}$. It is easy to see that the functor Γ^H, restricted to $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w,\lambda}$, extends to a functor

$$\mathrm{R}\,\Gamma^H : D^+(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod})^{H,w,\lambda} \to D^+(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

Assume now that λ satisfies the following conditions:

$$\begin{cases} \langle \lambda + \rho, \check{\alpha} \rangle \notin \mathbb{Z}^{\geq 0} \text{ for } \alpha \in \Delta^+ \\ \pm\langle \lambda + \rho, \check{\alpha} \rangle + 2n \frac{k + h^\vee}{\kappa_{\mathrm{can}}(\alpha,\alpha)} \notin \mathbb{Z}^{\geq 0} \text{ for } \alpha \in \Delta^+ \text{ and } n \in \mathbb{Z}^{>0}. \end{cases}$$

Following [BD], Sect. 7.15, we will prove:

Theorem 5.2.
(1) *For $\mathcal{F} \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{H,w,\lambda}$ the higher cohomologies $\mathrm{R}^i\,\Gamma^H(\widetilde{\mathrm{Fl}}_G, \mathcal{F})$, $i > 0$, vanish.*
(2) *The resulting functor $\mathrm{R}\,\Gamma^H : D^b(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod})^{H,w,\lambda} \to D^b(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is fully-faithful.*

5.3. Let $\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod}^{I^0,H,w,\lambda} \subset \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{–mod})^{H,w,\lambda}$ be the full subcategory, consisting of twisted D-modules, equivariant with respect to the I^0-action on the left. Our present goal is to describe its image under the above functor Γ.

Consider the category $\mathcal{O}_{\mathrm{aff}} := \widehat{\mathfrak{g}}_\kappa\text{–mod}^{I^0}$. This is a version of the category $\mathcal{O}$ for the affine Lie algebra $\widehat{\mathfrak{g}}_\kappa$. Its standard (resp., co-standard, irreducible) objects are numbered by weights $\mu \in \mathfrak{h}^*$, and will be denoted by $M_{\kappa,\mu}$ (resp., $M^\vee_{\kappa,\mu}$, $L_{\kappa,\mu}$). Since κ was assumed to be negative, every finitely generated object of $\mathcal{O}_{\mathrm{aff}}$ has finite length.

The extended affine Weyl group $W_{\mathrm{aff}} := W \ltimes \Lambda$ acts on $\mathfrak{h}^*$, with $w \in W \subset W_{\mathrm{aff}}$ acting as

$$w \cdot \mu = w(\mu + \rho) - \rho,$$

and $\check{\lambda} \in \check{\Lambda} \subset W_{\mathrm{aff}}$ by the translation by means of $(\kappa - \kappa_{\mathrm{crit}})(\check{\lambda}, \cdot) \in \mathfrak{h}^*$.

For a W_{aff}-orbit υ in $\mathfrak{h}^*$ let $(\mathcal{O}_{\mathrm{aff}})_\upsilon$ be the full-subcategory of $\mathcal{O}_{\mathrm{aff}}$, consisting objects that admit a filtration, such that all subquotients are isomorphic to $L_{\kappa,\lambda}$ with $\lambda \in \upsilon$.

The following assertion is known as the linkage principle (see [DGK]):

Proposition 5.4. *The category $\mathcal{O}_{\mathrm{aff}}$ is the direct sum over the orbits υ of the subcategories $(\mathcal{O}_{\mathrm{aff}})_\upsilon$.*

For λ as in Theorem 5.2 let $\upsilon(\lambda)$ be the W_{aff}-orbit of λ. (Note that by assumption, the stabilizer of λ in W_{aff} is trivial.)

We shall prove the following: [5]

Theorem 5.5. *The functor Γ^H defines an equivalence*

$$\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod}^{I^0,H,w,\lambda} \to (\mathcal{O}_{\mathrm{aff}})_{\upsilon(\lambda)}.$$

5.6. **Proofs.** To prove point (1) of Theorem 5.2, it suffices to show that $\mathrm{R}^i\,\Gamma^H(\widetilde{\mathrm{Fl}}_G, \mathcal{F}) = 0$ for $\mathcal{F} \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod}^{H,\lambda}$ and $i > 0$. However, this follows immediately from [BD], Theorem 15.7.6.

To prove point (2) of Theorem 5.2 and Theorem 5.5 we shall rely on the following explicit computation, performed in [KT]:

For an element $\widetilde{w} \in W_{\mathrm{aff}}$ let $j_{\widetilde{w},*,\lambda} \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod}^{I^0,H,\lambda}$ (resp., $j_{\widetilde{w},!,\lambda}$) be the *-extension (resp, !-extension) of the unique I^0-equivariant irreducible twisted D-module on the preimage of the corresponding I^0-orbit in Fl_G. We have:

Theorem 5.7. *We have:*

$$\Gamma(\mathrm{Fl}_G, j_{\widetilde{w},*,\lambda}) \simeq M^\vee_{\kappa,\widetilde{w}\cdot 0} \text{ and } \Gamma(\mathrm{Fl}_G, j_{\widetilde{w},!,\lambda}) \simeq M_{\kappa,\widetilde{w}\cdot 0}.$$

Let us now proceed with the proof of Theorem 5.2(2). Clearly, it is enough to show that for two finitely generated objects $\mathcal{F}, \mathcal{F}_1 \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod}^{H,\lambda}$ the map

$$\mathrm{R}\,\mathrm{Hom}_{D(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod})^{H,\lambda}}(\mathcal{F}, \mathcal{F}_1) \to \mathrm{R}\,\mathrm{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})}(\Gamma^H(\widetilde{\mathrm{Fl}}_G, \mathcal{F}), \Gamma^H(\widetilde{\mathrm{Fl}}_G, \mathcal{F}))$$

is an isomorphism.

By adjunction (see [FG2], Sect. 22.1), the latter is equivalent to the map

$$\mathrm{R}\,\mathrm{Hom}_{D(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod})^{I,\lambda}}(j_{1,!,\lambda}, \mathcal{F}^{op} \star \mathcal{F}_1) \to$$
$$\to \mathrm{R}\,\mathrm{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^{I,\lambda}}(\Gamma^H(\widetilde{\mathrm{Fl}}_G, j_{1,!,\lambda}), \mathrm{R}\,\Gamma^H(\mathrm{Fl}_G, \mathcal{F}^{op} \star \mathcal{F}_1))$$

being an isomorphism, where $\mathcal{F}^{op} \in \mathrm{D}(G((t))/K)\text{-mod}^{I,\lambda}$ is the dual D-module, where K is a sufficiently small open-compact subgroup of $G[[t]]$.

Using the stratification of $\widetilde{\mathrm{Fl}}_G$ by I-orbits, we can replace $\mathcal{F}^{op} \star \mathcal{F}_1$ by its Cousin complex. In other words, it is sufficient to show that

$$\mathrm{R}\,\mathrm{Hom}_{D(\mathrm{D}(\widetilde{\mathrm{Fl}}_G)_\kappa\text{-mod})^I}(j_{1,!,\lambda}, j_{\widetilde{w},\lambda,*}) \to \mathrm{R}\,\mathrm{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^I}(\Gamma^H(\widetilde{\mathrm{Fl}}_G, j_{1,!,\lambda}), \Gamma^H(\widetilde{\mathrm{Fl}}_G, j_{\widetilde{w},\lambda,*}))$$

is an isomorphism, for all $\widetilde{w}$ such that $j_{\widetilde{w},\lambda,*}$ is (I,λ)-equivariant.

Note that the LHS is 0 unless $\widetilde{w} = 0$, and is isomorphic to $\mathbb{C}$ in the latter case. Hence, taking into account Theorem 5.7, it remains to prove the following:

[5] This theorem is not due to the authors of the present paper. The proof that we present is a combination of arguments from [BD], Sect. 7.15, and [KT].

Lemma 5.8.
(1) $\mathrm{R}\operatorname{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^{I,\lambda}}(M_{\kappa,\lambda}, M^\vee_{\kappa,\mu}) = 0$ *for* $\lambda \neq \mu \in \mathfrak{h}^*$ *but such that* $M^\vee_{\kappa,\mu} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{I,\lambda}$ *is* (I,λ)*-equivariant.*
(2) *The map* $\mathbb{C} \to \mathrm{R}\operatorname{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^{I,\lambda}}(M_{\kappa,\lambda}, M^\vee_{\kappa,\lambda})$ *is an isomorphism.*

Proof. For any $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{I,\lambda}$,

$$\mathrm{R}\operatorname{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^{I,\lambda}}(M_{\kappa,\lambda}, \mathcal{M}) \simeq \mathrm{R}\operatorname{Hom}_{I\text{-mod}}(\mathbb{C}, \mathcal{M} \otimes \mathbb{C}^{-\lambda}).$$

Since $M^\vee_{\kappa,\mu}$ is co-free with respect to I^0, we obtain

$$\mathrm{R}\operatorname{Hom}_{I\text{-mod}}(\mathbb{C}, M^\vee_{\kappa,\mu} \otimes \mathbb{C}^{-\lambda}) \simeq \mathrm{R}\operatorname{Hom}_{H\text{-mod}}(\mathbb{C}, \mathbb{C}^{\mu} \otimes \mathbb{C}^{-\lambda}),$$

implying the first assertion of the lemma.

Similarly,

$$\mathrm{R}\operatorname{Hom}_{D(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})^{I}}(M_{\kappa,\lambda}, M^\vee_{\kappa,\lambda}) \simeq \mathrm{R}\operatorname{Hom}_{I\text{-mod}}(\mathbb{C}, M^\vee_{\kappa,\lambda}) \simeq \mathrm{R}\operatorname{Hom}_{H\text{-mod}}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C},$$

implying the second assertion.

□

Finally, let us prove Theorem 5.5. Taking into account Theorem 5.2, and using Lemmas 3.10 and 3.12, it remains to show that for every $\mathcal{M} \in (\mathcal{O}_{\mathrm{aff}})_{\upsilon(\lambda)}$ there exists an object $\mathcal{F} \in \mathrm{D}(\widetilde{\mathrm{Fl}}_G)\text{-mod}^{I^0,H,w,\lambda}$ with non-zero map

$$\Gamma^H(\widetilde{\mathrm{Fl}}_G, \mathcal{F}) \to \mathcal{M}.$$

It is clear that for every $\mathcal{M} \in (\mathcal{O}_{\mathrm{aff}})_{\upsilon(\lambda)}$ there exists a Verma module $M_{\kappa,\mu} \in (\mathcal{O}_{\mathrm{aff}})_{\upsilon(\lambda)}$ with a non-zero map $M_{\kappa,\mu} \to \mathcal{M}$. Hence, the required property follows from Theorem 5.7.

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